A Research Program for a Big Picture of Mathematics

I share 23 research projects that would help me understand the big picture in mathematics.

Overview and Compare Similar Efforts

Most working mathematicians have not spent a few hundred hours in their life searching for a key by which to understand all of mathematics. Indeed, it has been said that Henri Poincare (1854-1912) was the last person to excel at all of the mathematics of his time.

1) One step is to overview the history of mathematics to glean insights from the research interests of the most profound mathematicans, such as Euclid, Descartes, Leibnitz, Pascal, Hilbert, von Neumann and more recently, Weyl, Atiyah, Conway, Grothendieck, Langlands, Lurie to imagine their perspectives on the big picture.

When I was a graduate student (1986-1993), an interest in the big picture was quite taboo, and moreover, quite impractical, given that the way to learn math was to take classes, read textbooks, do exercises, and read journal articles. However, since then, much has changed which has made it possible to learn advanced mathematics much more personally, intuitively, selectively and comprehensively. I can read a vast mathematical encyclopedia (Wikipedia), watch video lectures (You Tube) by expert thinkers on the most advanced subjects, and ask questions and get answers at Math Stack Exchange or Math Overflow.

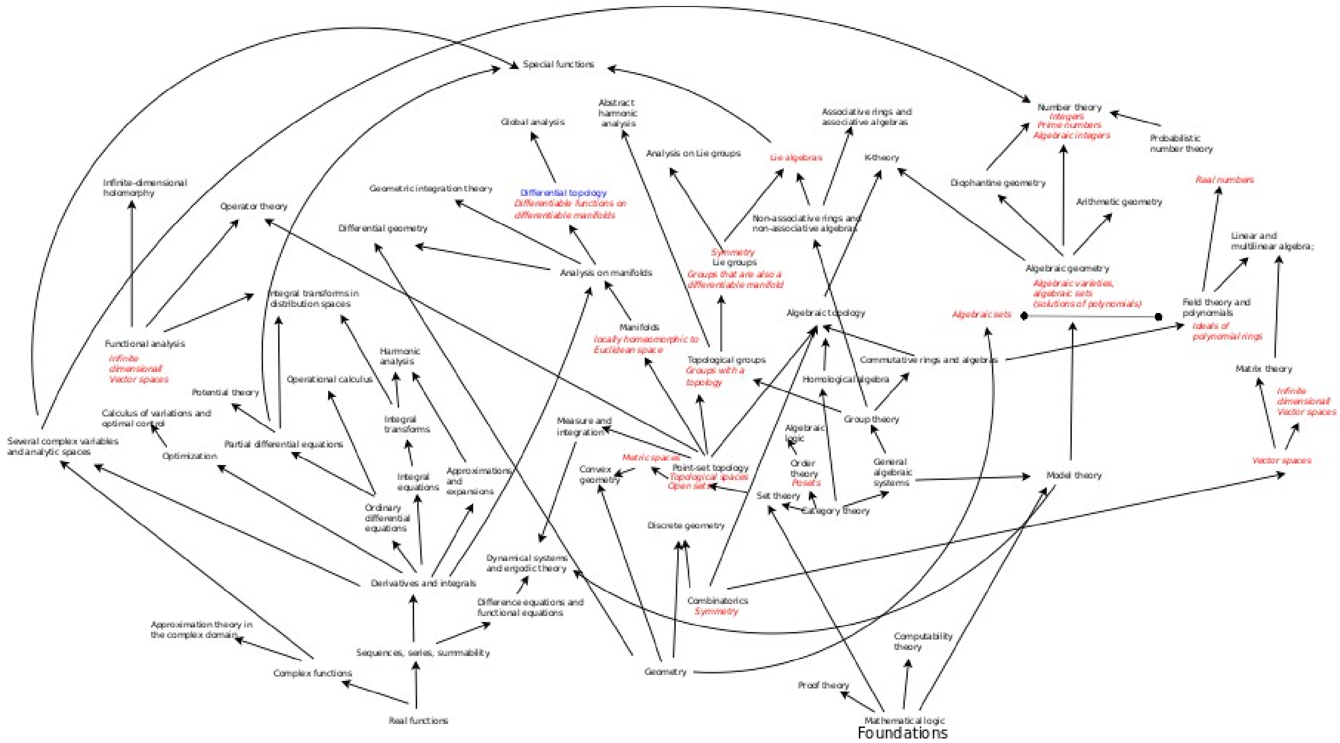
Of special importance are math bloggers who are sharing their personal intuitions regarding math. It is most strange that intuition is acknowledged as the key to learning and furthering math, and yet articulating, documenting and studying that intuition is considered out of bounds, as can be seen from the little space devoted to it in any article or textbook. The reason, I suppose, is that we would have to reveal our general ignorance. It is particularly refreshing and encouraging to read blog posts by John Baez, Urs Schreiber, Terrence Tao, Qiaucho Yuan and others who do seem to grapple with the big picture.

2) A further step is to note the areas and structures which such bold thinkers believe to be fruitful. Succinctly, as I learn from thinkers such as Olivia Caramello, Urls Schreiber, John Baez, Roger Penrose, Lou Kauffman, Vladimir Voevodsky, Saunders MacLane, William Lawvere, John Isbell, Harvey Friedmann, Joseph Goguen, Robert May, Kirby Urner, Maria Droujkova it seems that they focus on particular areas, such as category theory, topoi, algebraic geometry, homology, homotopy, string theory, network theory but also that they are intrigued by particular structures which seem exceptionally rich, such as the octonions...

Of course, I do not intend to master these subjects in the usual way. Instead, I hope to be clever enough to find a new way of looking at math which shares and yields mathematical intuition much more readily.

Make a Map of Mathematics

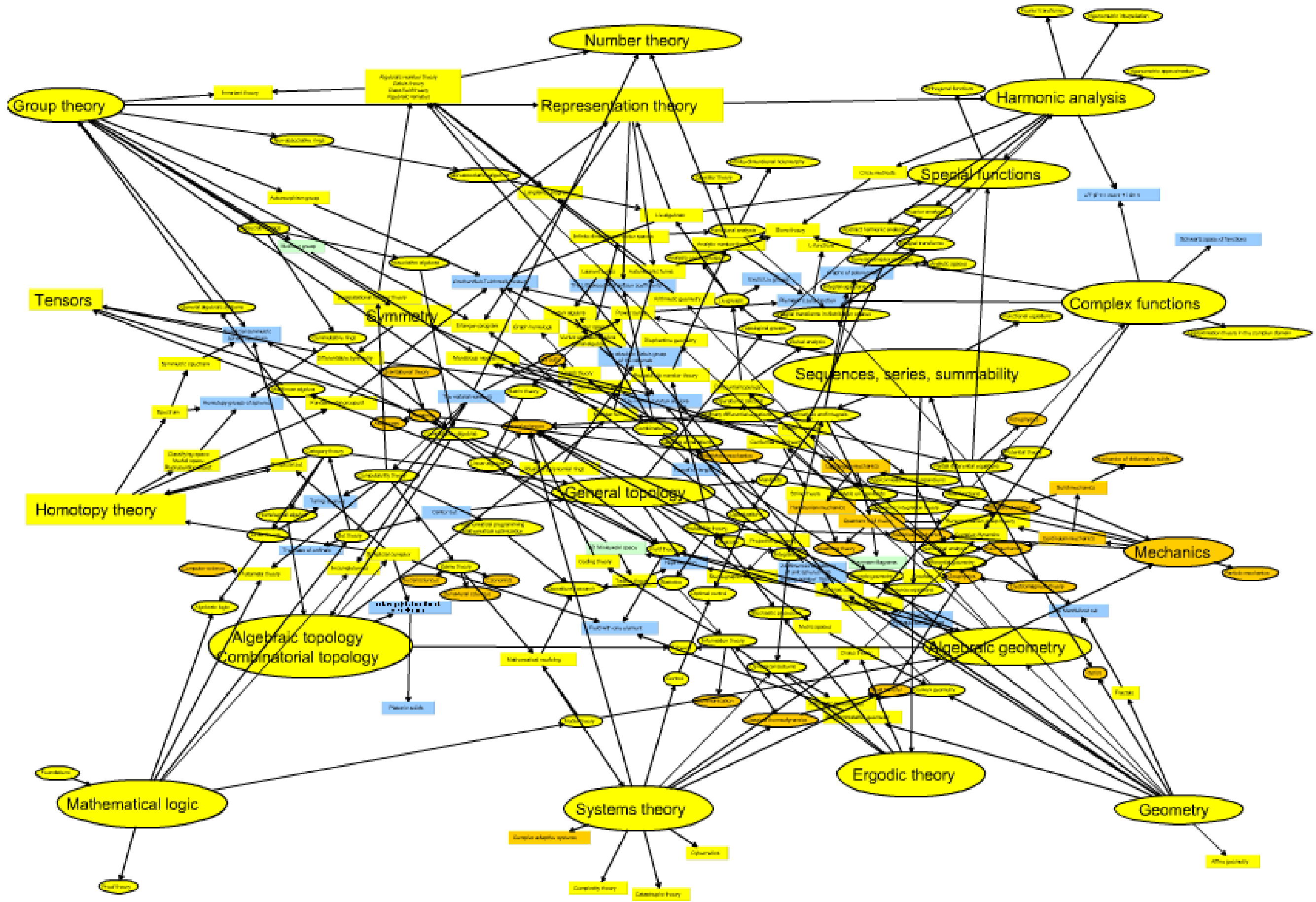
3) One of my goals is to be able to make a map of how mathematical subjects, concepts and objects become relevant. Such a map would systematize existing mathematics, identify overlooked mathematics, and show the directions in which math can evolve in the future.



I started by organizing the subjects listed in the Mathematics Subject Classification by trying to show which areas depend on which other areas. Acknowledging my general ignorance, I was able to draw several conclusions.

As expected, there do seem to be two major areas, algebra and analysis. The capstone of math seems to be number theory, which makes use of tools from all of math. Lie theory seems especially central as a bridge between algebra and analysis.

Surprisingly for me, geometry seems to be a well spring for math. I studied algebraic combinatorics as "the basement of math" from which I thought mathematical objects arose. Geometry thus seemed rather idiosyncratic. But from the map it seems that geometry is a key ingredient in math, in terms of its content, perhaps in the way that logic is, in terms of its form.

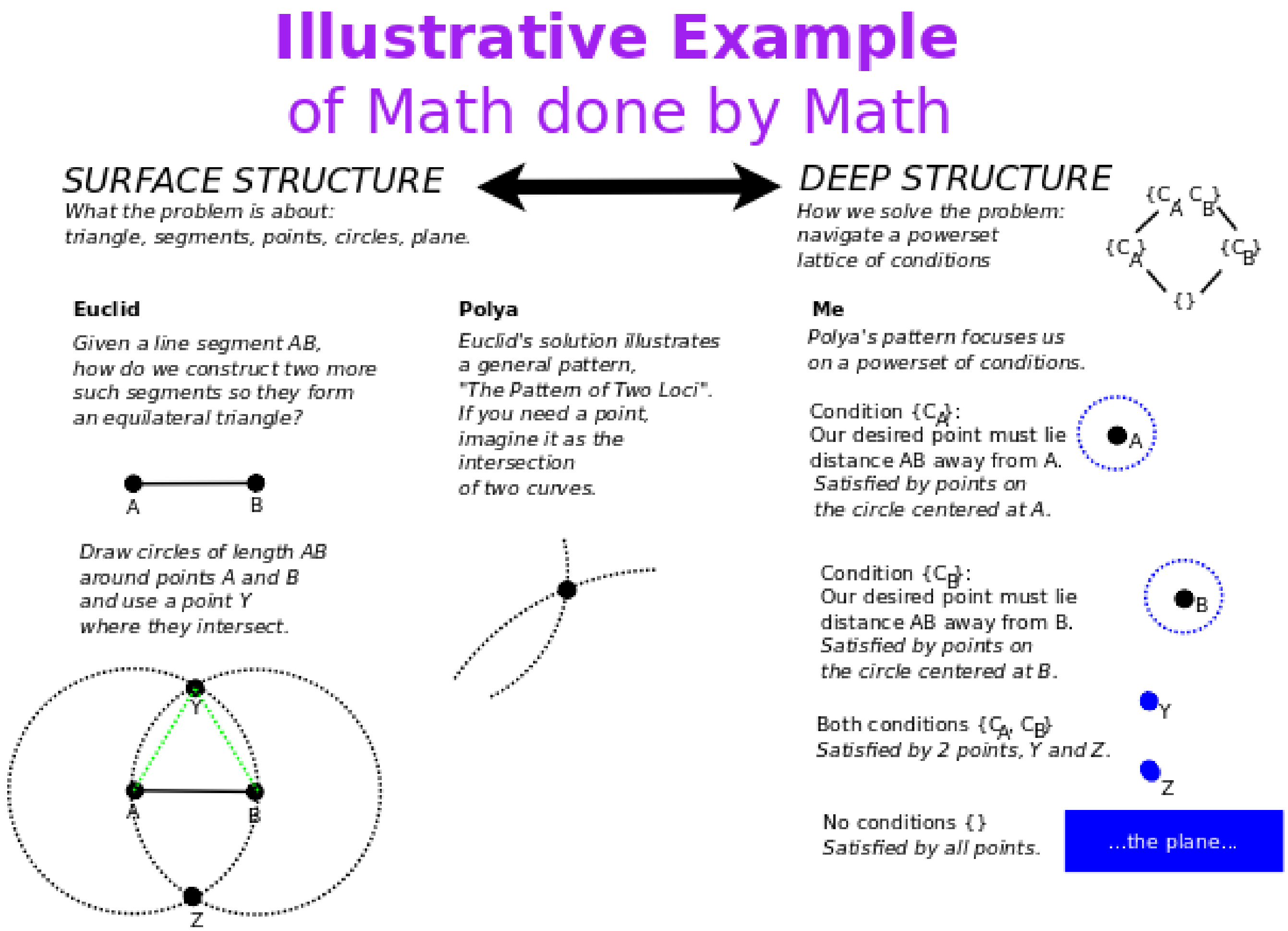


I then tried to improve my map by adding more detail. I used the graphic editor yEd. This simply yielded a spaghetti diagram. However, I am hopeful that ultimately it should be possible to discover principles for making a meaningful map and collaborating with others to make it a comprehensive resource related to Wikipedia and MathStackExchange/MathOverflow.

4) Another way to build a map is to use the tags from MathOverflow. The idea is to make a list of the, say, X=100 most popular tags, and also to make a list of the most popular pairs of tags, where pairs are created for any two tags that are used for the same post. In the map, for each popular tag, I would show a link to its most popular pair, and also include, say, the most popular 2X links overall.

Study the Ways of Figuring Things Out in Mathematics

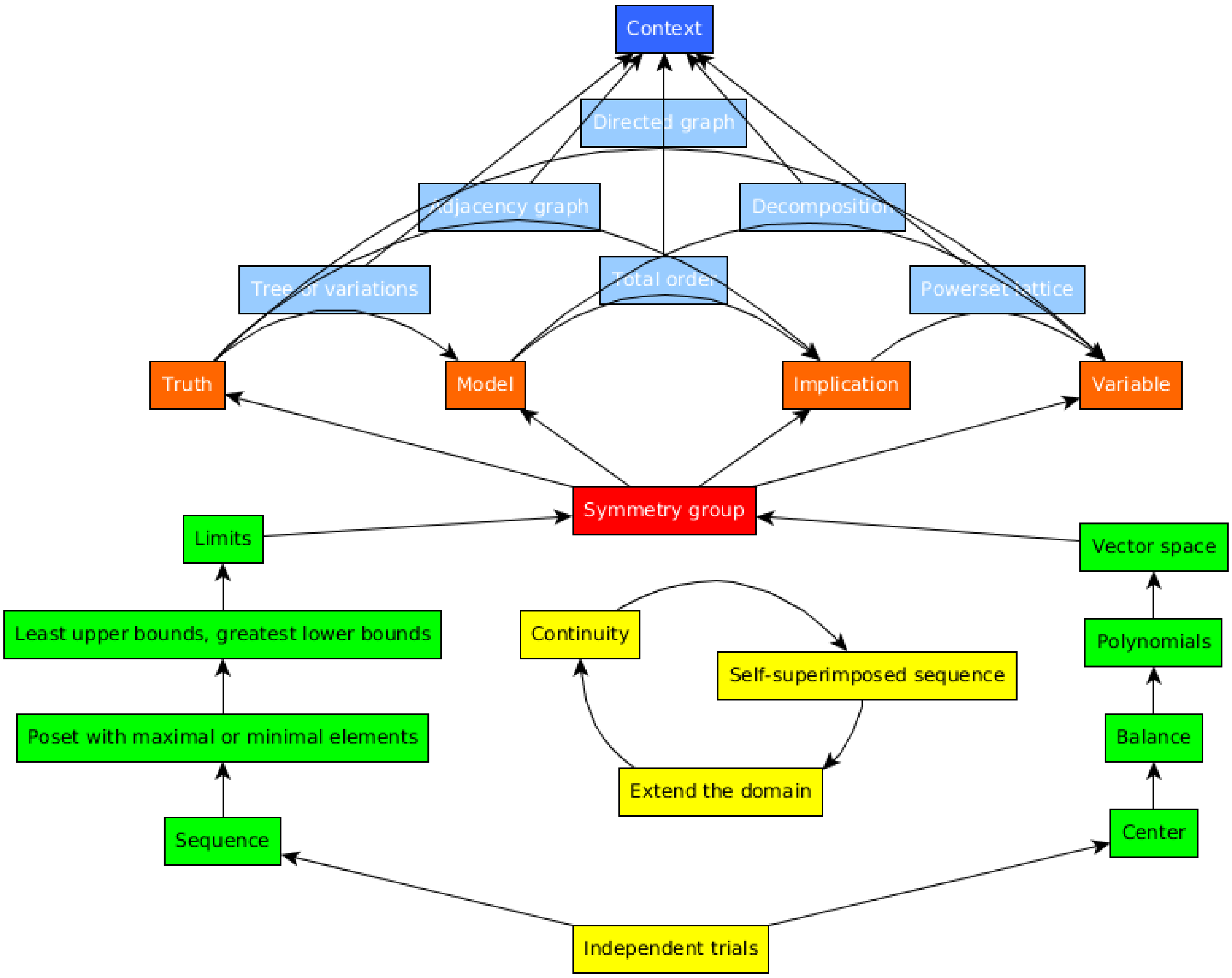
I am most encouraged by my study in 2011 of the ways of figuring things out in mathematics which I shared in this letter to the Math Future online group. Here is an extended version of the results which I presented in 2016 at the Lithuanian Mathematics Association Conference: Discovery in Mathematics: A System of Deep Structure.
The basic idea came in considering George Polya's "pattern of 2 loci" by which he solves Euclid's first problem of how to construct an equilateral triangle. We solve this problems in our minds by constructing a lattice of conditions. Given two points, the third point that we want to construct must satisfy two conditions, namely, it must be on both of two circles centered on the two other points, thus it must be at their intersection. The solution is clear as soon as our minds apply the relevant structure, namely, the lattice of conditions.



This example suggests a distinction between deep structure - the natural mathematical structures which we use in our mind to solve problems, and surface structure - the contrived math by which we describe problems on paper. I thus surveyed the problem solving patterns taught by Paul Zeitz, George Polya and others in their books.

5) Collecting and analyzing such examples could be a collaborative effort. Here is a database I made of almost 200 examples of figuring things out in math.

6) I used my philosophical structures to systematize the recurring patterns. This yielded the following diagram. I would like to sharpen the results.



The lower half of the diagram grounds the mathematical thinking which is pre-systemic. The upper half of the diagram grounds that which takes place within a mathematical system.

7) The axioms of Zermelo Frankel set theory (except for the Axiom of Infinity) and the Axiom of Choice are all present in the above system and so I would like to work further to clarify their role.

Of special interest to me, currently, is to study the four concepts (in orange) that seem to ground logic but also geometry. These methods apply the concepts of truth (argument by contradiction), model (solving an easier version), implication (working backwards) and variable (classifying the problem).

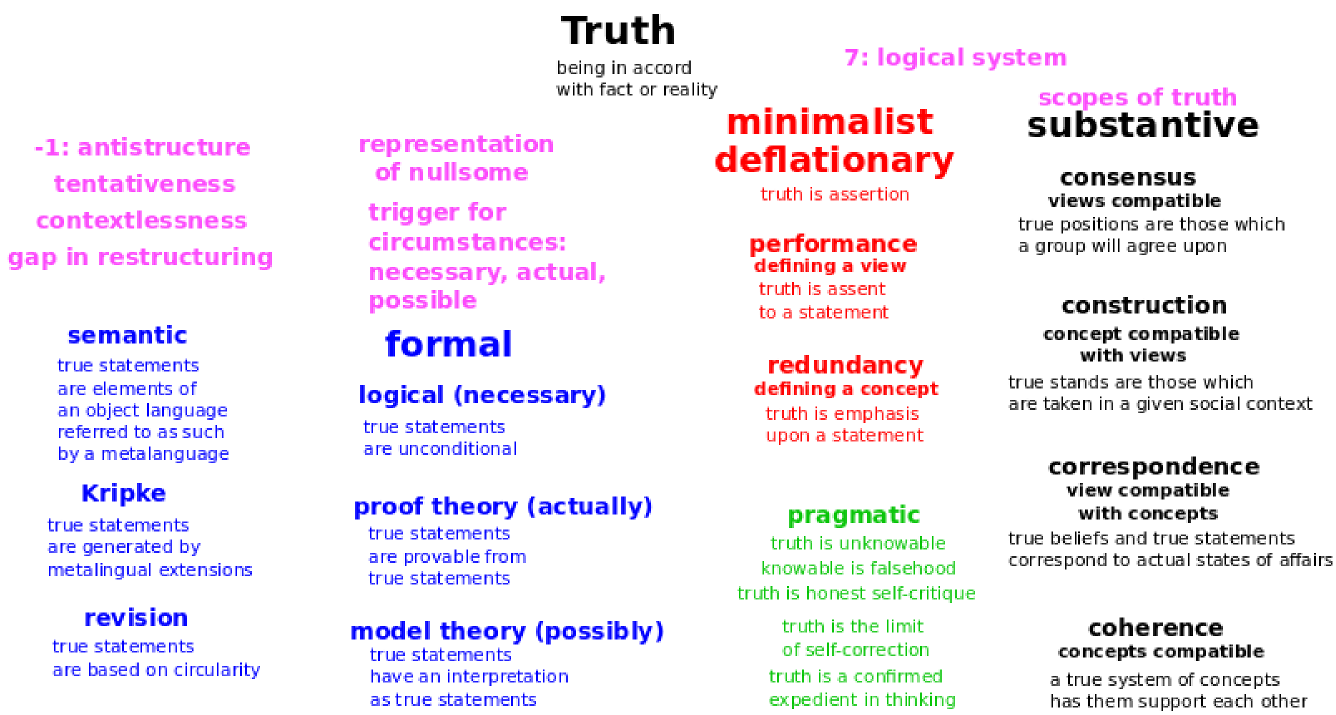
Understand the Basics of Logic and Truth

I would like to learn more what logic is all about, in practice. I have taken the mathematical logics course, am familiar with Goedel's theorems and have done graduate study in recursive function theory.

8) However, I want to be able to describe the cognitive foundations that account for logic.

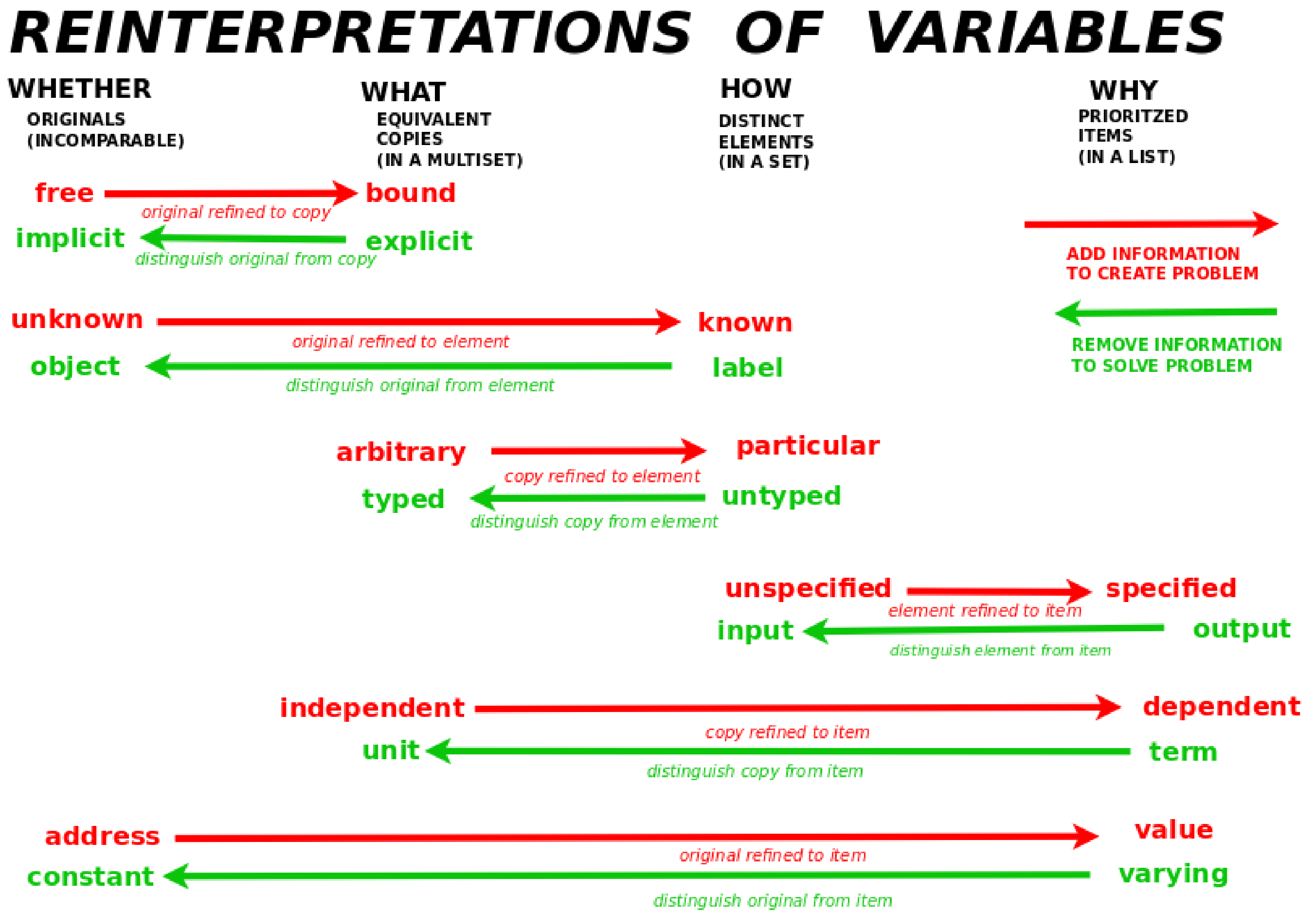
9) I have made some progress in describing such foundations for truth: Truth as the Admission of Self-

[Contradiction](). Which is to say, truth is inherently unstable and tentative, the relation of a level with a metalevel.



Analyze How We Use Variables

My understanding is that there are four levels of knowledge (whether, what, how, why) and that in thinking in a mathematical system we establish a level (surface structure) and a metalevel (deep structure). Our use of variables plays off this distinction. I thus made a diagram of the roles that we imagine variables to play.



10) I need to validate this further. I expected there to be six kinds of variables but instead found evidence for twelve kinds. It seems that, on the one hand, variables are used to solve problems, but on the other hand, variables are used to create problems.

Indeed, this points to a key aspect missing in my research so far. Much of advanced mathematics is about abstraction, the creation of frameworks. But it is clear that this process of abstraction is not arriving at cognitive foundations but rather is growing ever more rich, complex and distant.

11) I would like to understand the various kinds of opposites in math and classify them.

12) I would like to learn more about the kinds of equivalences in math - I know that Voevodsky, etc. have studied that deeply - and draw on that and perhaps contribute.

Study the Process of Abstraction

13) Thus it is important to study the process of abstraction. One approach is to try to describe, in an elegant way, a theory that is practically complete, such as the geometry of triangles in the plane. Norman Wildberger's book and videos are very helpful for this. It may be that a matrix approach might be insightful. Having stated a theory it may be possible to see in what directions it develops further.

14) Another approach is to identify classic theorems in the history of mathematics and consider how

abstraction and generalization drove them to arise and develop further.

Abstraction may relate to the disembodying mind. Lakoff, Nunez and others have collected much evidence to show the importance of "the embodied mind". However, this same evidence can be used to think about a "disembodying mind". Evolutionary processes are favoring central nervous systems which have been developing to live in increasingly abstract worlds: first icons (sensory images), then indices (models of attention, as noted by Graziano) but ultimately symbols (which function by dividing up the global workspace).

15) One place to look for the cognitive foundations of mathematics is to develop models of attention, for example, in terms of category theory.

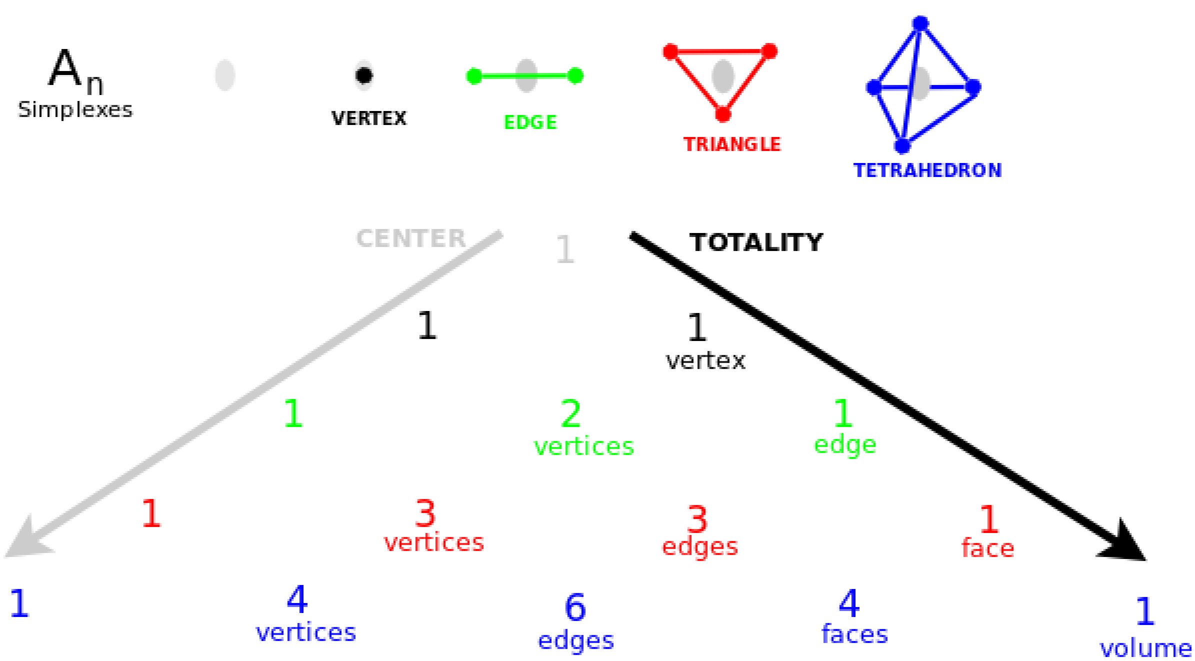
Discover Cognitive Foundations for the Classical Lie Groups/Algebras

16) It is surprising that in mathematics there is a small collection of structures which seem most rich in content. This is a point that Urs Schreiber keeps returning to. Thus one task is to make a list of such structures and try to relate them with a map, and indeed, understand how they fit in a map of all math.

In particular, John Baez and others have pointed out that the classical Lie algebras ground different geometries. I would like to learn the basics of affine, projective, conformal and symplectic geometries so that I could understand how they relate to the four classical groups.

17) In particular, I am interested in understanding, intuitively, [the cognitive foundations for the four classical Lie groups/algebras]. I have been learning about the classification through the Dynkin diagrams. But that does not explain intuitively the qualitative distinctions. So instead I have been working backwards, from the Cartan diagrams, trying to understand concretely how to imagine the growth of a chain (how it ever adds a dimension via an angle of 120 degrees) and the possible ways that chain might end.

18) I am

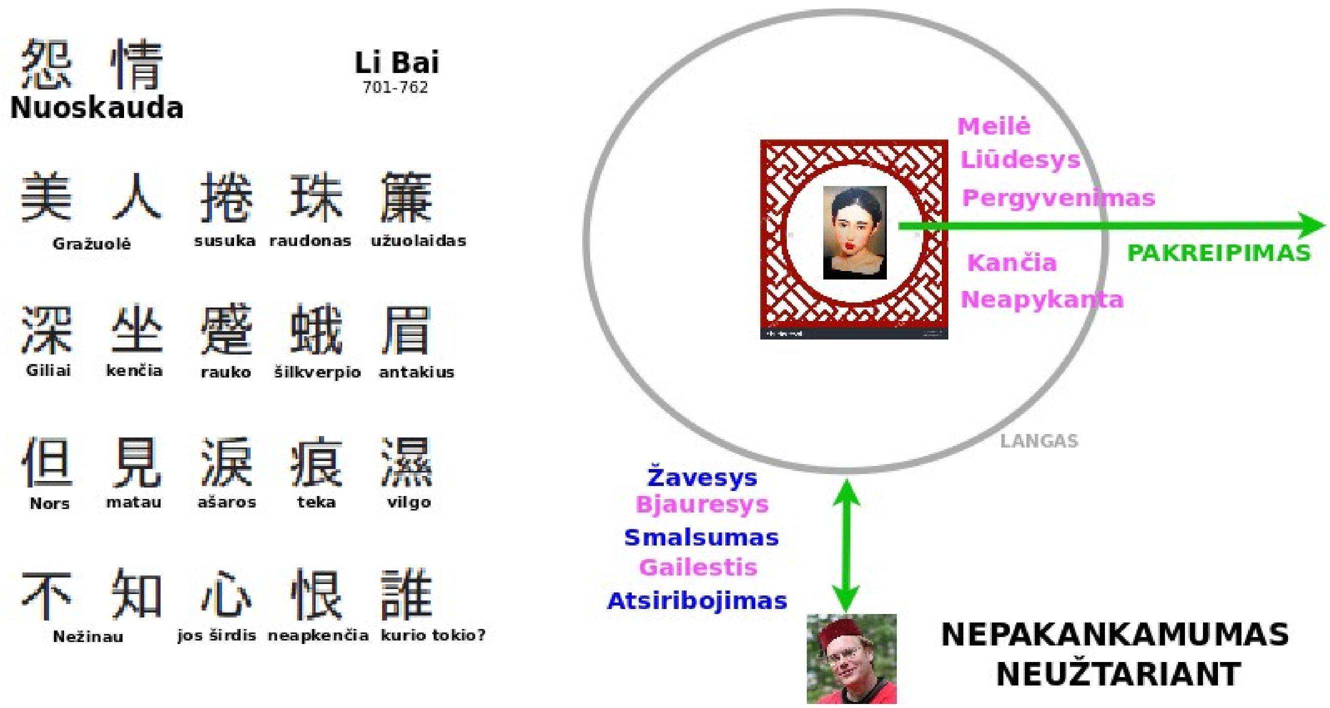


encouraged that I myself have made some mathematical discoveries by focusing on these questions. I have thought a lot about the regular polytopes which the Weyl groups are symmetries of. In particular, I was able to come up with [an interpretation for the -1 simplex] and a novel [q-analogue of the simplex].

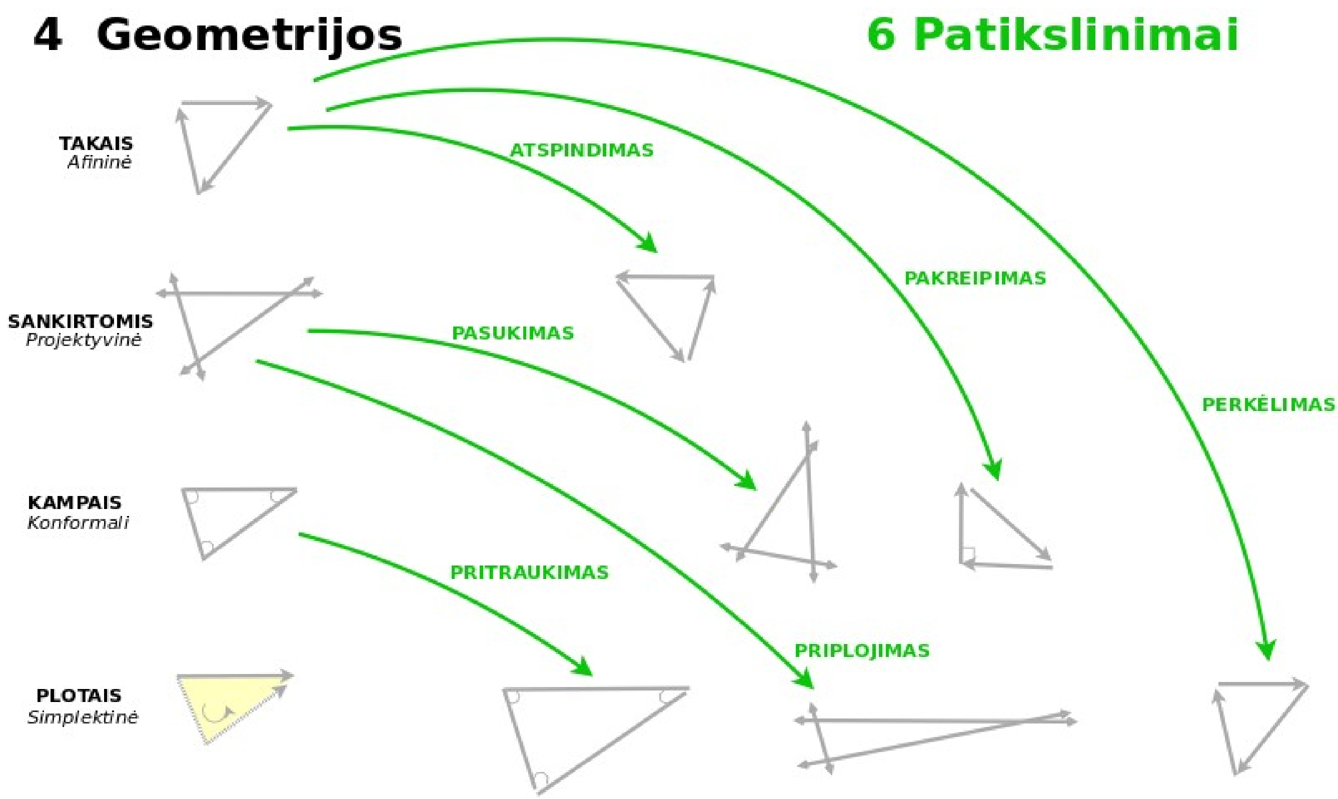
19) I am also seeing how the polytopes can be thought to arise by a "center" which ever generates vertices (for simplices), pairs of vertices (for cross-polytopes), planes (for hypercubes) and "coordinate systems" (for demicubes). This type of process is very relevant for my theological ideas, see: God's Question: Is God Necessary? In particular, I think about the "field with one element" as being interpretable as 0, 1 and infinity.

Study the Geometry of Moods

In my study of emotions and moods, I have successfully linked my philosophical and mathematical research. My model of basic emotions is based on whether our expectations are satisfied. Of special importance is the boundary between self and world. For example, if we discover that we are wrong about the world, or anything peripheral, then we may feel surprised, but if we learn that we are wrong about ourselves, or something deeply important, then we may feel distraught. See my talk: A Research Program for a Taxonomy of Moods.



I did a study of some thirty classic Chinese poems from the Tang dynasty to explain the moods they evoked. (In Lithuanian: Nuotaikų aplinkybės: Tang dinastijos poezija ir šiuolaikinė geometrija.) I discovered that the mood depended on how the poem transformed the boundary between self and world. Each of them applied one of six transformations (reflection, shear, rotation, dilation, squeeze, translation) which shifted the geometry from a cognitively simpler one to a cognitively richer one (path geometry - affine, line geometry - projective, angle geometry - conformal, area geometry - symplectic).



20) I would like to better understand these geometries by learning about the math but also by seeing what they should be given the data from intepreting such poems. I made a related post at Math Stack Exchange: [Is this set of 6 transformations fundamental to geometry?]

21) This emotional theory describes beauty as arising upon the disappearance of one's inner self whereby disgust becomes impossible. It would be meaningful to study what is beautiful in mathematics and why.

Pursuing Connections Between Philosophical Structures and Mathematical Structures

22) Most interesting, most fruitful and most speculative would be for me to look for connections between the philosophical structures I work with and what seem to be related mathematical structures.

Such connections include:
* The state of contradiction <=> God
* The Field with One Element <=> God
* The center of a regular polytope <=> God
* The totality of a regular polytope <=> Everything
* Exact sequences of length n <=> Divisions of everything into N perspectives
* Bott Periodicity <=> The eight-cycle of divisions of everything
* The Snake Lemma <=> The eightfold way.
* The octonions <=> The eightfold way.
* 4 geometries & 6 transformations between them <=> The Ten Commandments (4 positive and 6 negative)

23) I would like to learn about the combinatorics of finite fields and consider what that might mean for F1^n. What might infinity mean for finite fields and F1? How do zero, one and infinity get differentiated?

A brief account of the role of "divisions of everything" in my metaphysics is: Divisions of Everything: Defining the Most Basic Definitions.

Conclusion

I have listed 23 research projects that reflect my current interests in my attempt to understand the big picture in mathematics. Certainly, one year from now, my list will have changed. Currently, I would be especially interested to make progress on the connections between philosophical and mathematical structures (#22), to learn more the four geometries - affine, projective, conformal, symplectic (#20), and to do some mathematical explorations regarding concrete foundations for the four classical Lie algebras/groups (#17) and finite fields and F1^n (#23). These problems should inspire me to work on the overall big picture including key ideas (#1 and #2), key structures (#16) and the overall map (#3).

My reflections above on the big picture in mathematics are based on about 600 hours of work since 2016 and about 100 hours in 2011 on identifying and systematizing the ways of figuring things out in mathematics.

Given the opportunity to devote a month of research on this agenda, I expect to make great advances during my visit but also before and after. I will also benefit from acknowledgement of my efforts by the philosophy of mathematics community.