UNIVERSITY OF CALIFORNIA, SAN DIEGO

Symmetric Functions of the Eigenvalues of a Matrix

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Mathematics

by

Andrius Kulikauskas

Committee in charge:

Professor Jeffrey B. Remmel, Chair
Professor Jim Agler
Professor Norman H. Anderson
Professor Ronald J. Evans
Professor Ramamohan Paturi

1993

To my grandmother,

Marija Kapačinskienė.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

ACKNOWLEDGEMENTS

I thank my advisor, Jeff Remmel, for taking me as his student. He spent many hours with me in conversation and encouraged me at key moments of the research. I am especially thankful that he accommodated my philosophical interests, not only when I pursued mathematics, but also when I spent time away from it.

I thank the many teachers who helped me along the way. Jim Agler, Norman Anderson, Ron Evans, and Ramamohan Paturi served on my committee. Adriano Garsia and Michael Saks likewise contributed their energy and time. Nolan Wallach answered many algebraic questions.

I thank the University of Vilnius, Lithuania, which provided me with a year of support for pursuing independent research. I am grateful to Rolandas Pavilionis, who not only served as my advisor, but also defended me from the KGB.

The greatest excitement that I have found in graduate school has come from my fellow students. Joaquin Carbonara guided me through many of the mysteries of the symmetric functions and Macdonald's book. John Harland and I had many discussions about the foundations of mathematics. I enjoyed conversations on many subjects of life with Shuhong Zhu, Žilvinas Beliauskas, and Giedrius Buračas. I also acknowledge the great impact of Regina Dringelienė on my studies, both in her presence and in her absence.

I thank the University of California, San Diego, for its consistent generosity throughout the seven years that it has taken for me to complete my thesis. I thank the Powell Foundation for supporting me during my first year, and I thank my parents for supporting me during my final year. Most of all, I thank my parents, Edmund and Rūta, for the love and care that they have shown me throughout each of the twenty-three years of my education.

VITA

December 17, 1964	Born, Santa Monica, California
1986	B.S., B.A., University of Chicago
1986 - 1992	Teaching Assistant, Department of Mathematics, University of California, San Diego.
1987	Powell Foundation Fellow
1988	M.A., University of California, San Diego
1988 - 1989	University of Vilnius, Lithuania
1992	Associate-in-Mathematics, University of California, San Diego
1993	Ph.D., Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

Symmetric Functions of the Eigenvalues of a Matrix

by

Andrius Kulikauskas

Doctor of Philosophy in Mathematics

University of California, San Diego, 1993

Professor Jeffrey B. Remmel, Chair

Consider a matrix $\mathbf{A} \in \mathrm{GL}_N(\mathbb{C})$ with eigenvalues $\xi_1,\dots,\xi_N$. This thesis provides combinatorial interpretations for six bases of the ring of symmetric functions when they are evaluated at $\xi_1,\dots,\xi_N$ in terms of the entries of $\mathbf{A} = \left(a_{ij}\right)_{1\le i,j\le N}$. For each basis the usual definition is then recovered upon setting $a_{ij} = 0$ whenever $i \ne j$, so that $\xi_1,\dots,\xi_N$ are given by $a_{11},\dots,a_{NN}$. The bases considered are the elementary $\{e_\lambda\}$, power $\{p_\lambda\}$, homogeneous $\{h_\lambda\}$, forgotten $\{f_\lambda\}$, monomial $\{m_\lambda\}$ symmetric functions, and the Schur functions $\{s_\lambda\}$.

The equation $h_n(\xi_1,\dots,\xi_N) = \frac{1}{n!}\sum_{\sigma\in S_n} p_\sigma(\xi_1,\dots,\xi_N)$ yields an original interpretation for $h_n(\xi_1,\dots,\xi_N)$ in terms of multisets of Lyndon words on $1,\dots,N$. The same method describes $f_\lambda(\xi_1,\dots,\xi_N)$ in terms of multisets of Lyndon words on an alphabet $\overline{1},\dots,\overline{N},1,\dots,N$ of upper and lower case letters such that λ records the distances between the upper case letters. A similar approach expresses $m_\lambda(\xi_1,\dots,\xi_N)$ as those terms from

the determinant of the walk matrix $\det(\mathbf{I}/(\mathbf{I}-\mathbf{A}))$ for which the lengths of the walks involved are given by λ.

Theorem 4.3.1 expresses $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux. Each hook R_i is associated with a Lyndon word ℓ_i, and if $\ell_i = \ell_j$, $i<j$, then the start of R_j is in a column to the right of the start of R_i. This generalizes the fact that s_λ generates column strict tableaux. The proof depends on lemmas which relate rim hook tableaux and special rim hook tabloids. Also included is a new formula $s_\lambda(\xi_1,\ldots,\xi_N) = \det\left(\left(\mathbf{A}^{\lambda_i+N-i}\right)_{jj}\right)_{1\le i,j\le N} \Big/ \det\left(\left(\mathbf{A}^{N-i}\right)_{jj}\right)_{1\le i,j\le N}$.

Evaluating symmetric functions at eigenvalues may provide a unifying framework for disparate results in combinatorial matrix algebra. In such a context, the techniques used to interpret $ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}$ are the same as those used to prove the Cayley-Hamilton theorem, and those used to interpret $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$ are those used to prove the MacMahon Master theorem. In a sense, evaluating symmetric functions at eigenvalues establishes a Master theory because $\mathbf{A}$ may be specialized in profitable ways. For example, if $\mathbf{A}$ is in rational canonical form, then the expansion of a symmetric function F in terms of $\{e_\lambda\}$ is embedded in the expression for $F(\xi_1,\ldots,\xi_N)$.

The Appendix contains a new and simplest known proof that $s_\lambda(x_1,\ldots,x_N) = \det\left(x_j^{\lambda_i+N-i}\right)_{1\le i,j\le N} \Big/ \det\left(x_j^{N-i}\right)_{1\le i,j\le N}$ generates column strict tableaux.

CHAPTER 1

INTRODUCTION

The determinant $\det(\mathbf{A})$ of an $N \times N$ matrix $\mathbf{A} \in GL_N(\mathbb{C})$ is the product of its eigenvalues $\xi_1,\ldots,\xi_N$. The trace $\mathrm{tr}(\mathbf{A})$ of $\mathbf{A}$ is the sum of its eigenvalues. These are symmetric functions $e_N(x_1,\ldots,x_N) = x_1 x_2 \cdots x_N$ and $e_1(x_1,\ldots,x_N) = x_1 + x_2 + \ldots + x_N$ that have been evaluated at the eigenvalues $\xi_1,\ldots,\xi_N$. Symmetric functions are very rich in combinatorics. This thesis evaluates six bases of the ring of symmetric functions at the eigenvalues $\xi_1,\ldots,\xi_N$ of an $N \times N$ matrix $\mathbf{A} = \{a_{ij}\}_{1 \le i,j \le N}$ and considers the combinatorial objects that they generate.

From the point of view of algebra, the problem is straightforward. A symmetric function $F(x_1,\ldots,x_N) \in \mathbb{Z}[x_1,\ldots,x_N]$ is a polynomial with the property that $F(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = F(x_1,\ldots,x_N)$ for all permutations $\sigma \in S_N$. The fundamental theorem of symmetric functions states that any symmetric function can be expanded in terms of elementary symmetric functions $e_\lambda = \prod_{i=1}^{r} e_{\lambda_i}$, $\lambda = (\lambda_1,\ldots,\lambda_r)$, $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r > 0$, where $e_n(x_1,\ldots,x_N) = \sum_{1 \le i_1 < \ldots < i_n \le N} x_{i_1} x_{i_2} \cdots x_{i_n}$. It therefore suffices for us to evaluate the elementary symmetric functions $e_n(\xi_1,\ldots,\xi_N)$, $1 \le n \le N$. But these are generated by the function $\det(\mathbf{I} + x\mathbf{A}) = \prod_{i=1}^{N}(1 + x\xi_i) = \sum_{n=0}^{N} x^n e_n(\xi_1,\ldots,\xi_N)$. The coefficient of x^n in

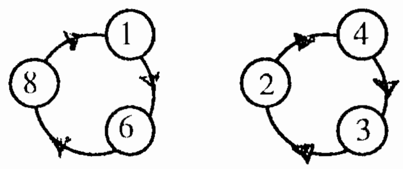
$$\det\begin{pmatrix} 1 + xa_{11} & xa_{12} & \cdots & xa_{1N} \\ xa_{21} & 1 + xa_{22} & \cdots & xa_{2N} \\ \vdots & \vdots & & \vdots \\ xa_{N1} & xa_{N2} & \cdots & 1 + xa_{NN} \end{pmatrix}$$

is the sum of the determinants of the $n \times n$ principal minors of $\mathbf{A}$. This allows us to calculate any symmetric function $F(x_1,\dots,x_N)$, so long as we know how to expand $F = \sum_{\lambda_1 \geq \dots \geq \lambda_r > 0,\ r \geq 0} M_{\lambda_1,\dots,\lambda_r} \prod_{i=1}^{r} e_{\lambda_i}$ in terms of elementary symmetric functions.

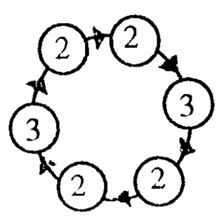
It is from the point of view of combinatorics that our problem becomes interesting. There are infinitely many bases of the ring of symmetric functions, but our attention is captured by six bases. They are the power $\{p_\lambda\}_{\lambda \vdash n}$, elementary $\{e_\lambda\}_{\lambda \vdash n}$, homogeneous $\{h_\lambda\}_{\lambda \vdash n}$, monomial $\{m_\lambda\}_{\lambda \vdash n}$, forgotten $\{f_\lambda\}_{\lambda \vdash n}$ symmetric functions, and finally, the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$, all of which we define in Section 1.1. The importance of these bases is made manifest by their role in the representation theory of the symmetric group S_N. This role is most succinctly expressed by the formula $b(x_1,\dots,x_N) = \frac{1}{n!}\sum_{\sigma \in S_n} B(\sigma) p_\sigma(x_1,\dots,x_N)$ where ch is the Frobenius characteristic map and B is the character of S_N for which $b = \mathrm{ch}(B)$. And yet at times it seems that something deeper hides within the personalities of these six bases, as if they reflect schemes by which our own minds organize variables. In any event, interpreting equations that relate symmetric functions is one of the richest areas of algebraic combinatorics.

This thesis contributes the observation that evaluating a basis $\{F_\lambda(\xi_1,\dots,\xi_N)\}_{\lambda \vdash n}$ for the symmetric functions at the eigenvalues $\xi_1,\dots,\xi_N$ of an $N \times N$ matrix $\mathbf{A}$ only serves to increase the wealth of combinatorial structure that it holds. That this is true can be seen from the fact that setting the off-diagonal elements of $\mathbf{A} = \{a_{ij}\}_{1 \leq i,j \leq N}$ equal to zero leaves us with a diagonal matrix with eigenvalues $a_{11},\dots,a_{NN}$, and $F_\lambda(x_1,\dots,x_N)$ becomes $F_\lambda(a_{11},\dots,a_{NN})$. For example, setting $a_{ij} = 0$, $i \neq j$, $1 \leq i,j \leq N$, eliminates all of the terms of $e_N(\xi_1,\dots,\xi_N) = \det(\mathbf{A})$ except $a_{11}a_{22}\cdots a_{NN}$, and setting $a_{ii} = x_i$, $1 \leq i \leq N$, recovers $e_N(x_1,\dots,x_N) = x_1 x_2 \cdots x_N$. The fact that $e_N(\xi_1,\dots,\xi_N) = \det(\mathbf{A})$ is a generating function for the elements of S_N only emphasizes the difference between the algebraic point of view, in which $F_\lambda(\xi_1,\dots,\xi_N)$ is a specialization of $F_\lambda(x_1,\dots,x_N)$, and the combinatorial point of view, in which $F_\lambda(\xi_1,\dots,\xi_N)$ is a generalization of $F_\lambda(x_1,\dots,x_N)$.

What kind of objects do symmetric functions $\{F_\lambda(\xi_1,\ldots,\xi_N)\}_{\lambda\vdash n}$ generate ? The elementary symmetric functions $\{e_\lambda(\xi_1,\ldots,\xi_N)\}_{\lambda\vdash n}$ help supply an answer. A term from $e_n(\xi_1,\ldots,\xi_N)$ can be described as a weight associated with a set of disjoint cycles, as in the example below, where $n=6$, $N=9$, and the weight is $a_{16}a_{24}a_{32}a_{43}a_{68}a_{81}$.



It follows that a symmetric function $F_\lambda(\xi_1,\ldots,\xi_N)$ generates terms that can be understood as weights associated with multisets of cycles. In practice, the multisets that arise are interpreted in terms of various combinatorial objects, including closed walks and Lyndon words, but these objects are always circular in nature. In greatest generality, the combinatorial phenomena that develop involve objects which resemble cycles, but may have letters that appear more than once, as in the example below.



In Section 1.3 we formalize these objects and call them walkalongs. Throughout this thesis we find that various operations on these walkalongs occur over and over again in a variety of settings.

However, the more significant results of this thesis rely on combinatorial tools that allow us to express the six bases in terms of each other. These are tableaux and tabloids that provide combinatorial interpretations for the entries of the transition matrices that relate the six bases. Best known are the column strict tableaux and rim hook tableaux. But we also make good use of brick tabloids, weighted brick tabloids, and special rim hook tabloids, recently introduced by Eğecioğlu and Remmel [ER1][ER2]. Both the tableaux and the tabloids are defined in Section 1.2.

Theorems 3.3.1 and 4.3.1 are the two main results of this thesis. Theorem 3.3.1 evaluates the forgotten symmetric functions $f_\lambda(\xi_1,\ldots,\xi_N)$. The terms that they generate involve Lyndon words on an alphabet $\overline{1}<\ldots<\overline{N}<1<\ldots<N$ of upper case and lower case letters. This is a surprising result because it suggests that $f_\lambda(\xi_1,\ldots,\xi_N)$ has arguably the most natural expression of any of the bases that we consider, whereas a combinatorial expression for the forgotten symmetric functions $f_\lambda(x_1,\ldots,x_N)$ was arrived at only recently. Theorem 4.3.1 is an expression for the Schur functions $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux that generalizes the fact that $s_\lambda(x_1,\ldots,x_N)$ is generates column strict tableaux. The rim hook tableaux are constructed so that each rim hook is associated with a Lyndon word. Rim hooks that are associated with the same Lyndon word are layed down in a descending order. Setting $a_{ij}=0$, $i\neq j$, $1\le i,j\le N$, $a_{ii}=x_i$, $1\le i\le N$, recovers the usual description of the Schur functions in terms of column strict tableaux.

In Chapter 2 we discuss the power $p_n(\xi_1,\ldots,\xi_N)$, elementary $e_n(\xi_1,\ldots,\xi_N)$, and homogeneous $h_n(\xi_1,\ldots,\xi_N)$ symmetric functions. We start by finding combinatorial interpretations for the three recursion relations that relate these bases, and they suggest that symmetric functions of eigenvalues provide a unifying framework for the combinatorics of matrix algebra. First, we relate Straubing's [Str] combinatorial proof of the Cayley-Hamilton theorem with an interpretation of $ne_n(\xi_1,\ldots,\xi_N)=\sum_{r=1}^{n}(-1)^{r-1}p_r(\xi_1,\ldots,\xi_N)e_{n-r}(\xi_1,\ldots,\xi_N)$. Second, we relate a combinatorial interpretation of $nh_n(\xi_1,\ldots,\xi_N)=\sum_{r=1}^{n}p_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)$ with the factorization of words into Lyndon words. Third, we relate Foata's [CF] combinatorial proof of the MacMahon

Master theorem with a combinatorial interpretation of $\sum_{r=0}^{n}(-1)^r e_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)=0$.

Most of our attention in the rest of chapter 2 is devoted to the homogeneous symmetric functions. We provide four different interpretations for $h_n(\xi_1,\ldots,\xi_N)$, including an original one in terms of multisets of Lyndon words. All of these are referred to in later chapters. We also study the walk matrix, the trace of which $\mathrm{tr}(\mathbf{I}/(\mathbf{I}-x\mathbf{A})) = \sum_{i=1}^{N}(1/(1-x\xi_i)) = \sum_{n=0}^{N}x^n p_n(\xi_1,\ldots,\xi_N)$ generates the power symmetric functions, and the determinant of which $\det(\mathbf{I}/(\mathbf{I}-x\mathbf{A})) = \prod_{i=1}^{N}(1/(1-x\xi_i)) = \sum_{n=0}^{N}x^n h_n(\xi_1,\ldots,\xi_N)$ generates the homogeneous symmetric functions.

Chapter 3 provides a combinatorial interpretation of the forgotten $f_\lambda(\xi_1,\ldots,\xi_N) = \frac{1}{n!}\sum_{\sigma\in S_n}(\mathrm{ch}^{-1} f_\lambda)(\sigma)p_\sigma(\xi_1,\ldots,\xi_N)$ and the monomial symmetric functions $m_\lambda(\xi_1,\ldots,\xi_N) = \frac{1}{n!}\sum_{\sigma\in S_n}(\mathrm{ch}^{-1} m_\lambda)(\sigma)p_\sigma(\xi_1,\ldots,\xi_N)$. Section 3.1 discusses an immanant formula $b(\xi_1,\ldots,\xi_N) = \frac{1}{n!}\sum_{w\in\mathbf{N}^n}\mathrm{Imm}_B\,\mathbf{A}_w$ from Littlewood's book, and shows that it is combinatorially equivalent to $b(\xi_1,\ldots,\xi_N) = \frac{1}{n!}\sum_{\sigma\in S_n}B(\sigma)p_\sigma(\xi_1,\ldots,\xi_N)$. Section 3.2 further prepares the way by showing how Eğecioğlu and Remmel [ER2] use weighted brick tabloids to interpret the character $(\mathrm{ch}^{-1} f_\lambda)(\sigma) = \mathrm{sgn}(\sigma)\cdot(\mathrm{ch}^{-1} m_\lambda)(\sigma)$. Our efforts lead to two combinatorial interpretations of $f_\lambda(\xi_1,\ldots,\xi_N)$, one of which is the Theorem 3.3.1 mentioned above. We also arrive at an expression for the monomial symmetric function in terms of the determinant $\det(\mathbf{I}/(\mathbf{I}-x\mathbf{A}))$ of the walk matrix. With the help of an involution from Section 2.3, this gives a striking interpretation of the equation $h_n(\xi_1,\ldots,\xi_N) = \sum_{\lambda\vdash n} m_\lambda(\xi_1,\ldots,\xi_N)$.

In Chapter 4 we consider two new expressions for the Schur functions $s_\lambda(\xi_1,\ldots,\xi_N)$. The chapter starts with Theorem 4.1.1, which presents an original formula

$$s_\lambda(\xi_1,\ldots,\xi_N) = \det\left(\left(\mathbf{A}^{\lambda_i+N-i}\right)_{jj}\right)_{1\le i,j\le N} \Big/ \det\left(\left(\mathbf{A}^{N-i}\right)_{jj}\right)_{1\le i,j\le N}$$

that brings to mind the usual quotient formula $s_\lambda(x_1,\ldots,x_N) = \det\left(x_j^{\lambda_i+N-i}\right)_{1\le i,j\le N} \Big/ \det\left(x_j^{N-i}\right)_{1\le i,j\le N}$. Next, Theorem 4.3.1, as mentioned above, presents an expression for $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux and Lyndon words. In Section 4.2 we prepare the way for the proof of this theorem by undertaking a study of the relation between rim hook tableaux and special rim hook tabloids.

Finally, in Section 4.4 we complete our picture of symmetric functions of eigenvalues by acknowledging perhaps the most important fact about them, which is that $s_\lambda(\xi_1,\ldots,\xi_N)$ is the trace of the irreducible representation of the general linear group $\mathrm{GL}_N(\mathbb{C})$ associated with the partition λ. We do not offer any new results, but simply present explicit representations due to Littlewood [L], and take their trace. The representations are straightforward when λ is a hook shape, and they bring to mind Foata's [CF] circuits, which we discuss in Section 2.2. In general, however, when λ is an arbitrary shape the representations are more complicated and unwieldy.

In the conclusion of this thesis we offer the theory of symmetric functions as a unifying framework for disparate results in matrix algebra. Many of the combinatorial techniques in the last chapter of Brualdi and Ryser's [BR] recent book can be understood in terms of symmetric functions evaluated at the eigenvalues $\xi_1,\ldots,\xi_N$ of an $N\times N$ matrix $\mathbf{A}$. Another idea presented in the conclusion is that specializing $\mathbf{A}$ in various ways may make the results of this thesis useful in a variety of algebraic situations. The thesis ends with our hope that the results may in some small way prove helpful in the study of algebraic structures such as the free Lie algebra, or the representations of the general linear group $\mathrm{GL}_N(\mathbb{C})$.

Also included in this thesis are laconic proofs of two classical results in the theory of symmetric functions. An Appendix presents a new combinatorial proof that the quotient of alternants $\det\left(x_j^{\lambda_i+N-i}\right)_{1\le i,j\le N} \Big/ \det\left(x_j^{N-i}\right)_{1\le i,j\le N}$ generates column strict tableaux. This proof is the shortest known and does not involve crossmultiplying by

$\det\left(x_j^{N-i}\right)_{1\le i,j\le N}$. A second result is to be found at the end of Section 4.3. It is a quick combinatorial proof of the fact that the usual calculation of the irreducible character $\chi^{\lambda}(\sigma)$ of S_N by way of rim hook tableaux does not depend on the order chosen for the lengths of the rim hooks. This proof depends on techniques for manipulating rim hooks that are discussed in Section 4.2.

SECTION 1.1 SYMMETRIC FUNCTIONS

In this section we define the six bases of the ring of the symmetric functions which are the subject of this thesis. We also review several of the equations that relate these bases. Many of these are of the form $b_\lambda = \sum_{\mu \succ n} \mathbf{M}(b,a)_{\lambda,\mu} a_\mu$, and we defer the combinatorial interpretation of the matrix entries $\mathbf{M}(b,a)_{\lambda,\mu}$ until Section 1.2. We do, however, make several remarks about equations $b_\lambda = \sum_{\mu \succ n} \mathbf{M}(b,p)_{\lambda,\mu} p_\mu$ that expand a symmetric function b_λ in terms of the power symmetric functions $\{p_\lambda\}_{\lambda \succ n}$. Such equations often arise in the form $\mathrm{ch}(B^\lambda) = \frac{1}{n!} \sum_{\mu \succ n} B^\lambda(\mu) C_\mu p_\mu$, where B^λ is a character or virtual character of the symmetric group S_N. Although the results of this thesis do not depend on concepts or facts from representation theory, it may be argued that features of the theory appear implicitly. Rather than recount this theory, we illustrate it by considering the case of 3×3 permutation matrices. Finally, we end this section with a discourse on the six bases defined above, with a sketch of their various personalities. It is our hope that the results in this thesis will shed more light on the nature of these bases.

Given a vector space, it is possible to express its elements in terms of a basis. Such expressions can be extremely practical to work with. From them we can tell the magnitude with which members of the basis are present in a given vector, and we can tell which members are not present at all. However, these expressions depend on the choice of basis, and in general there are infinitely many bases to choose from. Through most of this century an attitude has held sway that to express mathematical ideas clearly it is best not to work with a concrete basis. In the words of Herstein,

> As a general principle, it is preferable to give proofs, whenever possible, which are basis-free. Such proofs are usually referred to as invariant ones. An invariant proof or construction has the advantage, other than the mere aesthetic one, over a proof or construction using a basis, in that one does not

> have to worry how finely everything depends on a particular choice of bases. [H, 187]

Even when one does work with a basis, it is often with a single basis that suggests itself from the way the vector space is defined, a "natural" coordinate system.

As we turn to study the ring of symmetric functions, we find ourselves in a very different situation. Our attention is captured by six of the infinitely many bases available. For each of these six bases there is a different context in which it seems natural and a different reason for why it is interesting. But the importance of all six of these bases has been borne out by the developments and achievements of representation theory and these bases have a special significance to the theory of the symmetric group.

Let $\mathbb{Z}[x_1,\ldots,x_N]$ denote the ring of polynomials in the commuting variables $x_1,\ldots,x_m$. Then $b(x_1,\ldots,x_N)$ is a *symmetric function* if and only if $b(x_1,\ldots,x_N)\in\mathbb{Z}[x_1,\ldots,x_N]$ and $b(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(N)})=b(x_1,x_2,\ldots,x_N)$ for all permutations $\sigma\in S_N$. The symmetric functions form a subring Λ_N of $\mathbb{Z}[x_1,\ldots,x_N]$ which is known as the *ring of symmetric functions.*

Λ_N is a $\mathbb{Z}$-module. In this thesis for most purposes it will be simplest to think of Λ_N as a vector space with coefficients in $\mathbb{Q}$ or $\mathbb{C}$. This merely requires that we suppose the coefficients to be not exclusively integers, but rational or complex numbers in general. When we think of Λ_N as either a $\mathbb{Z}$-module or a vector space, then within Λ_N we tend to work with Λ_N^n, which consists of the *symmetric functions homogeneous of degree* n, that is, symmetric functions whose every term is of degree n. The zero polynomial is also included in Λ_N^n. In general, we assume that $N\geq n$.

The dimension of Λ_N^n as a $\mathbb{Z}$-module or as a vector space is the number of partitions of n. We define a *partition* λ of n to be a sequence $\lambda_1,\ldots,\lambda_n$ of n non-negative integers written in decreasing order $\lambda_1\geq\ldots\geq\lambda_n\geq 0$ such that $\lambda_1+\ldots+\lambda_n=n$. The length $\ell(\lambda)$ of λ is the number of nonzero integers in the sequence, and therefore

$\lambda_j = 0$ when $\ell(\lambda) < j \leq n$. If $j > n$, then it is always understood that $\lambda_j = 0$. We write $\lambda \succ n$ to emphasize that λ is a partition of n.

The members of any basis of Λ_N^n can be indexed by partitions, as can the rows and columns of any transition matrix. In writing down the entries of transition matrices there is a need for an ordering of the partitions, although it plays no role in our results. The *reverse lexicographic ordering* is used throughout. With regard to this ordering λ precedes μ if there exists an i such that $\lambda_j = \mu_j$ for all $j < i$ and $\lambda_i > \mu_i$. Partitions are then written in decreasing order. For example, the partitions of 5 are written as: 5, 41, 32, 311, 221, 2111, 11111.

The subject of this thesis is the combinatorics of the six bases of Λ_N^n that follow. Although the definitions all depend on the number of variables N, the bases are all defined so that equations relating their members hold for all N, so long as it is fixed and $N \geq n$.

The *power symmetric functions* $\{p_\lambda\}_{\lambda \succ n}$ are defined by $p_\lambda = \prod_{\lambda_i > 0} p_{\lambda_i}$ where $p_r = x_1^r + \ldots + x_N^r$, $r > 0$, and $p_0 = 1$.

The *elementary symmetric functions* $\{e_\lambda\}_{\lambda \succ n}$ are defined by $e_\lambda = \prod_{\lambda_i > 0} e_{\lambda_i}$ where $e_r = \sum_{j_1 < j_2 < \ldots < j_r} x_{j_1} x_{j_2} \ldots x_{j_r}$, $r > 0$, and $e_0 = 1$.

The *homogeneous symmetric functions* $\{h_\lambda\}_{\lambda \succ n}$ are defined by $h_\lambda = \prod_{\lambda_i > 0} h_{\lambda_i}$ where $h_r = \sum_{j_1 \leq j_2 \leq \ldots \leq j_r} x_{j_1} x_{j_2} \ldots x_{j_r}$, $r > 0$, and $h_0 = 1$.

The *monomial symmetric functions* $\{m_\lambda\}_{\lambda \succ n}$ are defined by $m_\lambda = \sum x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(N)}^{\lambda_N}$ where the sum is taken over all permutations σ of the numbers $1, 2, \ldots, N$ such that $\sigma(i) < \sigma(j)$ if both $i < j$ and $\lambda_i = \lambda_j$.

The *Schur functions* $\{s_\lambda\}_{\lambda \succ n}$ are defined by

$$s_\lambda = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{1 \leq i, j \leq N}}{\det\left(x_i^{N-j}\right)_{1 \leq i, j \leq N}}.$$

A definition in terms of column strict tableaux is given in Section 1.2.

The *forgotten symmetric functions* $\{f_\lambda\}_{\lambda \succ n}$ are defined by $f_\lambda = \omega(m_\lambda)$ where ω is the involution on Λ_N which maps e_λ into h_λ and vice versa. A definition in terms of brick tabloids is given in Section 1.2.

We elaborate on the involution ω mentioned above. There exists a ring homomorphism ω on Λ_N such that $\omega(e_n) = h_n$ for all n. From the symmetry of the h's and the e's in the recursion relation $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$ it follows that $\omega(\omega(e_n)) = e_n$, which says that ω is an involution. It can be shown that $\omega(p_\lambda) = \text{sgn}(\lambda)p_\lambda$, where $\text{sgn}(\lambda) = \prod_{1 \le i \le n}(-1)^{\lambda_i - 1}$ is the usual sign of the cycle structure λ. Moreover, $\omega(s_\lambda) = s_{\lambda'}$, where λ' is the shape conjugate to λ, as defined at the end of Section 1.2. Finally, suppose that Λ_N is endowed with an inner product defined by declaring $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, with $\delta_{\lambda\lambda} = 1$ and $\delta_{\lambda\mu} = 0$ if $\lambda \ne \mu$. Then ω is an isometry with regard to this inner product, which is to say that $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$ for all $f, g \in \Lambda_N$. It can be shown that $\langle f_\lambda, e_\mu \rangle = \delta_{\lambda\mu}$, $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$, and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$, where $z_\lambda = \frac{n!}{C_\lambda}$ and C_λ is the number of permutations for which the cycles have lengths given by $\lambda \succ n$. The usual reference to the theory of symmetric functions is Macdonald's book, Symmetric Functions and Hall Polynomials. [M]

Equations that relate the six bases defined above do not depend on N, so long as N is fixed and $N \ge n$, as we have already remarked. The simplest of these equations are the recursion relations $ne_n = \sum_{r=1}^{n}(-1)^{r-1} p_r e_{n-r}$, $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$, $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$, which we interpret in Section 2.1. We also interpret the equations $h_n = \sum_{\lambda \succ n} m_\lambda$ in Section 3.4 and $e_n = \sum_{\lambda \succ n} f_\lambda$ in Section 3.3. In general, most of the equations take the form $b_\lambda = \sum_{\mu \succ n} \mathbf{M}(b,a)_{\lambda,\mu} a_\mu$, where $\mathbf{M}(b,a)_{\lambda,\mu}$ is the *transition matrix* from the a_μ's to the b_λ's. Combinatorial interpretations exist for the entries of these transition matrices, and we review them in Section 1.2.

The *Jacobi-Trudi identity* $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1\le i,j\le m}$, $m \ge \ell(\lambda)$, where $h_j = 0$ whenever $j < 0$, is of special importance in Chapter 4, which is devoted to evaluating the Schur functions $s_\lambda(\xi_1,...,\xi_N)$. Theorem 4.1.1 is inspired by an algebraic proof of the Jacobi-Trudi identity. Theorem 4.3.1 may be understood as a combinatorial interpretation of the Jacobi-Trudi identity, where we write it as the transition matrix equation $s_\lambda = \sum_{\mu\succ n} \mathbf{M}(s,h)_{\lambda,\mu} h_\mu$ that expresses s_λ in terms of the h_μ's.

The symmetric functions $f_\lambda(\xi_1,...,\xi_N)$ and $m_\lambda(\xi_1,...,\xi_N)$ are evaluated in chapter 3 using the equations $f_\lambda = \sum_{\mu\succ n} \mathbf{M}(f,p)_{\lambda,\mu} p_\mu$ and $m_\lambda = \sum_{\mu\succ n} \mathbf{M}(m,p)_{\lambda,\mu} p_\mu$, and it seems that a similar approach can work for the Schur functions $s_\lambda(\xi_1,...,\xi_N)$. With regard to our techniques, we find that there are special advantages to expressing symmetric functions in terms of the basis $\{p_\lambda\}_{\lambda\succ n}$ of power symmetric functions.

The statements and methods of this thesis do not depend on anything from the theory of symmetric functions other than what we have mentioned above. Nor do they depend on any concepts or results from the representation theory of the symmetric group S_n with which the theory of symmetric functions is usually associated. This having been said, it is true that some of our most effective approaches make use of equations $b_\lambda = \sum_{\mu\succ n} \mathbf{M}(b,p)_{\lambda,\mu} p_\mu$ in which the transition matrix takes on the form $\mathbf{M}(b,p)_{\lambda,\mu} = \frac{1}{n!}\sum_{\sigma\in U_\mu} B^\lambda(\sigma)$. Here U_μ is the set of permutations $\sigma \in S_n$ with cycle structure $\mu \succ n$ and $B^\lambda(\sigma)$ is the trace of a representation of the symmetric group, or a linear combination of such traces. Our constructions use the permutations $\sigma \in S_n$ to add labels and use the factor $\frac{1}{n!}$ to remove these labels. They suggest that the functions $B^\lambda(\sigma)$ are at work, albeit implicitly, and in fairness to them we devote some pages of this section to remarks on their role in representation theory. We also point them out in various places throughout this thesis.

Two permutations τ, $\sigma \in S_N$ belong to the same *conjugacy class* if and only if there exists a $g \in S_n$ such that $\tau = g^{-1}\sigma g$. In particular, τ and σ belong to the same conjugacy class if and only if they have the same cycle structure. This means that there is exactly one conjugacy class of S_N for each partition $\lambda \succ n$. The size of the conjugacy class associated with the cycle structure λ is $C_\lambda = n! \Big/ \left(\prod_{1 \le i \le n} i^{m_i} \cdot m_i! \right)$ where $m_1, \ldots, m_n$ are the integers for which $\lambda = n^{m_n} \cdots 2^{m_2} 1^{m_1}$.

Let $B: S_n \to \mathbb{Z}$ be a *class function*, that is, a function that is constant on conjugacy classes. We write either $B(\sigma)$ or $B(\lambda)$ to indicate the value of B at a permutation σ with cycle structure λ. Define class functions Z^λ such that $Z^\lambda(\lambda) = 1$ and $Z^\lambda(\mu) = 0$ for all $\mu \neq \lambda$. Then any class function B can be thought of as a linear combination $\sum_{\lambda \succ n} B(\lambda) Z^\lambda$, and with this in mind we let R_n be the module over $\mathbb{Z}$ whose elements are these linear combinations.

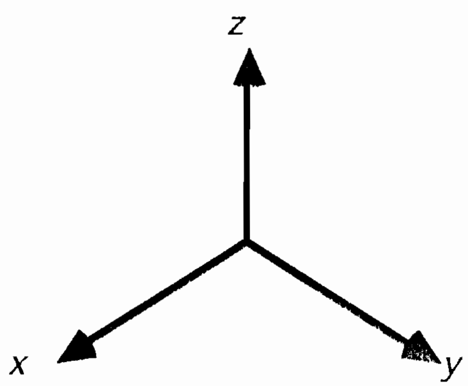
The *Frobenius characteristic map* $\mathrm{ch}: R_n \to \Lambda_n$ is defined by $\mathrm{ch}(B) = \frac{1}{n!} \sum_{\lambda \succ n} B(\lambda) C_\lambda p_\lambda$. We make frequent mention of this formula, especially in Chapter 3, but we usually write it in the form $\mathrm{ch}(B) = \frac{1}{n!} \sum_{\sigma \in S_n} B(\sigma) p_\sigma$, where the index σ of p_σ is understood as the partition which gives the cycle structure of σ. This is compatible with combinatorial constructions in which we think of a permutation σ as providing labels $\sigma(1), \sigma(2), \ldots, \sigma(n)$. The Frobenius characteristic map serves as a key to understanding the marriage between the theory of symmetric functions and the representation theory of the symmetric group S_n.

The Frobenius characteristic map is an isomorphism, and we may speak of its inverse, ch^{-1}. Therefore the functions $\{\chi^\lambda\}_{\lambda \succ n}$ for which $\mathrm{ch}(\chi^\lambda) = s_\lambda$ are a basis for R_n. They are the irreducible characters of S_n, and of central importance to the representation theory of S_n. A *representation* ρ of S_n of degree d is a group homomorphism $\rho: S_n \to \mathrm{GL}_d(\mathbb{C})$ which sends each permutation $\sigma \in S_n$ to a $d \times d$ matrix $\rho(\sigma)$ with complex entries, and $\rho(\sigma)\rho(\tau) = \rho(\sigma\tau)$ is the usual matrix multiplication. For example, the homomorphism ψ_{21} which maps a permutation $\sigma \in S_3$ into the associated 3×3 permutation matrix is a representation of degree 3. A *character* of S_n is the trace of a representation of S_n. A *reducible* representation is one that is isomorphic to a direct product of representations. The representation ψ_{21} is reducible. An *irreducible* representation is one that is not reducible. Finally, an *irreducible character* is the trace of an irreducible representation.

We want to touch on the role of irreducible characters in representation theory, if only so as to make plausible the significance of the Schur functions $s_\lambda = \mathrm{ch}(\chi^\lambda)$ and the transition matrix $\mathbf{M}(s,p)_{\lambda,\mu} = \frac{1}{n!}C_\mu \chi^\lambda(\mu)$. With this in mind, let us summarize some of the achievements of representation theory as they apply to S_n [Se]. Say that two representations ρ and ρ' are *equivalent* if $\rho(\sigma) = D\rho'(\sigma)D^{-1}$ for some $D \in \mathrm{GL}_d(\mathbb{C})$ and for all $\sigma \in S_n$, that is, they are equivalent if and only if they differ by a change in coordinates. The number of irreducible representations of S_n, up to equivalence, equals the number of conjugacy classes of S_n, which as we have observed equals the number of partitions of n. The trace χ^λ of any such representation $\{\lambda\}$ is constant on conjugacy classes and has integer values, so that $\chi^\lambda \in R_n$ for all $\lambda \succ n$. If two representations have the same trace, then they are equivalent. In fact, a representation is a direct sum $\rho(\sigma) = \rho_1(\sigma) \oplus \cdots \oplus \rho_s(\sigma)$ if and only if the corresponding character is a sum $\mathrm{tr}(\rho(\sigma)) = \mathrm{tr}(\rho_1(\sigma)) + \cdots + \mathrm{tr}(\rho_s(\sigma))$. This means, in particular, that a class function $B \in R_n$ is a character $B(\sigma) = \mathrm{tr}(p(\sigma))$ of S_n if and only if it is a linear combination $B(\sigma) = \sum_{\lambda \succ n} m_\lambda \chi^\lambda(\sigma)$ where $\{\chi^\lambda\}_{\lambda \succ n}$ are the irreducible characters $\chi^\lambda(\sigma) = \mathrm{tr}(\{\lambda\}(\sigma))$ of S_n, $\{\{\lambda\}\}_{\lambda \succ n}$ are the irreducible representations of S_n, and the *multiplicities* m_λ are the nonnegative integers which indicate the number of copies of $\{\lambda\}$ in the direct product $p = \bigoplus_{\lambda \succ n} \{\lambda\}^{m_\lambda}$.

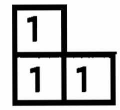
It is possible to give an explicit description of the representations $\{\{\lambda\}\}_{\lambda \succ n}$. For example, the representations of S_n of degree one are the trivial representation $\{n\}(\sigma) = (1)$ and the alternating representation $\{1^n\}(\sigma) = (\mathrm{sgn}(\sigma))$. In fact, in Section 4.4 we provide explicit descriptions for the representations of $\mathrm{GL}_n(\mathbb{C})$ from Littlewood's [L] book, and specializing these gives Young's natural representations of S_n. There is also a combinatorial interpretation of $\chi^\lambda(\mu) = \frac{n!}{C_\mu}\mathbf{M}(s,p)_{\lambda,\mu}$ in terms of rim hook tableaux which we state in Section 1.2 and use in Section 4.3. However, in order to make the representation theory of S_n more concrete, we present an example of the decomposition of a representation of S_3 into irreducible representations.

Let ψ_{21} be the representation of S_3 which maps $\sigma \in S_3$ into the 3×3 permutation matrix $\left(\delta_{\sigma(i)j}\right)_{1\le i,j\le 3}$. If we observe how these matrices act on vectors $(x,0,0)$, $(0,y,0)$, $(0,0,z)$, then we find that ψ_{21} leaves fixed the line $x=y=z$. We imagine this line as passing down our line of sight into the coordinate system shown below.



We therefore consider the equivalent representation $D\psi_{21}(\sigma)D^{-1}$ which is gotten by a change in the coordinate system $(x,0,0)\to(x,x,x)$, $(0,y,0)\to(y,-y,0)$, $(0,0,z)\to(0,z,-z)$, so that D is the matrix

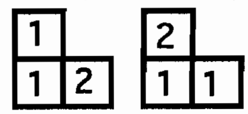
$$D=\begin{pmatrix}1&1&1\\1&-1&0\\0&1&-1\end{pmatrix} \qquad D^{-1}=\begin{pmatrix}1&2&1\\1&-1&1\\1&-1&-2\end{pmatrix}$$



$$\lambda = 3$$

$$\sigma = (123) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$\sigma = (132) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right)$$

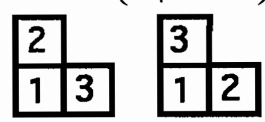


$$\lambda = 21$$

$$\sigma = (12)(3) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right)$$

$$\sigma = (13)(2) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right)$$

$$\sigma = (1)(23) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right)$$



$$\lambda = 111$$

$$\sigma = (1)(2)(3) \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

Figure 1: The decomposition of a representation of S_3.

Let ψ_{21} signify the familiar representation of S_3 by 3×3 permutation matrices. We illustrate the decomposition $\{3\}(\sigma) \times \{21\}(\sigma)$ by calculating $D\psi_{21}(\sigma)D^{-1}$ for all $\sigma \in S_3$, with D chosen as above. Moreover, we express $\chi^{21}(\sigma) = \mathrm{tr}(\{21\}(\sigma))$ in terms of rim hook tableaux, as described in Section 1.2.

In Figure 1 we present the values of $D\psi_{21}(\sigma)D^{-1}$ for all $\sigma \in S_3$. The representation ψ_{21} is isomorphic to the direct product $\{3\}\times\{21\}$, where $\{3\}$ is the trivial representation of S_3, and $\{21\}$ is an irreducible representation of S_3 of degree two. Note that χ^{21} is constant on conjugacy classes. The characters $\chi^3 = Z^3 + Z^{21} + Z^{111}$, $\chi^{21} = -Z^3 + Z^{111}$, $\chi^{111} = Z^3 - Z^{21} + Z^{111}$ constitute a basis of R_3.

For all F, $G \in R_n$, define the inner product $\langle F,G\rangle_{R_n} = \frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma)G(\sigma)$. With respect to this inner product the irreducible characters are orthonormal, that is $\langle \chi^\lambda,\chi^\mu\rangle_{R_n} = \delta_{\lambda\mu}$. In contrast, the class functions $\{Z^\lambda\}_{\lambda\vdash n}$, which are a "natural" basis for R_n, are merely orthogonal, so that $\langle Z^\lambda, Z^\mu\rangle_{R_n} = \frac{C_\lambda}{n!}\delta_{\lambda\mu}$. The fact that the inner product on R_n does not coincide with the basis $\{Z^\lambda\}_{\lambda\vdash n}$ means that the $\mathbb{Z}$-module R_n has two "natural" bases. The bases $\{\chi^\lambda\}_{\lambda\vdash n}$ and $\{Z^\lambda\}_{\lambda\vdash n}$ in R_n are related to each other in the same way as are the bases $\{s_\lambda\}_{\lambda\vdash n}$ and $\left\{\frac{C_\lambda}{n!}p_\lambda\right\}_{\lambda\vdash n}$ in Λ_N^n. What makes this all the more true is that the Frobenius characteristic map is an isometry, so that $\langle \mathrm{ch}(F),\mathrm{ch}(G)\rangle = \langle F,G\rangle_{R_n}$ for all F, $G \in R_n$.

Having described how Λ_N^n and R_n are related by the Frobenius characteristic map, let us turn back to the six bases $\{e_\lambda\}_{\lambda\vdash n}$, $\{m_\lambda\}_{\lambda\vdash n}$, $\{s_\lambda\}_{\lambda\vdash n}$, $\{p_\lambda\}_{\lambda\vdash n}$, $\{h_\lambda\}_{\lambda\vdash n}$, $\{f_\lambda\}_{\lambda\vdash n}$, that are the subject of this thesis, and for each basis provide a reason for its significance.

The elementary symmetric functions $\{e_\lambda\}_{\lambda\vdash n}$ are familiar even to people with no special interest in the theory of symmetric functions. This is because the coefficients of any polynomial are functions $(-1)^n e_n(x_1,\ldots,x_N)$ of its roots $x_1,\ldots,x_N$, which is to say that $(t-x_1)(t-x_2)\cdots(t-x_N) = \sum_{0\le i\le N} t^{N-i}(-1)^i e_i(x_1,\ldots,x_N)$. From the point of view of combinatorics, this means that $e_n(x_1,\ldots,x_N)$ is a generating function for subsets of $1,\ldots,N$ of size n. As we shall see, this interpretation changes when $e_n(x_1,\ldots,x_N)$ is evaluated at the eigenvalues $\xi_1,\ldots,\xi_N$ of an $N\times N$ matrix $\mathbf{A}$, so that $e_n(\xi_1,\ldots,\xi_N)$ generates terms with both positive and negative sign. This new context does not diminish the importance of the elementary symmetric functions, as they give the coefficients of the characteristic polynomial $\det(t\mathbf{I}-\Lambda)=\prod_{1\le i\le N}(t-\xi_i)=\sum_{0\le i\le N} t^{N-i}(-1)^i e_i(\xi_1,\ldots,\xi_N)$ of the matrix Λ.

The monomial symmetric functions $\{m_\lambda\}_{\lambda\succ n}$ are the "natural" coordinate system for Λ_N^n. The basis $\{m_\lambda\}_{\lambda\succ n}$ is the one which most single-mindedly expresses the symmetry in the variables $x_1,\ldots,x_N$. According to our definition, $m_\lambda(x_1,\ldots,x_N)$ $=\sum_\sigma x_{\sigma(1)}^{\lambda_1}x_{\sigma(2)}^{\lambda_2}\cdots x_{\sigma(N)}^{\lambda_N}$ where the summation takes place over all distinct permutations σ of $\lambda_1\ge\ldots\ge\lambda_n$. In other words, $m_\lambda(x_1,\ldots,x_N)$ consists of the term $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_N^{\lambda_N}$ and no other terms except those implied by the fact that m_λ is a symmetric function. This last description is very helpful, as we can see if we multiply together $e_2(x_1,x_2,x_3)$ $=x_1x_2+x_1x_3+x_2x_3$ and $e_1(x_1,x_2,x_3)=x_1+x_2+x_3$. The product $e_{21}(x_1,x_2,x_3)$ $=e_2(x_1,x_2,x_3)e_1(x_1,x_2,x_3)$ contains two copies of the term $x_1^2x_2$, and three of the term $x_1x_2x_3$, and so we quickly conclude that $e_{21}=e_2e_1=2m_{21}+3m_{111}$, where $m_{21}(x_1,x_2,x_3)=x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2$ and $m_{111}(x_1,x_2,x_3)=x_1x_2x_3$.

The Schur functions $\{s_\lambda\}_{\lambda\succ n}$ are most readily introduced in the context of the ring of antisymmetric functions. A function $f(x_1,\ldots,x_N)$ is *antisymmetric* if $f\left(x_{\sigma(1)},\ldots,x_{\sigma(N)}\right)=\operatorname{sgn}(\sigma)f(x_1,\ldots,x_N)$ for all $\sigma\in S_N$. If $c_\mu x_1^{\mu_1}x_2^{\mu_2}\cdots x_N^{\mu_N}$ is a term in a antisymmetric function $f(x_1,\ldots,x_N)$ with $\mu_1\ge\mu_2\ge\ldots\ge\mu_N$ and $c_\mu\ne 0$, then it must be that $\mu_1>\mu_2>\ldots>\mu_N$ because otherwise there exists a $\sigma\in S_N$ such that $\operatorname{sgn}(\sigma)=-1$ and $c_\mu x_1^{\mu_1}x_2^{\mu_2}\cdots x_N^{\mu_N}=-c_\mu x_1^{\mu_1}x_2^{\mu_2}\cdots x_N^{\mu_N}$, contradicting $c_\mu\ne 0$. Let $\delta\succ\binom{N}{2}$ with $\delta_j=N-j$ for all j, $1\le j\le N$, and let $a_{\lambda+\delta}(x_1,\ldots,x_N)=\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)x_{\sigma(1)}^{\lambda_1+\delta_1}x_{\sigma(2)}^{\lambda_2+\delta_2}\cdots x_{\sigma(N)}^{\lambda_N+\delta_N}$ $=\det\left(x_i^{\lambda_j+\delta_j}\right)_{1\le i,j\le N}$. We say that $\{a_{\lambda+\delta}\}_{\lambda\succ n}$ are the *monomial antisymmetric functions* of degree $n+\binom{N}{2}$. They form a basis for the $\mathbb{Z}$-module A_N^n of antisymmetric functions in variables $x_1,\ldots,x_N$ of degree $n+\binom{N}{2}$. Observe that setting $x_i=x_j$ for any $f\in A_N^n$ gives $f=-f=0$, which shows that x_i-x_j divides f. Therefore the product $\prod_{1\le i<j\le N}(x_i-x_j)$ divides any $f\in A_N^n$. In particular, $\prod_{1\le i<j\le N}(x_i-x_j)$ divides $a_\delta(x_1,\ldots,x_N)=\det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$. These two polynomials have the same degree and a check of the coefficients shows that

they are equal. We define the Schur functions $\{s_\lambda\}_{\lambda\vdash n}$ to be the quotients $s_\lambda(x_1,\dots,x_N) = a_{\lambda+\delta}(x_1,\dots,x_N)/a_\delta(x_1,\dots,x_N)$. The Schur functions are quotients of two antisymmetric polynomials and therefore must be symmetric functions. In fact, they form a basis of Λ_N^n because $\{a_{\lambda+\delta}\}_{\lambda\vdash n}$ is a basis of A_N^n. The Schur functions may be thought of as the basis of Λ_N^n which arises from the "natural" basis of A_N^n. [M, 24]

The power symmetric functions $\{p_\lambda\}_{\lambda\vdash n}$ may be thought of as being given by $p_\lambda = \mathrm{ch}\left(\frac{n!}{C_\lambda}Z^\lambda\right)$, where $\{Z^\lambda\}_{\lambda\vdash n}$ is the "natural" basis of R_n, as we have already seen.

The homogeneous symmetric functions $\{h_\lambda\}_{\lambda\vdash n}$ become especially significant when they are identified with the characters $\{\eta_\lambda\}_{\lambda\vdash n}$ of S_n for which $\eta_\lambda = \mathrm{ch}^{-1}(h_\lambda)$. These characters are the traces of some of the most important representations of S_n. Let S_λ be a subgroup of S_n that is isomorphic to the group $S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_{\ell(\lambda)}}$. Then η_λ is the trace of the representation ψ_λ of S_n that is induced from the trivial representation of S_λ. We provide several examples of this induction, starting with $\lambda = n-1,1$. Let $S_{n-1,1}$ be the subgroup of S_n whose elements permute the letters $2,\dots,n$ but keep 1 fixed. Then S_n has $|S_n|/|S_{n-1,1}| = \frac{n!}{(n-1)!1!} = \binom{n}{n-1} = n$ cosets of $S_{n-1,1}$. In particular, S_n is a disjoint union of the cosets $S_{n-1,1}, (12)S_{n-1,1}, (13)S_{n-1,1},\dots,(1n)S_{n-1,1}$, where, for example, $(12)S_{n-1,1} = \{g\in S_n \mid g=(12)h, h\in S_{n-1,1}\}$. Note that $g[(1i)S_{n-1,1}] = [(1j)S_{n-1,1}]$ if and only if $(1j)g(1i)\in S_{n-1,1}$ if and only if g maps j to i. Therefore g may be understood to act on the cosets as an $n\times n$ permutation matrix $\left(\delta_{\sigma(j),i}\right)_{1\le j,i\le n}$, and this is the representation $\psi_{n-1,1}$ of S_n of which $\eta_{n-1,1}$ is the trace. It is the usual representation of S_n in terms of $n\times n$ permutation matrices, with one matrix for each of the $n!$ elements of S_n. Its trace $\eta_{n-1,1}(\sigma)$ counts the number of letters i for which $\sigma(i)=i$.

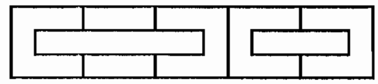
If $\lambda = 1^n$, then S_{1^n} has a single element, the identity, and there is one coset of S_{1^n} for each element of S_n. Note that $g[\sigma S_{1^n}] = [\tau S_{1^n}]$ if and only if $g=\tau\sigma^{-1}$. Therefore the induced representation ψ_{1^n} is that which maps g into the $n!\times n!$ permutation matrix $\left(\delta_{g\sigma,\tau}\right)_{\sigma,\tau\in S_n}$. It is the *regular* representation and associates each element of S_n with a table that records the effect of its multiplication on the elements of S_n. Its trace $\eta_{1^n}(g)$ counts the number of permutations for which $g\tau = \tau$, and therefore $\eta_{1^n}(g) = 0$ for $g \neq id$, and $\eta_{1^n}(id) = n!$. If $\lambda = n$, then S_λ is all of S_n, and there is but one coset. Therefore ψ_n is the trivial representation, and $\eta_n(g) = 1$.

The several examples that we have considered are very important, and it can be argued that the characters $\{\eta_\lambda\}_{\lambda \vdash n}$ are traces of representations which in practice arise more frequently and naturally than do the irreducible representations. The multiplicities of $\{\chi^\lambda\}_{\lambda \vdash n}$ in η_μ have an interpretation in terms of column strict tableaux of shape μ and type λ, which are defined in Section 1.2. From them we can tell, for example, that $\eta_{21} = \chi^{21} + \chi^3$, where ψ_{21} is the representation of S_3 which we portray in Figure 1.

The forgotten symmetric functions $\{f_\lambda\}_{\lambda \vdash n}$ are given by $f_\lambda = \omega(m_\lambda)$, and only recently received a combinatorial interpretation, due to Eğecioğlu and Remmel [ER2]. Ostensibly they are a "missing" basis that is included so as to make the picture complete. However, in evaluating different symmetric functions at the eigenvalues $\xi_1, \ldots, \xi_N$ of an $N \times N$ matrix $\mathbf{A}$, we argue in the conclusion of this thesis that it is the forgotten symmetric functions which have the most elegant and satisfying combinatorial interpretation.

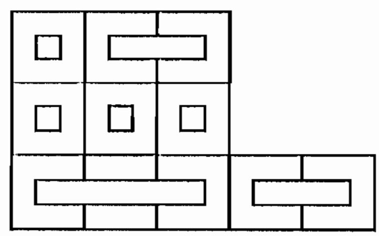
We conclude this section by deriving Eğecioğlu and Remmel's interpretation of $f_\lambda(x_1, \ldots, x_N)$ in terms of brick tabloids. Recall the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le m}$, where $h_j = 0$ whenever $j < 0$. In the special case $\lambda = 1^n$, we have $e_n = \det\left(h_{1-i+j}\right)_{1 \le i,j \le n}$. A composition of n is a sequence $\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}$ of positive integers which add up to n. The terms generated by $\det\left(h_{1-i+j}\right)_{1 \le i,j \le n}$ correspond to the compositions of n, and each such term has weight $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}}$, as illustrated by the example below.

$$\begin{pmatrix} h_1 & h_2 & \underline{\mathbf{h}_3} & h_4 & h_5 \\ \underline{1} & h_1 & h_2 & h_3 & h_4 \\ 0 & \underline{1} & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 & \underline{\mathbf{h}_2} \\ 0 & 0 & 0 & \underline{1} & h_1 \end{pmatrix}$$



We say that the n squares drawn above belong to bricks of length α_j, $1 \le j \le \ell(\alpha)$. The fact that $e_\lambda = \prod_{\lambda_i > 0} e_{\lambda_i}$ means that e_λ is a generating function for sequences of compositions of $\lambda_1, \dots, \lambda_n$, where each brick of length α_j has weight h_{α_j}, and each term has sign $(-1)^{\ell(\alpha)}$. Such objects may be drawn as brick tabloids, as illustrated below, and defined in Section 1.2.

$$\begin{pmatrix} h_1 & h_2 & \underline{\mathbf{h}_3} & h_4 & h_5 \\ \underline{1} & h_1 & h_2 & h_3 & h_4 \\ 0 & \underline{1} & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 & \underline{\mathbf{h}_2} \\ 0 & 0 & 0 & \underline{1} & h_1 \end{pmatrix} \begin{pmatrix} \underline{\mathbf{h}_1} & h_2 & h_3 \\ 1 & \underline{\mathbf{h}_1} & h_2 \\ 0 & 1 & \underline{\mathbf{h}_1} \end{pmatrix} \begin{pmatrix} \underline{\mathbf{h}_1} & h_2 & h_3 \\ 1 & h_1 & \underline{\mathbf{h}_2} \\ 0 & \underline{1} & h_1 \end{pmatrix}$$

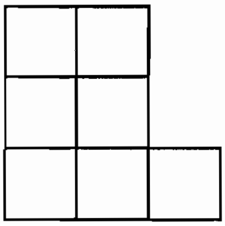


Alternatively, the coefficient of h_μ in e_λ is given by $\operatorname{sgn}(\mu) B_{\mu,\lambda}$, where $B_{\mu,\lambda}$ is the number of brick tabloids with rows of length $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ and for which the partition μ records the lengths of the bricks. Recall that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$, and note that $f_\mu = \sum_{\lambda \vdash n} \langle f_\mu, h_\lambda \rangle m_\lambda$ and $e_\lambda = \sum_{\mu \vdash n} \langle e_\lambda, m_\mu \rangle h_\mu$. The observation $\langle f_\mu, h_\lambda \rangle = \langle \omega(f_\mu), \omega(h_\lambda) \rangle$ $= \langle e_\mu, m_\lambda \rangle = \langle e_\lambda, m_\mu \rangle = \operatorname{sgn}(\mu) B_{\mu,\lambda}$ provides us with an interpretation of $f_\mu(x_1, \dots, x_N)$ in terms of brick tabloids. In Chapter 3 we find a satisfying generalization of this interpretation, and in the conclusion of this thesis we speculate as to whether there is a context in which the forgotten symmetric functions $\{f_\lambda\}_{\lambda \vdash n}$ may be found to be a "natural" basis.

SECTION 1.2 TABLEAUX AND TABLOIDS

Any two bases $\{b_\lambda\}_{\lambda\succ n}$ and $\{a_\mu\}_{\mu\succ n}$ of the symmetric functions can be related by the transition matrix $\mathbf{M}(b,a)_{\lambda,\mu}$ from the a_μ's to the b_λ's, so that $b_\lambda = \sum_{\mu\succ n}\mathbf{M}(b,a)_{\lambda,\mu}a_\mu$ for all $\lambda \succ n$. Thirty nontrivial transition matrices relate the six bases $\{e_\lambda\}_{\lambda\succ n}$, $\{m_\lambda\}_{\lambda\succ n}$, $\{s_\lambda\}_{\lambda\succ n}$, $\{p_\lambda\}_{\lambda\succ n}$, $\{h_\lambda\}_{\lambda\succ n}$, $\{f_\lambda\}_{\lambda\succ n}$, that we introduced in the previous section. We use them in Chapter 3 of this thesis to express $f_\lambda(\xi_1,\ldots,\xi_N)$ and $m_\lambda(\xi_1,\ldots,\xi_N)$ in terms of $p_\lambda(\xi_1,\ldots,\xi_N)$, and we use them in Chapter 4 of this thesis to express $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of $h_\lambda(\xi_1,\ldots,\xi_N)$. Our methods depend on having combinatorial means of expressing the entries of the transition matrices $\mathbf{M}(f,p)_{\lambda,\mu}$, $\mathbf{M}(m,p)_{\lambda,\mu}$, $\mathbf{M}(s,h)_{\lambda,\mu}$, as well as $\mathbf{M}(f,m)_{\lambda,\mu}$, $\mathbf{M}(s,m)_{\lambda,\mu}$, $\mathbf{M}(p,s)_{\lambda,\mu}$. Such entries are indexed by pairs of partitions, as are the combinatorial objects used to express them. In this section we describe how we use shapes to depict partitions. We then define various combinatorial objects indexed by pairs of partitions, namely, brick tabloids, weighted brick tabloids, ordered brick tabloids, column strict tableaux, rim hook tableaux, special rim hook tabloids. We put the brick tabloids to use in Chapter 3, and the others in Chapter 4.

Given a partition $\lambda \succ n$, we adhere to the French convention of depicting it as a set of n squares with $\ell(\lambda)$ rows, for which the lengths of the rows are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)}$ from bottom to top, and the leftmost square in each row is in the same column. We refer to such a set as the Ferrers diagram of shape λ, or simply, the *shape* λ. For example, below, on the left, is the shape 322.



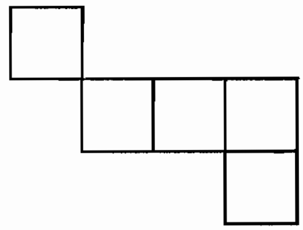
We orient ourselves in the shape λ with the aid of the directions of the compass: north (up), south (down), east (right), west (left). We endow the squares of the shape λ with a coordinate system $E \times N$, east by north, as shown below.

$(1,3)_{E\times N}$	$(2,3)_{E\times N}$	
$(1,2)_{E\times N}$	$(2,2)_{E\times N}$	$(3,2)_{E\times N}$
$(1,1)_{E\times N}$	$(2,1)_{E\times N}$	$(3,1)_{E\times N}$

We may also think of the shape λ as embedded in a plane of squares and use the $E \times N$ coordinate system to refer to squares outside of the shape λ.

The tableaux and tabloids that we define in this section are indexed by pairs of partitions. One of these gives the shape of the tableau or tabloid. The other dictates the sizes of disjoint subsets that make up the shape, and is called the *type*. In some constructions we imagine these subsets as being laid down in a certain order. An order $\sigma \in S_N$ is specified, so that in each of the objects that results, if the type is $\nu \succ n$, $n \le N$, then for all i, $1 \le i \le N$, the ith subset has length $\nu_{\sigma(i)}$. With this in mind, we define a *weak composition* $V = (V_1,\dots,V_N)$ of n to be a sequence of nonnegative integers such that $V_1 + \dots + V_N = n$. We write $V \to \nu$ or $(V_1,\dots,V_N) \to \nu$ if the positive integers of the partition $\nu \succ n$ are those of V. In particular, $m_\nu(x_1,\dots,x_N) = \sum_{(V_1,\dots,V_N)\to\nu} x_1^{V_1} \cdots x_N^{V_N}$.

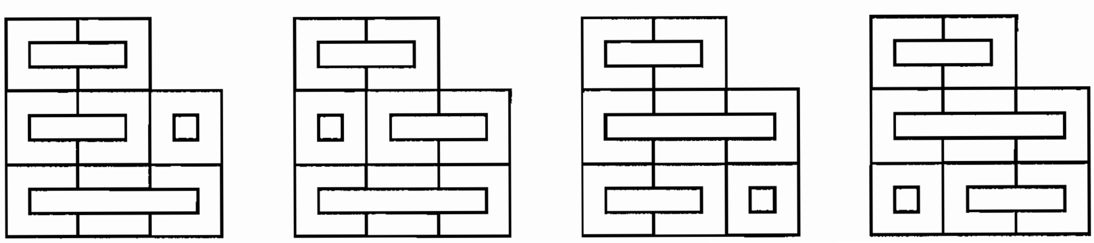
Given shapes $\lambda \succ (n+x)$, $\mu \succ n$, with $\lambda_i - \mu_i \ge 0$ for all i, $1 \le i \le n$, the *skew shape* $\lambda - \mu$ is the set of squares $(E,N)_{E\times N}$ for which $(E,N)_{E\times N} \in \lambda$ and $(E,N)_{E\times N} \notin \mu$. For example, the skew shape $441-31$ is the difference of the shape 441 and the shape 31 and is drawn as below.



The ith *row* of a shape λ or a skew shape $\lambda - \mu$ is the set of its squares $(E,N)_{E\times N}$ for which $N = i$. The jth *column* of a shape λ or a skew shape $\lambda - \mu$ is the set of its squares $(E,N)_{E\times N}$ for which $E = j$.

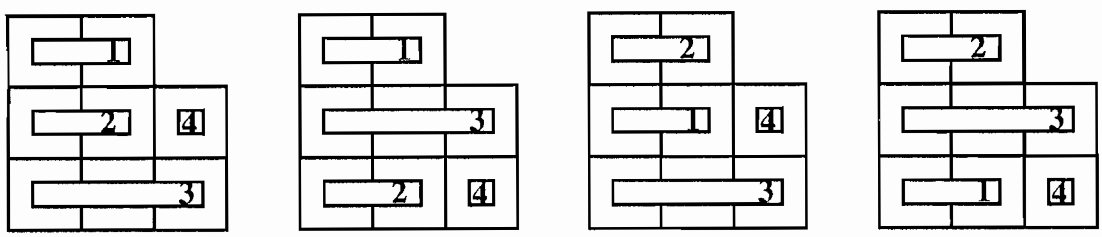
Two squares of the form $(i,j)_{E\times N}$, $(i+1,j)_{E\times N}$ or $(i,j)_{E\times N}$, $(i,j+1)_{E\times N}$ are said to be *adjacent*. A set of squares Ξ is *connected* if given any two squares $A,B \in \Xi$ there exists a sequence of squares $\{C_1,C_2,\ldots,C_r\} \subset \Xi$ such that $A = C_1$ and $B = C_r$ and C_i is adjacent to C_{i+1} for all $0 < i < r$. A *brick* of length r is a connected set $\{(i,j+k)_{E\times N} | 1 \le k \le r\}$, with $i,\ j \in \mathbb{Z}$ and is said to start at $(i+1,j)_{E\times N}$ and end at $(i,j+r)_{E\times N}$. We now define three kinds of brick tabloids introduced by Eğecioğlu and Remmel [ER2]. They play a central role in Chapter 3.

A *brick tabloid* $Y_{\nu,\mu}$ of shape μ and type ν is an object that consists of a shape $\mu \succ n$ and a decomposition of it as a disjoint union of bricks of size $\nu_1,\ldots,\nu_{\ell(\nu)}$. For example, the brick tabloids of shape 332 and type 3221 are:



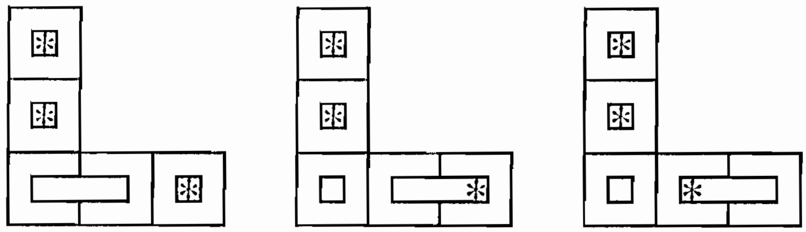
As mentioned in Section 1.1, f_ν can be defined in terms of brick tabloids as $f_\nu = \operatorname{sgn}(\nu)\sum_{\mu \vdash n} B_{\nu,\mu} m_\mu$, where $B_{\nu,\mu}$ is the number of brick tabloids of shape μ and type ν.

Fix a weak composition $(V_1,\dots,V_N) \to \nu$. Suppose that $Y_{\nu,\mu}$ is a brick tabloid of shape μ and type ν. Assign to each brick of $Y_{\nu,\mu}$ one of the letters $1,\dots,\ell(\nu)$ so that for all i, $1 \le i \le \ell(\nu)$, the letter i is assigned to a brick of length V_i. If in each row of $Y_{\nu,\mu}$ the bricks are ordered so that the letters increase from west to east, then we say that it is an *ordered brick tabloid* $Y_{\nu,\mu}$ of shape μ and type ν. For example, if the weak composition is 2231, then the ordered brick tabloids of shape 332 and type 3221 are:



Ordered brick tabloids give a combinatorial interpretation of $\operatorname{ch}^{-1}(h_\mu)(\nu) = O_{V,\mu}$, where $O_{V,\mu}$ is the number of such tabloids of shape μ and type ν with bricks of length $V_1,\dots,V_N$. Ours is a slight modification of the original definition that Eğecioğlu and Remmel give for ordered brick tabloids. They assume that $V_1 = \nu_1,\dots,V_N = \nu_N$, but we note that any fixed $(V_1,\dots,V_N) \to \nu$ yields the same number $O_{V,\mu}$ of tabloids. This property may seem uninteresting with regard to ordered brick tabloids, but in chapter 4 we derive from it the analogous but not obvious property for rim hook tableaux.

Again, suppose that $Y_{\nu,\mu}$ is a brick tabloid of shape μ and type ν. Within each row distinguish a square in the rightmost brick by placing an asterisk inside of it. The resulting object is a *weighted brick tabloid* $Y_{\nu,\mu}$ of shape μ and type ν. For example, the weighted brick tabloids of shape 311 and type 2111 are



Eğecioğlu and Remmel define weighted brick tabloids as brick tabloids that come with a weight $\prod_{i=1}^{\ell(\mu)} b_i$ where b_i is the number of squares in the rightmost brick of the ith row. Our definition is equivalent to theirs: our asterisks play the role of their weights. In Chapter 3 we make use of the fact that $\mathrm{ch}^{-1}(f_\nu)(\mu) = \mathrm{sgn}(\nu)W_{\nu,\mu}$, where $W_{\nu,\mu}$ is the number of weighted brick tabloids of shape μ and type ν. Most of Section 3.2 is devoted to further discussion of these tabloids because they play a prominent role in our proofs of Theorems 3.3.1 and 3.4.1.

Just as the brick tabloids are central to our work in Chapter 3, so are column strict tableaux, rim hook tableaux, and special rim hook tabloids in Chapter 4. We review the construction of each of the latter three objects. In what follows, let $X_{\mu,V} = \left(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(N)}\right)$ be an object that consists of a sequence of shapes $\varnothing = \mu^{(0)} \subset \mu^{(1)} \subset \mu^{(2)} \subset \ldots \subset \mu^{(N)} = \mu$ such that for all i, $1 \le i \le N$, the skew shape $\mu^{(i)} - \mu^{(i-1)}$ has V_i squares, and $V \to \nu$.

$X_{\mu,V}$ is a *column strict tableau* of shape μ and type ν if for all i, $1 \le i \le N$, each square of $\mu^{(i)} - \mu^{(i-1)}$ is the northernmost square of a column of $\mu^{(i)}$. If $\mu^{(i)} - \mu^{(i-1)}$ has r squares, then it is called a horizontal r-strip. In drawing a column strict tableau the squares of the skew shapes $\mu^{(i)} - \mu^{(i-1)}$ are filled with the letter i. For example, the column strict tableaux of shape 321 and type 2211 are:

3		
2	2	
1	1	4

4		
2	2	
1	1	3

3		
2	4	
1	1	2

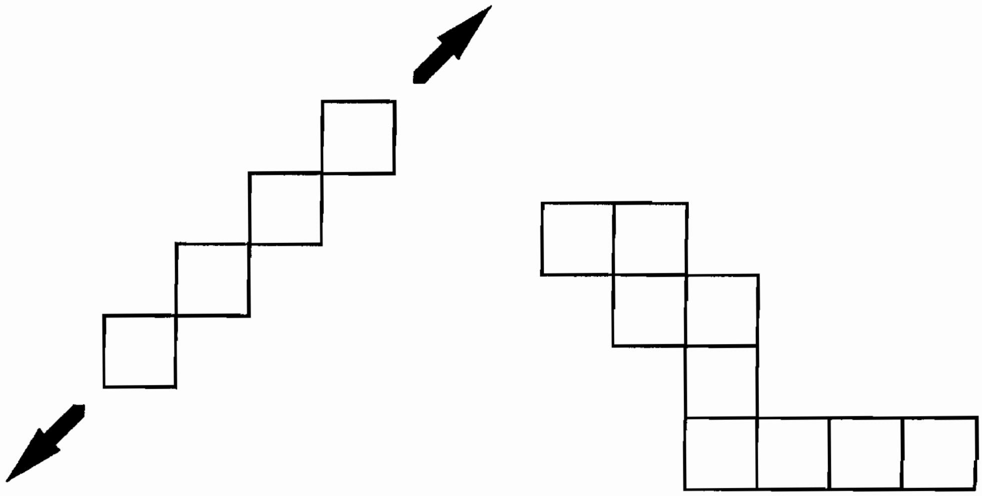
4		
2	3	
1	1	2

$X_{\mu,V}$ is a *standard tableau* if it is a column strict tableau and $\nu = 1^n$.

Alternatively, a column strict tableau of shape μ may be defined as a shape μ with a letter assigned to each square so that the letters increase strictly (<) as one moves north along any column whereas the letters increase weakly ($\leq$) as one moves east along any row. A column strict tableau is said to have *content* $1^{V_1}2^{V_2}\cdots N^{V_N}$ if for all i, $1 \leq i \leq N$, it has V_i occurrences of the letter i, and is said to have type ν, where $V \to \nu$. A standard tableau has type $\nu = 1^n$, and therefore its letters increase strictly, both as one moves north, and as one moves east.

If we fix a weak composition $V \to \nu$, and let C_V^μ be the set of column strict tableaux $X_{\mu,V}$, then the Kostka matrix $K_{\mu,\nu} = \mathbf{M}(s,m)_{\mu,\nu}$ has the combinatorial interpretation $K_{\mu,\nu} = \sum_{X_{\mu,V} \in C_V^\mu} 1$. Therefore the Schur function has the combinatorial interpretation $s_\mu(x_1,\ldots,x_N) = \sum_{X_{\mu,V} \in C_V^\mu,\ V \to \nu} x_1^{V_1} \cdots x_N^{V_N}$. One of the two main result of this thesis is Theorem 4.3.1, which generalizes this interpretation by considering what happens when $x_1,\ldots,x_N$ are the eigenvalues $\xi_1,\ldots,\xi_N$ of an $N \times N$ matrix $\mathbf{A}$. This generalization involves rim hook tableaux, which we now define.

A *northeasterly line* is a set of squares $\{(i+k,k)_{K\times N} \mid k \in \mathbb{Z}\}$, $i \in \mathbb{Z}$, as illustrated below, on the left. A set of squares R is a *rim hook* if it is connected and consists of at most one square per northeasterly line, as illustrated below, on the right.



The *start* of the rim hook R is its northwesternmost square, and the *end* of R is its southeasternmost square. We may also use these two words as verbs, and therefore prefer them to the words 'tail' and 'head', which appear in the literature. The length of R is the total number of its squares. If a skew shape $\mu - \tilde{\mu}$ is a rim hook, then we say that it is an inner rim hook of μ and an outer rim hook of $\tilde{\mu}$.

The northeasterly lines reveal much about the properties of rim hooks, and in Section 4.2 we introduce a second coordinate system that makes these properties apparent. But it is also helpful to keep in mind three remarks that elaborate on the fact that rim hooks are connected sets. First, if a rim hook R starts on one northeasterly line and ends on another, then every northeasterly line in between contains exactly one square of R. Second, if a square C belongs to a rim hook R, then either R ends at C, R steps east from C, or R steps south from C. Third, disjoint rim hooks cannot cross over each other.

As before, let $X_{\mu,\nu} = \left(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(N)}\right)$ be an object that consists of a sequence of shapes $\varnothing = \mu^{(0)} \subset \mu^{(1)} \subset \mu^{(2)} \subset \ldots \subset \mu^{(N)} = \mu$ such that for all i, $1 \le i \le N$, the skew shape $\mu^{(i)} - \mu^{(i-1)}$ has V_i squares, and $\left(V_1, \ldots, V_N\right) \to \nu$. Then $X_{\mu,\nu}$ is a *rim hook tableau* of shape μ and type ν if for all i, $1 \le i \le N$, $\mu^{(i)} - \mu^{(i-1)}$ is a rim hook. In drawing a rim hook tableau the squares of the skew shape $\mu^{(i)} - \mu^{(i-1)}$ are filled with the letter i. For example, the rim hook tableaux of shape 332 and type 332 are:

1	2	
1	2	3
1	2	3

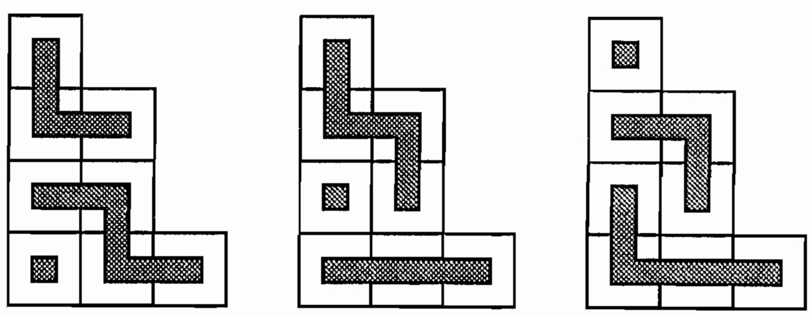
2	2	
1	2	3
1	1	3

3	3	
1	2	2
1	1	2

3	3	
2	2	2
1	1	1

The sign of a rim hook tableau is an important factor when these tableaux are used in counting. The *sign of a rim hook* that starts at $(E,N)_{E\times N}$ and ends at $(E',N')_{E\times N}$ is $(-1)^{N-N'}$. The *sign of a rim hook tableau* is the product of the signs of its rim hooks. In the example above, the rim hook tableaux, from left to right, have sign $(-1)^{2+2+1}$, $(-1)^{1+1+1}$, $(-1)^{1+1+0}$, and $(-1)^{0+0+0}$.

If a rim hook is a subset of μ, then it is a *special rim hook* if it starts in the leftmost column of μ. A rim hook tableau of shape μ and type ν is a *special rim hook tabloid* $X_{\mu,\nu}$ of shape μ and type ν if for all i, $1 \le i \le N$, its rim hook $\mu^{(i)} - \mu^{(i-1)}$ is either empty or starts at $(1,i)_{E\times N}$. In drawing a special rim hook tabloid the skew shapes $\mu^{(i)} - \mu^{(i-1)}$ are indicated by solid lines. For example, the special rim hook tabloids of shape 3221 and type 431 are:

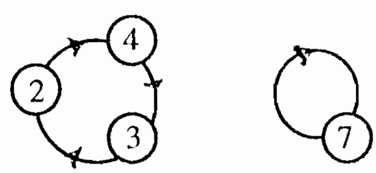


The *sign of a special rim hook tabloid* is defined to be the product of the signs of its rim hooks.

Given a shape μ, the *conjugate shape* μ' is the set of squares for which $(i,j)_{E\times N} \in \mu'$ if and only if $(j,i)_{E\times N} \in \mu$. This is an important concept in working with the Schur functions because $\omega(s_\mu) = s_{\mu'}$, but our only use for it will be the identity $s_{\mu'} = \det\left(e_{\mu_i'-i+j}\right)_{1\le i,j\le N}$, $\mu' \succ n$, $n \le N$, which is the dual of the Jacobi-Trudi identity $s_\mu = \det\left(h_{\mu_i-i+j}\right)_{1\le i,j\le N}$ and follows from the fact that $\omega(h_r) = e_r$. In Chapter 4 the Jacobi-Trudi identity also takes the form $s_\mu = \sum_{\nu\succ n} K^{-1}_{\nu,\mu} h_\nu$, where the set $S_{\mu,\nu}$ of special rim hook tabloids of shape μ and type ν is used to calculate $K^{-1}_{\nu,\mu} = \sum_{X_{\mu,\nu}\in S_{\mu,\nu}} \operatorname{sgn}(X_{\mu,\nu})$. Section 4.2 is especially devoted to the study of the relation between special rim hook tabloids and rim hook tableaux.

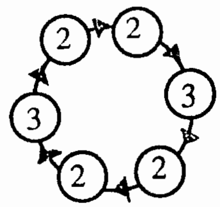
SECTION 1.3 WALKALONGS

If we evaluate a symmetric function $b(x_1,...,x_N)$ at the eigenvalues $\xi_1,...,\xi_N$ of an $N \times N$ matrix $\mathbf{A}$, then $b(\xi_1,...,\xi_N)$ can be expressed as a polynomial in the edges of $\mathbf{A}$. Our purpose in this thesis is to provide a description for the terms of such a polynomial as weights of combinatorial objects generated by $b(\xi_1,...,\xi_N)$. For example, from the equation $\sum_{r=0}^{N} t^r e_r(\xi_1,...,\xi_N) = \prod_{i=1}^{N}(1+t\xi_i) = \det(\mathbf{I}+t\mathbf{A})$ we know that the elementary symmetric function $e_r(\xi_1,...,\xi_N)$ is the sum of the determinants of the $r \times r$ principal minors of $\mathbf{A}$. If $r=4$ and $N=8$, then $e_4(\xi_1,...,\xi_8)$ has a term $a_{24}a_{32}a_{43}a_{77}$ which we may think of as a weight $\mathrm{W}_{\mathbf{A}}(s)$ associated with the following set s of disjoint cycles.



The fact that every symmetric function b of degree n has an expansion $b = \sum_{\lambda \vdash n} C_\lambda e_\lambda$ in terms of the elementary symmetric functions means that each term $a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$ of $b(\xi_1,...,\xi_N)$ may be thought of as a weight associated with a multiset of cycles. The purpose of this section is to define the combinatorial objects that we will have cause to associate with a term $a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$. These are objects which can be constructed out of cycles, among them closed walks, circular walks, and even Lyndon words. One obstacle that we try to overcome is the decidedly linear nature of mathematical notation. We counter this prejudice by introducing the walkalong, which allows us to define a variety of combinatorial objects in a context that keeps us aware of the circular nature that they share in common.

A walkalong is a structure very much like a cycle except that the letters involved may appear not only once, but several times, as in the example below.

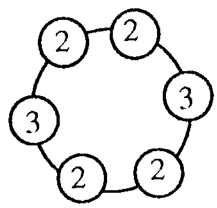


The structure shown above very much resembles a closed walk, except that there is no distinguished vertex at which the walk would start and end. Indeed, at times we want to concentrate on this aspect, and we speak of such a structure as a circular walk. However, at other times we want to refer to various combinatorial objects with a name that reminds us of what they have in common. We want to make use of operations that apply to all such objects, but we also want to formalize concepts that will allow us to generate new combinatorial objects. For all of these reasons we introduce the walkalong.

If $X = \{x_1, x_2, x_3, \ldots\}$ is a set, then we say that $x_1, x_2, x_3, \ldots$ are its elements. A multiset is a set in which the same element may appear several times. For example, the prime factors of 28 are given by the multiset $\{2,2,7\}$. More formally, a *multiset* $Y = \{y_1, y_2, y_3, \ldots\}$ is a set $\{y_1, y_2, y_3, \ldots\}$ together with an equivalence relation on its elements. Any equivalence relation establishes equivalence classes, for example, $\{2,2\}$, $\{7\}$. A *letter* is a representative of an equivalence class. Therefore a letter may refer to an element or to an equivalence class, and which is actually meant is usually made clear by the context.

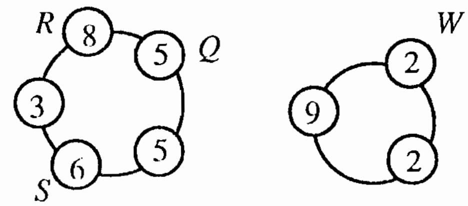
Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ be a set of r elements on which we do *not* define a total order. Let σ be the bijection defined by $\sigma(x_{i_1}) = x_{i_2}$, $\sigma(x_{i_2}) = x_{i_3}, \ldots, \sigma(x_{i_r}) = x_{i_1}$. Given a set $\mathbf{R}$ of letters, say that two functions $F: \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \to \mathbf{R}$ and $F': \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \to \mathbf{R}$ are equivalent if and only if $F'(\sigma^k(x_{i_j})) = F(x_{i_{j+1}})$ for some k, $1 \le k \le r$, and for all j,

$1 \le j \le r$. Let $C(\mathbf{R})$ be the set of equivalence classes on the functions $F:\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \to \mathbf{R}$. Then a *walkalong* on $\mathbf{R}$ with a total of r places is an element of $C(\mathbf{R})$. Less formally, a walkalong on $\mathbf{R}$ of length r is a way of filling the places of a cycle of length r with letters from the set $\mathbf{R}$. In drawing a walkalong w we fix a representative $w \in C(\mathbf{R})$ and we draw the letters $F(x_{i_1}), \ldots, F(x_{i_r})$ in a circle so that $F(x_{i_1})$ always appears clockwise of $F(x_{i_r})$, and in general, $F(x_{i_{j+1}})$ *always appears clockwise of* $F(x_{i_j})$, as in the example below, where $\mathbf{R} = \{1,2,3\}$ and $r = 6$.



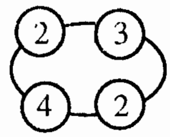
The resulting drawings are in one-to-one correspondence with the walkalongs from which they arise. We think of each letter as occurring at a *place* in the walkalong, which we draw ○. We emphasize that there is no first or last place in the walkalong.

Given a walkalong of length r, specify a place Q within it. We then refer to the other places of the walkalong as $Q+1, Q+2, \ldots, Q+r-1$ as we go around clockwise, with $Q+i+r = Q+i$ for all $i \in \mathbb{Z}$. If places $R_1, \ldots, R_k$ of a walkalong appear in that order as we go around clockwise, then we define the *circular distances* between them to be the lengths $r_1, \ldots, r_k$, which are the smallest positive integers such that for all j, $1 \le j \le k-1$, $R_j + r_j = R_{j+1}$, and $R_k + r_k = R_1$. The concept of circular distance is important in our combinatorial interpretation of $f_\lambda(\xi_1, \ldots, \xi_N)$, where we apply it to a multiset of walkalongs, such as the one below.

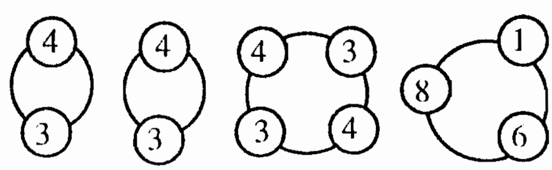


In the example above the circular distances between Q, R, S, W are given by the partition 3221.

Given a walkalong w on $\mathbf{R}$ of length $r > 0$, distinguish a place Q. Fix an $R \times R$ matrix $\mathbf{A} = (a_{ij})_{i,j \in \mathbf{R}}$ with entries from a commutative ring with identity, and for all i, $0 \le i \le r$, let $s(Q+i)$ be the letter at the place $Q+i$. We define the weight $\mathrm{W}_{\mathbf{A}}(w)$ to be $\mathrm{W}_{\mathbf{A}}(w) = \prod_{0 \le i \le r-1} a_{s(Q+i),s(Q+i+1)}$ and the sign $\mathrm{sgn}(w)$ to be $(-1)^{r-1}$. For example, if w is the walkalong shown below, then $\mathrm{W}_{\mathbf{A}}(w) = a_{23}a_{32}a_{24}a_{42}$ and $\mathrm{sgn}(w) = (-1)^3$.



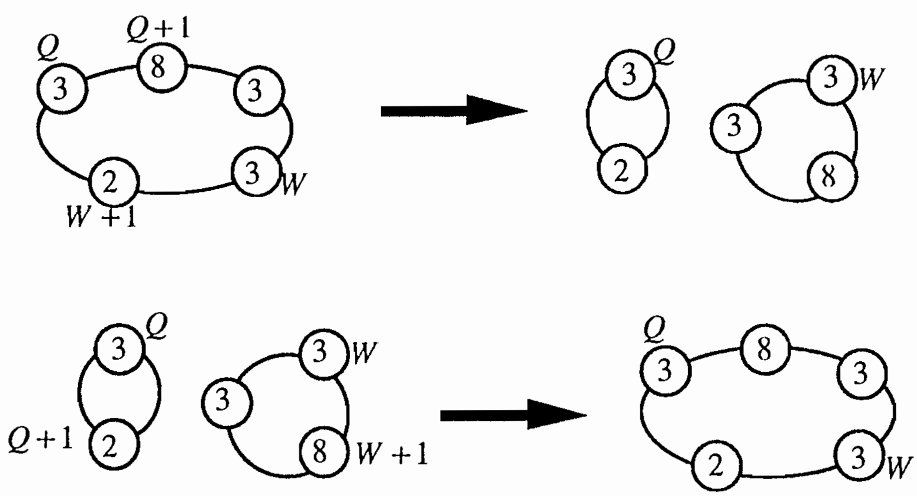
If $s = (w_1, \ldots, w_k)$ is a sequence of walkalongs, then we define $\mathrm{W}_{\mathbf{A}}(s) = \prod_{i=1}^{k} \mathrm{W}_{\mathbf{A}}(w_i)$ and $\mathrm{sgn}(s) = \prod_{i=1}^{k} \mathrm{sgn}(w_i)$. Likewise, if $m = (w_1, \ldots, w_k)$ is a multiset of walkalongs, then we define $\mathrm{W}_{\mathbf{A}}(m) = \prod_{i=1}^{k} \mathrm{W}_{\mathbf{A}}(w_i)$ and $\mathrm{sgn}(m) = \prod_{i=1}^{k} \mathrm{sgn}(w_i)$. For example, the weight of the multiset of walkalongs shown below



is $a_{16}a_{34}^4a_{43}^4a_{68}a_{81}$, and the sign is $(-1)^{1+1+3+2} = -1$.

Given a walkalong w on $\mathbf{R}$ of length $r > 0$, we say that each place Q in w is a *directed place*, by which we emphasize that there is a single place $Q+1$ to which Q is directed, and a single place $Q-1$ which is directed to Q. In contrast, we establish a convention that a walkalong w on $\mathbf{R}$ of length zero is an *undirected* place with an associated letter $i \in \mathbf{R}$, which we draw $\textcircled{i}$ and for which we define $W_A(w) = 1$ and $\operatorname{sgn}(w) = 1$. This distinction plays a role in our proof of Theorem 2.3.1.

Structures like walkalongs arise often in the study of the combinatorics of a matrix $\mathbf{A}$, as in Zeilberger's [Z] paper "A combinatorial approach to matrix algebra", and also the last chapter of Brualdi and Ryser's [BR] book Combinatorial Matrix Theory. A common technique that arises is one that we call *redirecting places*. Let $Q \neq W$ be two places in a multiset m of walkalongs such that the letter at Q is the letter at W. Then we may speak of redirecting Q to $W+1$ and W to $Q+1$, as in the two examples below.



If Q and W are in the same walkalong, then redirecting places in this way creates one walkalong from two, and if Q and W are in different walkalongs, then it creates two walkalongs from one. A consequence of this is the following lemma.

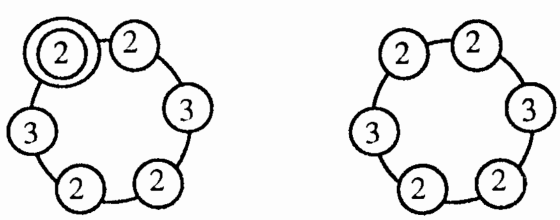
LEMMA 1.3.1 *Fix* $\mathbf{A}=(a_{ij})_{i,j\in\mathbf{R}}$. *Let* $Q \neq W$ *be two places in a multiset* m *of walkalongs on* $\mathbf{R}$ *such that the letter at* Q *is the letter at* W. *Let* $\tilde{m}$ *be the multiset of walkalongs on* $\mathbf{R}$ *that arises from* m *upon redirecting* Q *to* $W+1$ *and* W *to* $Q+1$. *Then* $\mathrm{W}_{\mathbf{A}}(\tilde{m})=\mathrm{W}_{\mathbf{A}}(m)$ *and* $\mathrm{sgn}(\tilde{m})=-\mathrm{sgn}(m)$.

We use this lemma in various involutions, especially in Chapter 2, because if the same places Q and W are redirected a second time, then we get back the original multiset m.

Another important technique arises in working with a matrix $-\mathbf{A}$. If we take account of the sign of the entries of $-\mathbf{A}$, then given a walkalong w of length r, we have that $\mathrm{sgn}(w)\mathrm{W}_{-\mathbf{A}}(w)=(-1)^{r-1}(-1)^{r}\mathrm{W}_{\mathbf{A}}(w)=-\mathrm{W}_{\mathbf{A}}(w)$. Therefore we have the following lemma.

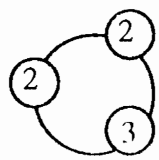
LEMMA 1.3.2 *Fix* $\mathbf{A}=(a_{ij})_{i,j\in\mathbf{R}}$. *Let* w *be a walkalong on* $\mathbf{R}$ *and let* m *be a multiset of walkalongs on* $\mathbf{R}$. *Then* $\mathrm{sgn}(\{w\}\cup m)\mathrm{W}_{-\mathbf{A}}(\{w\}\cup m)=$
$\mathrm{sgn}(m)\mathrm{sgn}(w)\mathrm{W}_{-\mathbf{A}}(m)\mathrm{W}_{\mathbf{A}}(w)$.

In some constructions we distinguish a unique place within a walkalong to serve as its *origin*. We draw such a place as ◎. We say that a walkalong with an origin is a *closed walk*, and that a walkalong without an origin is a *circular walk*. Below we exhibit an example of each.



The existence of an origin makes it straightforward to depict the walkalong in linear fashion as a word by listing the letters at its places from left to right, starting with the letter at the origin. In general, a *word* is a sequence of letters from left to right, such as 233124.

In this thesis we define a *Lyndon word* to be a circular walk without rotational symmetry, as in the example below.



We may compare our definition of Lyndon word with the traditional one. Suppose that there is a total order on the elements of $\mathbf{R}=\{r_1,\ldots,r_R\}$, so that $r_1<\ldots<r_R$. Let $\mathbf{R}^+$ be the semigroup of all words $r_{i_1}r_{i_2}\cdots r_{i_s}$, $s\geq 1$, $1\leq i_1,i_2,\ldots,i_s\leq R$, with the concatenation product $\left(r_{i_1}r_{i_2}\cdots r_{i_s}\right)\left(r_{j_1}r_{j_2}\cdots r_{j_t}\right)=\left(r_{i_1}r_{i_2}\cdots r_{i_s}r_{j_1}r_{j_2}\cdots r_{j_t}\right)$. The total order on $\mathbf{R}$ is extended to the *lexicographic order* on $\mathbf{R}^+$ by declaring that for all $r_{i_1}r_{i_2}\cdots r_{i_s}$, $r_{j_1}r_{j_2}\cdots r_{j_t}\in\mathbf{R}^+$ we write $r_{i_1}r_{i_2}\cdots r_{i_s}<r_{j_1}r_{j_2}\cdots r_{j_t}$ if and only if either $r_{i_k}=r_{j_k}$ for all k, $1\leq k\leq s$, and $s<t$, or there exists a $u<s,t$ such that $r_{i_k}=r_{j_k}$ for all k, $1\leq k\leq u$, and $r_{i_{u+1}}<r_{j_{u+1}}$. Then $r_{i_1}r_{i_2}\cdots r_{i_s}$ is a Lyndon word if and only if $r_{i_1}r_{i_2}\cdots r_{i_s}<r_{i_k}r_{i_{k+1}}\cdots r_{i_s}$ for all k, $1<k\leq s$. For example, if $\mathbf{R}=\{1,2,3,4,5\}$, then 22523 is a Lyndon word, but 2252 and 225225 are not. It follows that if $r_{i_1}r_{i_2}\cdots r_{i_s}$ is a Lyndon word, then for any rotation $\theta\in S_s$, $\theta\notin id$, $r_{\theta(i_1)}r_{\theta(i_2)}\cdots r_{\theta(i_s)}$ is not a Lyndon word. Moreover, a Lyndon word cannot consist of a repeated pattern. Consequently, if we write out the letters of a Lyndon word in a circle, we can always recover the first letter. In the same way, if a circular walk of length r has no rotational symmetry, then there is a unique place Q such that the letters at the places

$Q, Q+1, Q+2, \ldots, Q+r-1$ form a Lyndon word, if read in this order. This shows that our 'circular' definition of Lyndon word coincides with the traditional 'linear' one.

A good reference for the combinatorics of words is Combinatorics on Words, [Lo], written collectively under the pseudonym Lothaire. The properties of Lyndon words are discussed in Chapter 5, written by Reutenauer, which includes a proof of the following result.

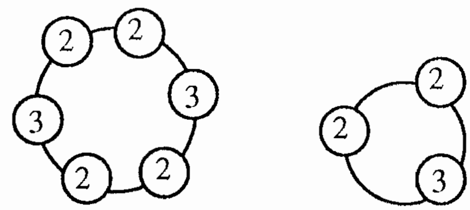
LEMMA 1.3.3 *Any word* $w \in \mathbf{R}^+$ *can be uniquely factored into Lyndon words* $w = w_1 w_2 \cdots w_k$ *where* $w_1 \geq w_2 \geq \ldots \geq w_k$.

The above mentioned factorization of a word w is gotten by finding the shortest word u such that $w = uw_1$ and w_1 is a Lyndon word, and then proceeding by induction. For example, $22136115 = (2)(2)(136)(115)$. This establishes a correspondence between words of length n and multisets of Lyndon words with a total of n letters. In Section 2.1 we will observe that if the word w and the Lyndon words $w_1, w_2, \ldots, w_k$ are coded as walkalongs, then this correspondence breaks down. The symmetric function $p_n(\xi_1, \ldots, \xi_N)$ generates all words on $1, \ldots, N$ of length n, whereas $h_n(\xi_1, \ldots, \xi_N)$ generates all multisets $w_1, w_2, \ldots, w_k$ of Lyndon words on $1, \ldots, N$ with a total of n places.

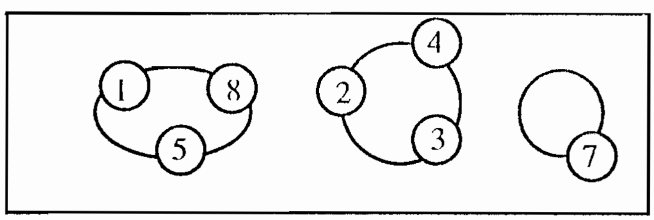
Lyndon words play a central role in the most important results of this thesis. One reason for this is the Lemma 1.3.3. Suppose that we have walkalongs $w_1 = w_2 = \ldots = w_k$. In each walkalong w_i choose a place Q_i, so that for all i, j, $1 \leq i, j \leq k$, the place Q_i in w_i is indistinguishable from the place Q_j in w_j. For all i, $1 \leq i < k$, redirect Q_i to Q_{i+1}, and redirect Q_k to Q_1. We say that the resulting walkalong w is a kth power of the walkalong w_1. The following lemma is then a consequence of our definition of a Lyndon word as a circular walk with no rotational symmetry.

LEMMA 1.3.4 *Suppose that there is a total order on* $\mathbf{R}$. *Then every circular walk on* $\mathbf{R}$ *is uniquely expressible as a power of a Lyndon word on* $\mathbf{R}$.

For example, the circular walk below, on the left, is a power of the Lyndon word below, on the right.

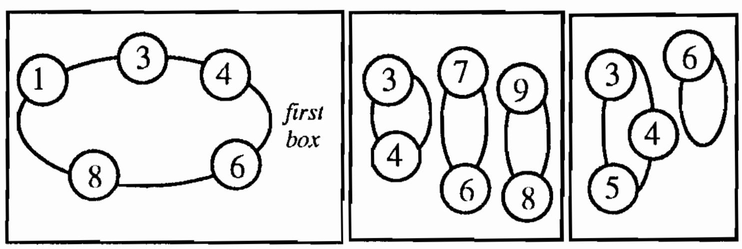


A *cycle* on $\mathbf{R}$ is a circular walk on $\mathbf{R}$ in which no two places have the same letter. If there is a total order on $\mathbf{R}$, then a cycle is necessarily a Lyndon word. A multiset m of cycles on $\mathbf{R}$ in which no two places of m have the same letter is usually known as a set of disjoint cycles. We will refer to such a multiset m as a *box of cycles* on $\mathbf{R}$, which we draw as shown below.



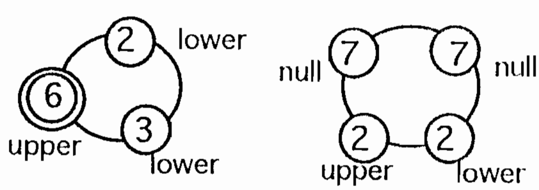
Note that $\mathrm{sgn}(m)$ is the same as it would be defined for a permutation with the cycle structure of m. If $\mathrm{sgn}(m)$ appears in a combinatorial construction, then we may speak of a *signed box of cycles*. For example, the expression $\sum_{r=0}^{N} t^r e_r(\xi_1,\ldots,\xi_N) = \det(\mathbf{I}+t\mathbf{A})$ shows that $e_r(\xi_1,\ldots,\xi_N)$ is a generating function $e_r(\xi_1,\ldots,\xi_N) = \sum_{b\in B} \mathrm{sgn}(b)\,\mathrm{W}_\Lambda(b)$ for the set B of signed boxes of cycles on $\{1,\ldots,N\}$ that use a total of r places.

If we think of cycles as elements of the symmetric group, then we can recall that two cycles commute if and only if they are disjoint. If we are interested in counting products of cycles, then we want to make sure that we count either $(12)(3)$ or $(3)(12)$, but not both. We say that two products of cycles are equivalent if they can be gotten from each other by interchanging adjacent disjoint cycles. Our problem is to select one representative from each equivalence class. One way to do this is to define a *cycle product* to be a sequence of cycles $C_1,\ldots,C_r$ such that for all i, $1 \le i < r$, if C_i and C_{i+1} are disjoint, then C_i has a smaller letter than any in C_{i+1}. Another way, which proves to be very appropriate in our work with walkalongs, is to define a *stack of boxes of cycles* as a sequence $B_1,\ldots,B_r$ of boxes of cycles such that for all i, $1 \le i < r$, every cycle in B_i has a letter that appears in B_{i+1}. Of special importance is the first box, which we usually indicate as such, as shown below.



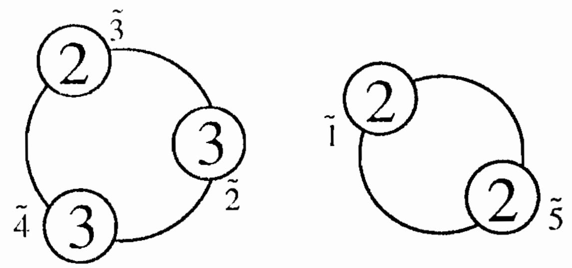
Given a stack $B_1,\ldots,B_r$ of boxes of cycles, we write out its cycles as a product of cycles by ordering the cycles $C_{i_1},\ldots,C_{i_s}$ in the ith box for all i, $1 \le i \le r$, so that the smallest label of each cycle increases as we go from left to right, and then writing $C_{1_1},\ldots,C_{1_s},\ldots,C_{r_1},\ldots,C_{r_s}$. We can recover the stack of boxes of cycles from such a sequence $C_1,\ldots,C_z$ by finding the largest k for which $C_1,\ldots,C_k$ are all disjoint, placing these cycles in the first box, and then proceeding by induction. We further claim that for each equivalence class of products of cycles on $\{1,\ldots,N\}$ there is exactly one stack of boxes of cycles on $\{1,\ldots,N\}$ such that the corresponding product of cycles is a representative of that equivalence class. For if there were two stacks s and s' of boxes of cycles for one equivalence class, then there must exist a smallest i, such that there exists a cycle C such that C is in the ith box of, say, s, but in the jth box of s', $j > i$. There must then be a cycle C' in the $j-1$th box of s' that shares at least one letter from C, and C' must appear in s in some kth box of cycles, $k > i$. The cycles C and C' do not commute, contradicting our assumption about s and s'. Therefore each equivalence class of products of cycles on $\{1,...,N\}$ has a unique representative which is associated with exactly one of the stacks of boxes of cycles on $\{1,...,N\}$. We therefore think of each stack of boxes of cycles as a canonical description of a cycle product.

Circular walks, closed walks, Lyndon words, and cycles are the basic walkalongs that we will work with in this thesis. However, there are certain ways of embellishing a walkalong that we will make use of in our combinatorial constructions. In this regard we introduce the concept of *case*, which is a feature added to a place of a walkalong that has no effect on the weight of the walkalong. In this thesis we make use of *upper case*, *lower case*, and *null case*. For example, in our proof of Theorem 2.3.1 we attribute one of these three cases to each of the places in a sequence of closed walks, and the involution that we perform depends on the cases of the various places. In such situations we indicate the case next to the place, as shown below.



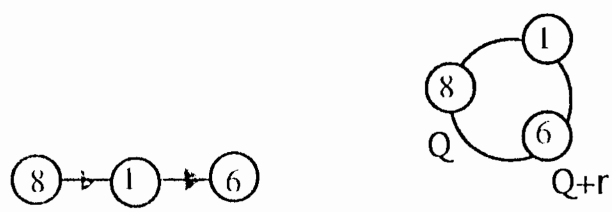
Labels $\tilde{1},...,\tilde{n}$ are another way of embellishing the places of a walkalong. They are very important in combinatorial constructions that are indexed by permutations of S_n, typically of the form $\sum_{\sigma \in S_n} F(\sigma) p_\sigma(\xi_1,...,\xi_N)$. In such a situation the index σ of

$p_\sigma = \prod_{1 \le i \le \ell(\sigma)} p_{\sigma_i}$ is understood to refer to the partition $\sigma_1, \ldots, \sigma_{\ell(\sigma)}$ that gives the cycle structure of the permutation σ. A cycle of length σ_i is associated with multisets of walkalongs generated by p_{σ_i}, and the letters of that cycle are understood as labels on the places of these walkalongs. The example below illustrates how labels are drawn next to the places that they are associated with.



As with case, labels do not affect the weight of a walkalong.

There are situations when the combinatorial objects that we wish to deal with are linear. Even some of these can be dealt with using walkalongs. For example, a walk w from i to j of length r may be understood as a closed walk w' from i to i of length $r+1$ with i at a place Q, j at a place $Q+r$, and for which the place $Q+r$ undirected. The weight of w is given by $\frac{1}{a_{ji}} \mathrm{W}_\Lambda(w')$. If w' is a cycle, then we say that w is a path, as drawn below on the left, along with the corresponding w' on the right.



At such times we may also fall back on the traditional way of describing walks in terms of vertices and edges, with the edge from i to j having weight a_{ij}.

Finally, there is one important situation in Section 2.2 where the walkalong fails us, and we need to make use of a different kind of weight $T_{\Lambda}(w)$. Given $\mathbf{N} = \{1,\ldots,N\}$, let $\mathbf{N}^n$ be the set of words on $\mathbf{N}$ of length n. Given $x_1x_2\cdots x_n \in \mathbf{N}^n$, list the letters of $x_1x_2\cdots x_n$ in order as $1^{m_1}2^{m_2}\cdots N^{m_N} = y_1y_2\cdots y_n$. We say that $y_1y_2\cdots y_n$ is the content of $x_1x_2\cdots x_n$ and we define $T_{\Lambda}(x_1x_2\cdots x_n) = \prod_{1\le i\le n} a_{y_i x_i}$.

CHAPTER 2

POWER, ELEMENTARY, AND HOMOGENEOUS SYMMETRIC FUNCTIONS

The object of this chapter is to evaluate the power p_n, elementary e_n, and homogeneous h_n symmetric functions at the eigenvalues $\xi_1,\ldots,\xi_N$ of an arbitrary $N\times N$ matrix **A**. We will devote almost all of our attention to the homogeneous symmetric functions. Indeed, we will record four different combinatorial interpretations for $h_n(\xi_1,\ldots,\xi_N)$. One of these is an original interpretation that describes $h_n(\xi_1,\ldots,\xi_N)$ as a generating function of multisets of Lyndon words.

The three bases $\{p_\lambda\}_{\lambda\vdash n}$, $\{e_\lambda\}_{\lambda\vdash n}$, $\{h_\lambda\}_{\lambda\vdash n}$ have in common the fact that they are multiplicative. A basis $\{b_\lambda\}_{\lambda\vdash n}$ of the symmetric functions of degree n is multiplicative if there is a sequence of symmetric functions $b_0,b_1,\ldots,b_n$ for which $b_0=1$ and each b_r has degree r and $b_\lambda=\prod_{i=1}^{\ell(\lambda)} b_{\lambda_i}$ for all $\lambda\vdash n$. In our case we find that the combinatorial description of such a function $b_\lambda(\xi_1,\ldots,\xi_N)$ yields nothing more than sequences of objects generated by $b_{\lambda_1}(\xi_1,\ldots,\xi_N)$, $b_{\lambda_2}(\xi_1,\ldots,\xi_N)$, ..., $b_{\lambda_{\ell(\lambda)}}(\xi_1,\ldots,\xi_N)$. This means that for the three bases $\{p_\lambda\}_{\lambda\vdash n}$, $\{e_\lambda\}_{\lambda\vdash n}$, $\{h_\lambda\}_{\lambda\vdash n}$ of this chapter we may as well restrict our attention to the functions p_n, e_n, h_n.

What helps make these three multiplicative bases interesting is that they can be expressed in terms of each other by way of recursion relations that happen to be very simple. Evaluating the functions p_n, e_n, and h_n at eigenvalues $\xi_1,\ldots,\xi_N$ generates fundamental combinatorial objects, and in Section 2.1 we are able to show how the recursion relations express relationships between these objects. In this context the recursion relations take the form of well known theorems about arbitrary matrices such as the Cayley-Hamilton theorem and the MacMahon Master theorem. This lays the groundwork for our argument that the theory of symmetric functions of the eigenvalues of an arbitrary matrix **A** serves as a unifying framework for the combinatorics of such a matrix.

The homogeneous symmetric functions $h_n(\xi_1,\ldots,\xi_N)$ stand out among the bases of the symmetric functions because of their many combinatorial interpretations. In this chapter they are described in terms of 1) Lyndon words, 2) cycle products, 3) circuits, which we define later, and 4) sequences of walks. We derive the descriptions 1) and 2) in Section 2.1 as consequences of the recursion relations $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$ and $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$. From 2) we then derive 3) and 4), which we discuss in Section 2.2 in the context of the MacMahon Master theorem. As we go along we relate the four descriptions in terms of each other in the following ways

circuits	$\leftrightarrow$	cycle products	Section	2.2
sequences of walks	$\leftrightarrow$	circuits	Section	2.2
sequences of walks	$\leftrightarrow$	cycle products	Section	2.4
Lyndon words	$\leftrightarrow$	sequences of walks	Section	3.3

The descriptions in terms of Lyndon words and sequences of walks are shown in Section 3.3 to be special cases of combinatorial interpretations of the forgotten symmetric functions $f_\lambda(\xi_1,\ldots,\xi_N)$.

Our discussion of the multiplicative bases concludes with an exploration of the combinatorics of the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. The power symmetric functions $p_n(\xi_1,\ldots,\xi_N)$ are generated by the trace of the walk matrix $\mathrm{tr}\left(\mathbf{I}/(\mathbf{I}-x\mathbf{A})\right) = \sum_{n=0}^{\infty} x^n p_n(\xi_1,\ldots,\xi_N)$, and the homogeneous symmetric functions $h_n(\xi_1,\ldots,\xi_N)$ are generated by the determinant of the walk matrix $\det\left(\mathbf{I}/(\mathbf{I}-x\mathbf{A})\right) = \sum_{n=0}^{\infty} x^n h_n(\xi_1,\ldots,\xi_N)$. In Sections 2.3 and 2.4 of this chapter we document the interplay between closed walks and cycle products. This culminates in a combinatorial proof of an identity due to Jacobi [Go]. Results from both of these sections will be referred to in later chapters.

SECTION 2.1 THREE RECURSION RELATIONS

The power, elementary, and homogeneous symmetric functions are related to each other by three recursion relations

$$ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}, \qquad nh_n = \sum_{r=1}^{n}p_r h_{n-r}, \qquad \sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0,$$

that have been known since the days of Newton. In this chapter we interpret the power symmetric function $p_n(\xi_1,\ldots,\xi_N)$ where $\xi_1,\ldots,\xi_N$ are the eigenvalues of an arbitrary matrix **A**. We then employ the recursion relations to verify interpretations of the elementary symmetric function $e_n(\xi_1,\ldots,\xi_N)$ and the homogeneous symmetric function $h_n(\xi_1,\ldots,\xi_N)$. This approach brings to light the combinatorial significance of the relations themselves.

A straightforward calculation shows that the power symmetric function $p_n(\xi_1,\ldots,\xi_N)$ is a generating function for closed walks of length n. We state this fact as a theorem in order to record its importance to this thesis. Recall from Section 1.3 that a closed walk is a walkalong with a distinguished place.

THEOREM 2.1.1 *Let C be the set of closed walks on $1,\ldots,N$ with a total of n places. Then*

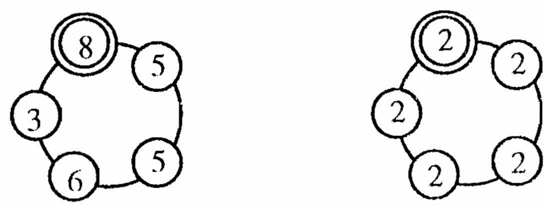
$$p_n(\xi_1,\ldots,\xi_N) = \sum_{c\in C} \mathrm{W}_{\mathbf{A}}(c)$$

The eigenvalues of $\mathbf{A}^n$ are $\xi_1^n,\ldots,\xi_N^n$. Therefore

$$p_n(\xi_1,\ldots,\xi_N) = \xi_1^n+\ldots+\xi_N^n = \mathrm{tr}(\mathbf{A}^n) = (\mathbf{A}^n)_{11}+\ldots+(\mathbf{A}^n)_{NN} = \sum_{w\in N^n} \mathrm{W}_{\mathbf{A}}(w)$$

The last equality is true because the matrix entry $\left(\mathbf{A}^n\right)_{ij}$ is the generating function of walks from i to j of length n. QED.

As we indicated in Chapter 1, one advantage of working with symmetric functions of eigenvalues is that we can recover the familiar expressions by specializing $a_{ij}=0$ for all $i \neq j$, and $a_{ii}=x_i$ for all i. With regard to the power symmetric functions this specialization gives $\left(\mathbf{A}^n\right)_{kk}=a_{kk}{}^n$ and $p_n(\xi_1,\ldots,\xi_N)=a_{11}{}^n+\ldots+a_{NN}^n=x_1{}^n+\ldots+x_N{}^n$. For purposes of comparison we juxtapose a typical closed walk generated by the general expression for $p_5(\xi_1,\ldots,\xi_9)$ with a typical closed walk generated by $a_{11}^5+\ldots+a_{99}^5$.



A consequence of Theorem 2.1.1 is that $\operatorname{tr}(\mathbf{I}/(\mathbf{I}-x\mathbf{A}))=\sum_{n=0}^{\infty} x^n p_n(\xi_1,\ldots,\xi_N)$ is a generating function for the power symmetric functions. Likewise, $\det(\mathbf{I}+x\mathbf{A})=\sum_{n=0}^{\infty} x^n e_n(\xi_1,\ldots,\xi_N)$ is a generating function for the elementary symmetric functions, and $\det(\mathbf{I}/(\mathbf{I}-x\mathbf{A}))=\sum_{n=0}^{\infty} x^n h_n(\xi_1,\ldots,\xi_N)$ is a generating function for the homogeneous symmetric functions. These last two facts follow from the equations $\prod_{i=1}^{N}(1+x\xi_i)=\sum_{n=0}^{\infty} x^n e_n(\xi_1,\ldots,\xi_N)$ and $\prod_{i=1}^{N}(1/(1-x\xi_i))=\sum_{n=0}^{\infty} x^n h_n(\xi_1,\ldots,\xi_N)$ and the existence of a Jordan canonical form for $\mathbf{A}$. In Section 1.1 we saw that expanding $\det(\mathbf{I}+x\mathbf{A})$ allows us to interpret $e_n(\xi_1,\ldots,\xi_N)$ in terms of boxes of cycles. The MacMahon Master theorem, which is the subject of Section 2.2, provides a rule for expanding $\det(\mathbf{I}/(\mathbf{I}-x\mathbf{A}))$. There is a combinatorial proof of the MacMahon Master theorem, due to Foata and Cartier [CF], which interprets $h_n(\xi_1,\ldots,\xi_N)$ as a generating function for circuits. A recounting of their proof at this point would satisfy the main goal of this chapter, which is to evaluate $p_n(\xi_1,\ldots,\xi_N)$, $e_n(\xi_1,\ldots,\xi_N)$, and $h_n(\xi_1,\ldots,\xi_N)$.

However, this chapter must also prepare us for work in the chapters to come, when we evaluate $f_\lambda(\xi_1,\ldots,\xi_N)$, $m_\lambda(\xi_1,\ldots,\xi_N)$, and $s_\lambda(\xi_1,\ldots,\xi_N)$. This requires that we appreciate the combinatorial relationships between the objects generated by $p_n(\xi_1,\ldots,\xi_N)$, $e_n(\xi_1,\ldots,\xi_N)$, and $h_n(\xi_1,\ldots,\xi_N)$. These relationships are most expressly captured upon interpreting the three recursion relations that relate these functions. In this section we interpret these recursion relations. One way of doing this would be to start with combinatorial interpretations for $p_n(\xi_1,\ldots,\xi_N)$, $e_n(\xi_1,\ldots,\xi_N)$, and $h_n(\xi_1,\ldots,\xi_N)$, and to give combinatorial proofs of the recursion relations. We choose the alternative course, which is to assume that a recursion relation is true, for example, $ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}$, and then to show that there are pairs of combinatorial objects which satisfy this relation. If we know that the symmetric functions p_r generate the first object in each pair, then we can conclude by induction on n that the symmetric functions e_{n-r} generate the second object.

With this in mind, we state the following theorem, the proof of which serves to illustrate the recursion relation $ne_n(\xi_1,\ldots,\xi_N) = \sum_{r=1}^{n}(-1)^{r-1}p_r(\xi_1,\ldots,\xi_N)e_{n-r}(\xi_1,\ldots,\xi_N)$. Recall from Section 1.3 that a box of cycles is a set of disjoint cycles.

THEOREM 2.1.2 *Let B be the set of boxes of cycles on $1,\ldots,N$ with a total of n places. Then*

$$e_n(\xi_1,\ldots,\xi_N) = \sum_{b\in B}\operatorname{sgn}(b)\mathrm{W}_\Lambda(b)$$

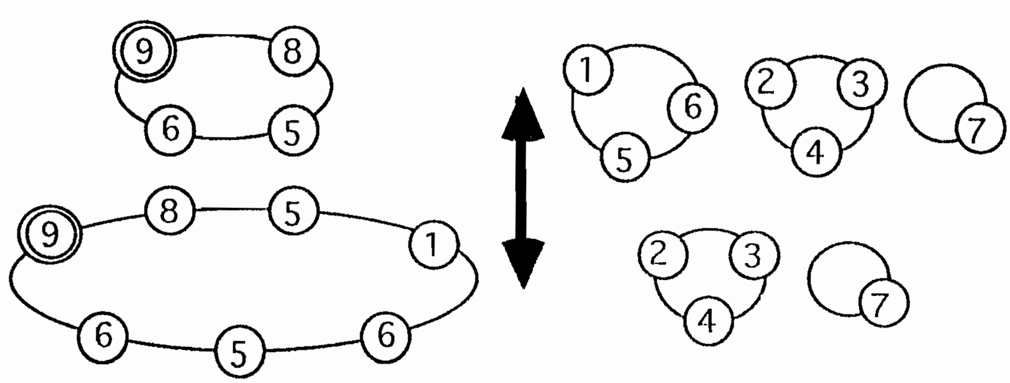
As discussed above, we demonstrate that this expression for $e_n(\xi_1,\ldots,\xi_N)$ satisfies the recursion relation $ne_n(\xi_1,\ldots,\xi_N) = \sum_{r=1}^{n}(-1)^{r-1}p_r(\xi_1,\ldots,\xi_N)e_{n-r}(\xi_1,\ldots,\xi_N)$ for all integers $n>0$. The result then follows by induction because it is true when $n=1$ in which case the box consists of a single cycle with a single edge and $e_1(\xi_1,\ldots,\xi_N) = a_{11} + a_{22} + \ldots + a_{NN} = p_1(\xi_1,\ldots,\xi_N)$.

By our claim the right hand side of the relation generates pairs consisting of a closed walk of length $r>0$, with sign, and a box of cycles, with sign, with $n-r\geq 0$ edges. The sign of the closed walk is $(-1)^{r-1}$, which is its sign as a walkalong. We define a weight preserving sign reversing involution on these pairs that proceeds by taking a walk along the closed walk. Starting at the coordinate Q, the origin of the walk, find the smallest $i\geq 0$ such that

either the letter at $Q+i$ also appears at some R in a cycle from the box,
or the letter at $Q+i$ equals the letter at some $R=Q+j$ for which $0\leq j<i$.

The two events are mutually exclusive. This is because the second event cannot occur at $Q+i$ unless the letters at $Q,Q+1,\ldots,Q+i$ are not to be found within the box. If $Q+i$ exists by virtue of the first event, then let the affected cycle consist of places $R,R+1,\ldots,R+\ell-1$. Remove this cycle from the box and insert it into the walk so that $Q+i$ is redirected to $R+1$ and R is redirected to $Q+i+1$. This preserves weight because R has the same value as $Q+i$. If $Q+i$ exists by virtue of the second event, then remove the cycle $Q+j+1,\ldots,Q+i$ from the walk and place it in the box. This involves redirecting $R=Q+j$ to $Q+i+1$ and $Q+i$ to $R+1=Q+j+1$.

If $Q+i$ exists, then this action describes a sign reversing weight preserving involution, as portrayed below, where $i=2$ and the letter 5 is at $Q+2$.



The sign of the term changes because all of the walkalongs have sign, and in the first event one walkalong is made from two, whereas in the second event one walkalong is made into two, as in Lemma 1.3.1. The action is reversible. If initially $Q+i$ is selected by virtue of the first event, then a cycle is added, but if the action is repeated, then $Q+i$ is selected by virtue of the second event, and the same cycle is removed, and vice versa.

The fixed points are those pairs for which $Q+i$ does not exist. For these pairs the closed walk is a cycle that is disjoint from the cycles in the box. Each pair is a box of cycles of length n, but one of the cycles has a letter that is marked: this cycle is the closed walk and the mark is at its origin. Note also that all of the cycles have the appropriate sign. Any one of the n letters may be marked and therefore the generating function for the fixed points is $ne_n(\xi_1,\ldots,\xi_N)$. This proves the theorem. QED.

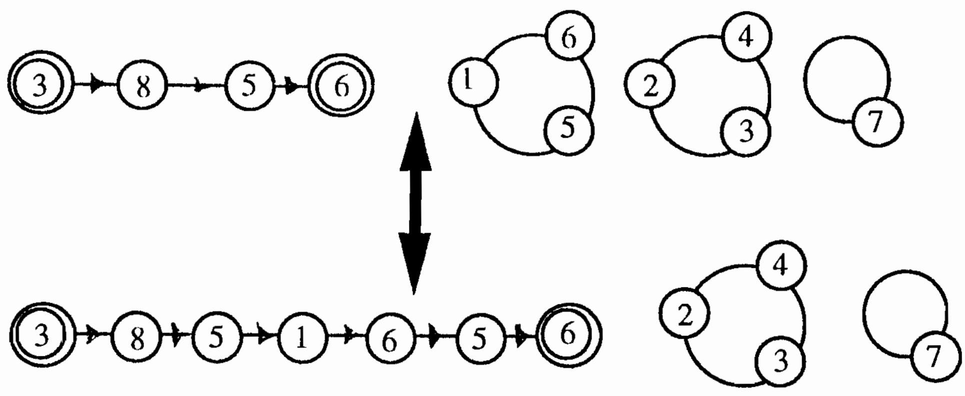
REMARK 2.1.3 Alternatively, in the special case that $n=N$ it is possible to define $p_0(\xi_1,\ldots,\xi_N)=N$ as the generating function of the closed walks of length zero. Then the recursion relation may be written $0=\sum_{r=0}^{N}(-1)^{r-1}e_{N-r}(\xi_1,\ldots,\xi_N)p_r(\xi_1,\ldots,\xi_N)$. The action may be altered so that $Q+i$ may equal $Q+r$, where r is the length of the closed walk, in which case the closed walk is a cycle of length r and is placed in the box, leaving behind a closed walk of length zero from and to the value of Q. The recursion relation may also be written $0=\sum_{r=0}^{N}(-1)^{r-1}e_{N-r}(\xi_1,\ldots,\xi_N)p_r(\xi_1,\ldots,\xi_N)$, in which case the edges in the box may be thought of as taken from $-\mathbf{A}$, so that the sign is changed upon adding or removing a cycle from the box. In any event there are no fixed points with the action defined in this way. QED.

We draw attention to the special case $n=N$ because then the recursion relation that we have interpreted is the trace of the matrix equation given by the historic Cayley-

Hamilton theorem. This equation states that evaluating the characteristic polynomial $\det(x\mathbf{I}-\mathbf{A})$ at the matrix $\mathbf{A}$ gives $\left(\det(x\mathbf{I}-\mathbf{A})\big|_{x=\mathbf{A}}\right)_{ij}=0$ for $1\le i,j\le n$. It was first given a combinatorial interpretation by Rutherford [R] [BR, 328] in 1964, and later by Straubing [Str] in 1983. They employ exactly the same action as we did in the proof above. The only difference is that the walk is in general not closed, but from a fixed i to a fixed j.

It may be claimed that symmetric functions cannot capture the combinatorics of those situations where symmetry is broken. If we think of symmetric functions as elements of a commutative algebra, then this is true. However, we may think of the symmetric functions evaluated at eigenvalues as described by the weights on the combinatorial objects that they generate. From this perspective, for example, we may take a power symmetric function $p_{r+1}(\xi_1,\dots,\xi_N)$ and in each term replace the first if any occurrence of a_{ji} with an x. Then if we take the coefficient of x we get walks from i to j of length r. In this light we see that the recursion relation is a generalization of the formula from the Cayley-Hamilton theorem, and not the other way around! Indeed, the Cayley-Hamilton theorem is only defined for the special case $n=N$.

This same argument proves the equation $(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{ij}\det(\mathbf{I}-\mathbf{A}) = \det(\mathbf{I}-\mathbf{A})^{(ij)}$ which is gotten by cross multiplying after using Cramer's rule to express the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$ as the inverse of the matrix $\mathbf{I}-\mathbf{A}$. Note that $(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{ij}$ is the generating function for all walks from i to j and $\det(\mathbf{I}-\mathbf{A})$ is the generating function for all boxes of cycles, with sign and with edges from $-\mathbf{A}$. Think of the walk from i to j as a closed walk from i to i in which the last edge is from j to i but the weight of this edge is set to equal 1. In the example below this missing edge is a_{63}.



Performing the action as before leaves fixed points that are boxes of cycles for which some cycle contains an edge from j to i but that edge has weight 1. The generating function for these fixed points is $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ and this proves the result. We will later encounter this result in Section 2.4 where we will prove it again using a different interpretation. In that case the interpretation of the fixed points will prove useful in deriving a new quotient formula for evaluating the Schur functions $s_\lambda(\xi_1,\ldots,\xi_N)$.

Zeilburger [Z] drew attention in 1985 to the connections between combinatorial techniques used to prove the Cayley-Hamilton theorem, the MacMahon Master theorem, the Matrix Tree theorem, as well as $\det(\mathbf{A})\cdot\det(\mathbf{B}) = \det(\mathbf{AB})$ and the identity $\det\left(e^{\mathbf{A}}\right) = e^{\operatorname{tr}(\mathbf{A})}$ due to Jacobi. These results depend on what we term "redirecting places" and on the effects of transporting a cycle with sign and with edges from $-\mathbf{A}$. The definitive survey at this date is the last and longest chapter of Brualdi and Ryser's [BR] book <u>Combinatorial Matrix Theory</u>, published in 1991. Indeed, the recursion relation $(-1)^k k\sigma_k = \sum_{i=1}^{k}(-1)^{i-1}\sigma_{k-i}\operatorname{tr}\left(\mathbf{A}^i\right)$, where "$\sigma_k$ is the coefficient of x^{n-k} in the characteristic polynomial of $\mathbf{A}$", appears as an exercise in their section on the Cayley-Hamilton theorem [BR, 334] and likewise in Zeilburger's paper. However, neither of these surveys suggests that symmetric functions play a role in the combinatorics of generic matrices.

In this thesis we accumulate evidence piece by piece to demonstrate that the symmetric functions of eigenvalues provide a natural framework for unifying disparate results about the combinatorics of an arbitrary matrix $\mathbf{A}$. The advantage of such a framework is that it makes it possible to approach the study of combinatorial phenomena in a systematic manner. The following is a new and surprising result that appears natural upon interpreting the recursion relation $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$. It says that $h_n(\xi_1,\ldots,\xi_N)$ is the generating function for words by which each word is coded up in terms of its factorization into Lyndon words.

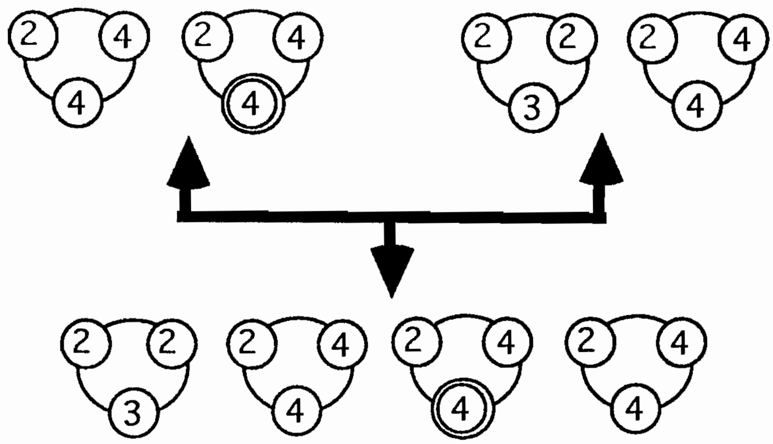
THEOREM 2.1.4 *Let M be the set of multisets of Lyndon words on $1,\ldots,N$ with a total of n places. Then*

$$h_n(\xi_1,\ldots,\xi_N) = \sum_{m\in M} \mathrm{W}_{\mathbf{A}}(m)$$

We show that the expression above satisfies the recursion relation $nh_n(\xi_1,\ldots,\xi_N) = \sum_{r=1}^{n} p_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)$ for all $n>0$, and therefore is true by induction as it is true for $n=1$, when $h_1(\xi_1,\ldots,\xi_N) = a_{11}+\ldots+a_{NN}$. Our argument depends on the fact that a circular walk is a power of a Lyndon word, and a closed walk is a circular walk with a distinguished place.

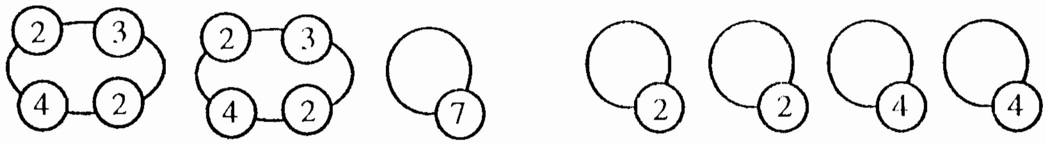
We think of $\sum_{r=1}^{n} p_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)$ as the generating function of pairs consisting of two objects having a total of n places. The first object is any closed walk of length r. But we think of it as any sequence of Lyndon words $w_1 = w_2 = \ldots = w_s$, all alike, such that there are r places in all, and one of the places X in the last word is distinguished. The second object is any sequence of Lyndon words $v_1 \le v_2 \le \ldots \le v_t$ in weakly increasing order such that there are $n-r$ places in all. Construct a single object from each pair by forming the weakly increasing sequence $v_1 \le v_2 \le \ldots < w_1 = w_2 = \ldots = w_s \le \ldots \le v_t$ and continuing to distinguish the place X.

For example, if $w_1 = w_2 = 244$ and $v_1 = 223$, $v_2 = 244$, then we have



We claim that the objects thereby generated are all of the weakly increasing sequences of Lyndon words $u_1 \le u_2 \le \ldots \le u_z$ with a total of n places and with one of these places distinguished. If the distinguished place is in a word, then removing this word and all like words that precede it recovers the original pair of objects. Each letter in each Lyndon word is ordered, and each Lyndon word in the sequence is ordered. This means that distinguishing any one of the n places yields a distinct object as illustrated above. Therefore by our claim the objects are generated by $nh_n(\xi_1,\ldots,\xi_N)$. The recursion relation is satisfied and the result follows. QED.

Our interpretation of $h_n(\xi_1,\ldots,\xi_N)$ is unexpected, but appears natural in the context of the recursion relation $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$. If we set $a_{ij} = 0$ for all $i \ne j$, and $a_{ii} = x_i$ for all i, then we recover the familiar expression for the homogeneous symmetric functions. In this case the only Lyndon words are $a_{11}, a_{22}, \ldots, a_{NN}$ and the multisets may be written as sequences $a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n}$ for which $1 \le i_1 \le i_2 \le \ldots \le i_n \le N$, so that $h_n(\xi_1,\ldots,\xi_N) = \sum_{1\le i_1 \le \ldots \le i_n \le N} a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n} = \sum_{1\le i_1 \le \ldots \le i_n \le N} x_{i_1}x_{i_2}\cdots x_{i_n}$. For purposes of comparison we juxtapose a typical multiset of Lyndon words generated by the general expression for $h_9(\xi_1,\ldots,\xi_8)$ with a typical multiset of Lyndon words generated by $\sum_{1\le i_1\le i_2 \le \ldots \le i_4 \le 9} a_{i_1i_1}a_{i_2i_2}a_{i_3i_3}a_{i_4i_4}$.



At this point we also remark that for both $p_n(\xi_1,\ldots,\xi_N)$ and $h_n(\xi_1,\ldots,\xi_N)$ all of the terms are positive and the number of terms equals the number of words of length n.

A more transparent way of obtaining Theorem 2.1.4 is to interpret the equation $h_r = \frac{1}{n!}\sum_{\sigma \in S_r} p_\sigma$, but we reserve this approach for later when we evaluate the forgotten basis. At this time we point out a more algebraic way of proving Theorem 2.1.4. In order to establish a context for this, we note that the power symmetric functions and homogeneous symmetric functions may be related by the following equations taken from Macdonald's book [M,16]:

$$\sum_{r\geq 1} p_r(x_1,\ldots,x_N)t^{r-1} = \sum_{i\geq 1}\frac{x_i}{1-x_i t} = \frac{d}{dt}\sum_{i\geq 1}\log\frac{1}{1-x_i t} = \frac{d}{dt}\log\prod_{i\geq 1}\frac{1}{1-x_i t} = \frac{d}{dt}\log\sum_{r\geq 0} h_r(x_1,\ldots,x_N)t^r$$

which depend on the fact that $\frac{d}{dt}\log\frac{1}{1-x_i t} = \frac{x_i}{1-x_i t}$. In an analogous manner we may write

$$\mathrm{tr}\left(\frac{\mathbf{A}}{\mathbf{I}-t\mathbf{A}}\right) = \mathrm{tr}\left(\frac{d}{dt}\log\frac{\mathbf{I}}{\mathbf{I}-t\mathbf{A}}\right) = \frac{d}{dt}\mathrm{tr}\,\log\left(\frac{\mathbf{I}}{\mathbf{I}-t\mathbf{A}}\right) = \frac{d}{dt}\log\det\left(\frac{\mathbf{I}}{\mathbf{I}-t\mathbf{A}}\right)$$

where we first use the fact that $\frac{\mathbf{I}}{\mathbf{I}-t\mathbf{A}} = \frac{d}{dt}\log\frac{\mathbf{I}}{\mathbf{I}-t\mathbf{A}}$, and then use the identity $e^{\mathrm{tr}\,\mathbf{B}} = \det e^{\mathbf{B}}$. Our interpretation of $h_n(\xi_1,\ldots,\xi_N)$ in terms of multisets of Lyndon words may be gotten from the equation

$$\operatorname{tr}\log\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right) = \log\det\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right)$$

upon expanding the power series $\log(\mathbf{I}/(\mathbf{I}-\mathbf{A})) = \mathbf{A}+\frac{1}{2}\mathbf{A}^2+\frac{1}{3}\mathbf{A}^3+\ldots$ and interpreting $\operatorname{tr}(\log(\mathbf{I}/(\mathbf{I}-\mathbf{A}))) = \sum_{n\geq 1}\frac{1}{n}\operatorname{tr}(\mathbf{A}^n)$ in a suitable way.

Consider a term from $\operatorname{tr}(\mathbf{A}^n)$. It corresponds to a closed walk of length n, and if we think of it as a circular walk w, then it is the rth power of a Lyndon word ℓ. This is to say that $\mathrm{W}_{\mathrm{A}}(\ell)^r = \mathrm{W}_{\mathrm{A}}(w)$. There are n/r closed walks that correspond to the circular walk w. Let L be the set of all Lyndon words, and let C_r be the set of circular walks that are the rth power of a Lyndon word. Then $\log\det(\mathbf{I}/(\mathbf{I}-\mathbf{A})) = \operatorname{tr}(\log(\mathbf{I}/(\mathbf{I}-\mathbf{A}))) =$
$\sum_{n\geq 1}\frac{1}{n}\operatorname{tr}(\mathbf{A}^n) = \sum_{r\geq 1}\sum_{w\in C_r}\frac{1}{r}\mathrm{W}_{\mathrm{A}}(w) = \sum_{r\geq 1}\sum_{\ell\in L}\frac{1}{r}\mathrm{W}_{\mathrm{A}}(\ell)^r = \sum_{\ell\in L}\sum_{r\geq 1}\frac{1}{r}\mathrm{W}_{\mathrm{A}}(\ell)^r =$
$\sum_{\ell\in L}\log(1/(1-\mathrm{W}_{\mathrm{A}}(\ell))) = \log\prod_{\ell\in L}(1/(1-\mathrm{W}_{\mathrm{A}}(\ell)))$. This gives us the equation

$$\prod_{\ell\in L}\left(\frac{1}{1-\mathrm{W}_{\mathrm{A}}(\ell)}\right) = \det\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right)$$

and Theorem 2.1.4 follows from the fact that $\det(\mathbf{I}/(\mathbf{I}-\mathbf{A})) = \prod_{i=1}^{N}(1/(1-\xi_i))$
$= \sum_{n\geq 1} h_n(\xi_1,\ldots,\xi_N)$.

Another combinatorial interpretation of $h_n(\xi_1,\ldots,\xi_N)$ comes from the last of the three recursion relations. In Section 1.3 we defined stacks of boxes of cycles and thereby established a canonical description for products of cycles in which cycles commute if they are both disjoint and adjacent. We now use the recursion relation $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$ to show that the homogeneous symmetric function $h_n(\xi_1,\ldots,\xi_N)$ generates all stacks of boxes of cycles that have n places.

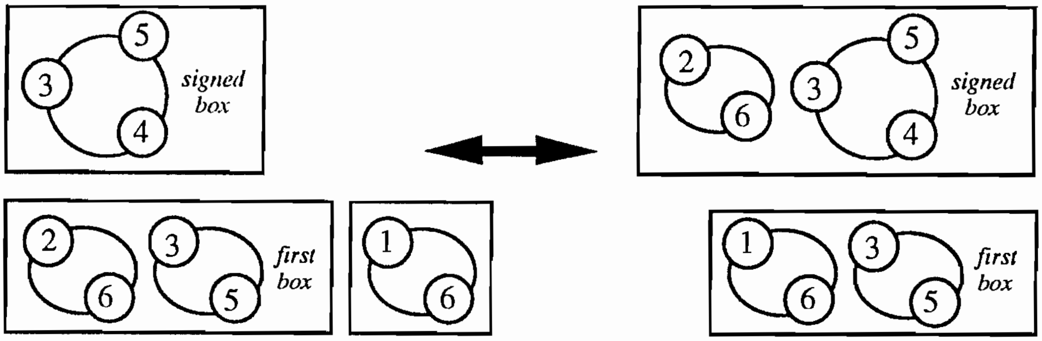
THEOREM 2.1.5 *Let S be the set of stacks of boxes of cycles on $1,\ldots,N$ with a total of n places. Then*

$$h_n(\xi_1,\ldots,\xi_N) = \sum_{s\in S} \mathrm{W}_{\mathbf{A}}(s)$$

As before, we show that the above interpretation for $h_n(\xi_1,\ldots,\xi_N)$ satisfies the recursion relation $\sum_{r=0}^{n}(-1)^r e_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N) = 0$, and then the result follows by induction. We think of $\sum_{r=0}^{n}(-1)^r e_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)$ as the generating function of pairs of objects with a total of n places. The first object is a signed box of cycles that has edges from $-\mathbf{A}$, r in all. The second object is a stack of box of cycles with $n-r$ edges in all. Given a pair of objects, say that a walkalong is a candidate if it is

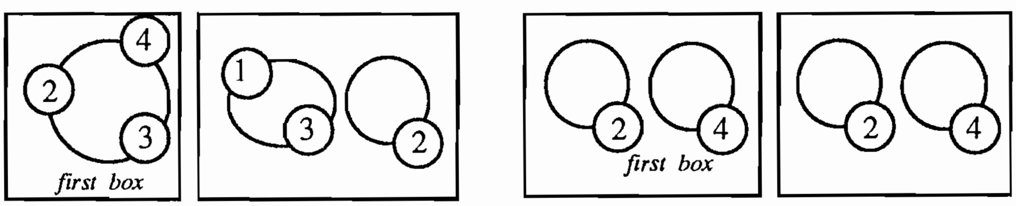
either a cycle in the signed box
or a cycle in the first box of the stack that is disjoint from all cycles
in the signed box.

There must be at least one candidate because otherwise both the signed box and the first box are both empty. The candidates are all disjoint and therefore there always exists a unique candidate C with a smallest letter k. Move C from the signed box to the first box (and the front of the stack), or vice versa. This results in a pair of objects that has opposite sign because a cycle has been added to or removed from the box of cycles, with sign, that has edges in $-\mathbf{A}$. This action defines an involution because the candidates do not change. If C was moved to the signed box, then any cycle that newly enters the first box must overlap with it and is not a candidate. If C was moved to the front of the stack, then any cycle not disjoint is pushed out of the first box and is not a candidate. The same candidate C is chosen upon a second application of the action, and it is returned to where it originally was. This describes a weight preserving sign reversing involution, as illustrated below, where $k=2$ and C is the cycle (26).



There are no fixed points because C always exists. QED.

Our first instinct after proving Theorem 2.1.4 is to check to see that setting $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = x_i$ for all i recovers the familiar expression for the homogeneous symmetric functions. In this case the only possible cycles are $a_{11}, a_{22}, \ldots, a_{NN}$ and therefore $h_n(\xi_1, \ldots, \xi_N) = \sum_{1 \le i_1 \le \ldots \le i_n \le N} a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_n i_n} = \sum_{1 \le i_1 \le \ldots \le i_n \le N} x_{i_1} x_{i_2} \cdots x_{i_n}$. We compare below a typical stack of boxes of cycles generated by the general expression for $h_6(\xi_1, \ldots, \xi_9)$ with a typical stack generated by $\sum_{1 \le i_1 \le i_2 \le \ldots \le i_4 \le 9} a_{i_1 i_1} a_{i_2 i_2} a_{i_3 i_3} a_{i_4 i_4}$.



We remark that the action described in the proof works even when $N < n$, in which case $e_r(\xi_1, \ldots, \xi_N) = 0$ for $r > N$ and there appears to be a new condition, that the index $n - N$ of $h_{n-r}(\xi_1, \ldots, \xi_N)$ cannot be less than $n - N$. However, any candidate from $h_{n-r}(\xi_1, \ldots, \xi_N)$ must be able to fit in the signed box, and therefore removing the cycle does not reduce the index by more than $N - r$ and does not violate the "new" condition.

The recursion relation $\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$ gives the terms of nth degree in the expansion

$$\left(\sum_{r=0}^{\infty}(-1)^r e_r\right)\left(\sum_{r=0}^{\infty} h_r\right) = 1$$

$$\prod_{i=0}^{\infty}(1-x_i)\prod_{i=0}^{\infty}\frac{1}{(1-x_i)} = 1$$

which if evaluated at $x_1 = \xi_1, \ldots, x_N = \xi_N$ is equal to

$$\det(\mathbf{I}-\mathbf{A})\cdot\det\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right) = 1$$

The action used in interpreting the recursion relation $\sum_{r=0}^{n}(-1)^r e_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N)$ $= 0$ can also be used to prove the MacMahon Master theorem. The MacMahon Master theorem provides an expression for the expansion of $1/\det(\mathbf{I}-\mathbf{A})$ and is the subject of the next section.

SECTION 2.2 THE MACMAHON MASTER THEOREM

The MacMahon Master theorem provides a means for expanding $1/\det(\mathbf{I}-\mathbf{A})$. Foata and Cartier [CF] have interpreted this expansion as the generating function for a kind of multigraph that they call circuit. They have used this interpretation to give the MacMahon Master theorem a combinatorial proof. In this section we first state the theorem and show how it may be interpreted in terms of circuits. We then present a weight preserving correspondence between circuits and cycle products. This makes it possible to state a proof of the MacMahon Master theorem which coincides with the interpretation of the recursion relation $\sum_{r=0}^{n}(-1)^r e_r(\xi_1,\ldots,\xi_N)h_{n-r}(\xi_1,\ldots,\xi_N) = 0$ that we gave in the previous section. We conclude this section with a weight preserving correspondence that relates circuits with sequences of closed walks.

The following theorem is known as the MacMahon Master theorem.

THEOREM 2.2.1 *Define the matrix* $\mathbf{AX}$ *as follows:*

$$\mathbf{X} = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \\ \vdots & & \ddots & \\ 0 & & & x_N \end{pmatrix} \qquad \mathbf{AX} = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1N}x_N \\ a_{21}x_1 & a_{22}x_2 & & \\ \vdots & & \ddots & \\ a_{N1}x_1 & & & a_{NN}x_N \end{pmatrix}$$

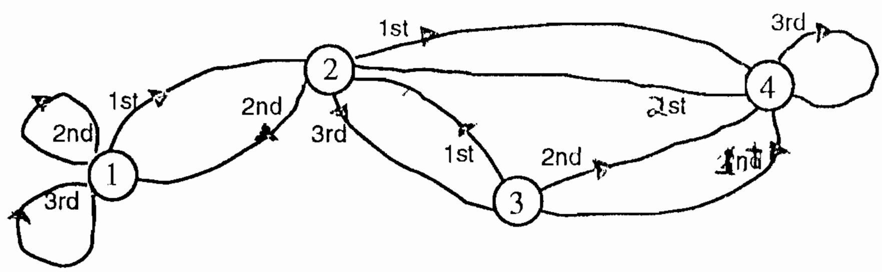
The coefficient of $x_1^{r_1}x_2^{r_2}\ldots x_N^{r_N}$ *in* $1/\det(\mathbf{I}-\mathbf{AX})$ *equals the coefficient of* $x_1^{r_1}x_2^{r_2}\ldots x_N^{r_N}$ *in* $(a_{11}x_1+a_{12}x_2\ldots+a_{1N}x_N)^{r_1}(a_{21}x_1+a_{22}x_2\ldots+a_{2N}x_N)^{r_2}\cdots(a_{N1}x_1+a_{N2}x_2\ldots+a_{NN}x_N)^{r_N}$.

This theorem will be a consequence of Theorem 2.2.2 and therefore we do not prove it separately. Instead we consider the objects that $\prod_{i=1}^{N}(a_{i1}x_1+a_{i2}x_2\ldots+a_{iN}x_N)^{r_i}$

generates. The observations that we make about these objects are originally due to Foata and Cartier [CF].

Fix $r_1,\ldots,r_N$. Any term in the product $(a_{i1}x_1 + a_{i2}x_2 \ldots + a_{iN}x_N)^{r_i}$ contributes a list of edges $a_{ij}x_j$ coming out of a vertex i such that there are r_i edges in all. From this point of view, any term in $\prod_{i=1}^{N}(a_{i1}x_1 + a_{i2}x_2 \ldots + a_{iN}x_N)^{r_i}$ constitutes a list of r_1 edges out of the vertex 1, a list of r_2 edges out of 2, and so on. This means that we may depict each term as a multigraph (a graph with multiple edges). There must be r_i edges coming out of any vertex i and these edges must be ordered with respect to each other. Furthermore, the fact that we take the coefficient $x_1^{r_1}x_2^{r_2}\ldots x_N^{r_N}$ means that there must be exactly r_j edges $a_{ij}x_j$ going into any vertex j.

The function $1/\det(\mathbf{I}-\mathbf{AX})$ generates multigraphs on the vertices $1,\ldots,N$ such that for all i, the edges coming out of a vertex i are all ordered, and their number equals the number of edges coming into the vertex i. Such multigraphs are called circuits. Consider the example below, which depicts the circuit that corresponds to the term $a_{12}a_{11}a_{11}a_{24}a_{21}a_{23}a_{32}a_{34}a_{43}a_{42}a_{44}\, x_1^3x_2^3x_3^2x_4^3$ in $\prod_{i=1}^{5}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}x_5)^{r_i}$ where $r_1 = 3$, $r_2 = 3$, $r_3 = 2$, $r_4 = 3$, $r_5 = 0$.



We may declare $x_j = 1$ so as to suppress the variable x_j, which can always be recovered by setting $a_{ij} \to a_{ij}x_j$. The MacMahon Master theorem may then be understood as stating that $1/\det(\mathbf{I}-\mathbf{A}) = \sum_{n\geq 0} h_n(\xi_1,\ldots,\xi_N)$ is a generating function for all circuits.

A circuit may be completely described in two line form by listing its edges in the order that the product $\prod_{i=1}^{N}(a_{i1}x_1 + a_{i2}x_2 ... + a_{iN}x_N)^{r_i}$ generates them. The top line lists the vertex from which the edge leaves, and the bottom line lists the vertex to which it goes. For example, the two line form of the circuit drawn above is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 1 & 1 & 4 & 1 & 3 & 2 & 4 & 3 & 2 & 4 \end{pmatrix}$$

Note that the letters in the top line and in the bottom line must appear with the same frequency. If the letters are $1,1,2,2,2,3,3,4,4,4$, then they must appear in the order $1,1,1,2,2,2,3,3,4,4,4$ in the top line, but there are no restrictions on the placement of these letters in the bottom line. The resulting two line form is said to be a rearrangement of the letters $1,1,1,2,2,2,3,3,4,4,4$. We define a weight $\mathrm{T}_{\mathbf{A}}(w)=$ $a_{12}a_{11}a_{11}a_{24}a_{21}a_{23}a_{32}a_{34}a_{43}a_{42}a_{44}$ on this rearrangement $w = 21141324324$ which is the same as the weight of the associated circuit. In general, if $x_1 x_2 \cdots x_n$ is a rearrangement of letters $y_1 y_2 \cdots y_n$ that belong to $\{1,...,N\}$, then $\mathrm{T}_{\mathbf{A}}(x_1 x_2 \cdots x_n) = \prod_{1 \le i \le n} a_{y_i x_i}$.

The two line form of a rearrangement may be compared with that of a permutation. A permutation is a bijection of the set $\{1,...,N\}$, whereas a rearrangement is a bijection of a multiset, in this case $\{1,1,1,2,2,2,3,3,4,4,4\}$. This is perhaps the point of view of the originator of the MacMahon Master theorem, which is that $1/\det(\mathbf{I}-\mathbf{A})$ is the generating function for all rearrangements. The theorem is useful because it allows one to make facts about permutations more general, or alternatively, to handle special cases. It is deservedly called a Master Theorem because every specialization of a_{ij} results in a new theorem.

Of all of the ways of interpreting $h_n(\xi_1,...,\xi_N)$, the fact that it generates circuits is arguably the most important, given that it is this description which extends to

$s_\lambda(\xi_1,\ldots,\xi_N)$ in Section 4.4 when we consider traces of the representations of the general linear group.

The circuit that we have been considering may also be described by a table in the following way:

a_{12}	a_{24}	a_{32}	a_{43}
a_{11}	a_{21}	a_{34}	a_{42}
a_{11}	a_{23}		a_{44}

Each column of such a table may be thought of as a computer stack. The idea is that new edges may be put down or old edges taken away from the top of any stack. This point of view was taken by Zeilburger [Z] in his recounting of Foata's proof of the MacMahon Master theorem. We will refer to such a table as a circuit table.

Two simple but important remarks can be made about reading information from a circuit table. The first remark, which we call the Horizontal fact, allows us to establish the weight preserving correspondence between circuits and cycle products that is the subject of Theorem 2.2.2. After we prove this theorem, we will make the second remark, which we call the Vertical fact. We use the Vertical fact to prove Theorem 2.2.3, which establishes a weight preserving correspondence between circuits and certain sequences of walks.

The Horizontal fact is the observation that *the edges from the top row of a table must form at least one cycle*. Moreover, *a circuit table is left upon removing any or all of these cycles*. This is true because in the top row there is one edge $a_{is(i)}$ for each letter i that appears in the circuit table. It is possible to generate a sequence of edges $a_{is(i)}, a_{s(i)s(s(i))}, \ldots, a_{s^r(i)s^{r+1}(i)}, \ldots$, all from the top row. These edges cannot all be distinct, and therefore there must be at least one cycle with edges $a_{s^k(i)s^{k+1}(i)}, \ldots, a_{s^r(i)s^{r+1}(i)}$. Note that there may be more than one such cycle, but that all such cycles are disjoint, because there is only one edge $a_{i s(i)}$ for each letter i. In the example that we have been considering there is only such cycle and it has edges a_{24}, a_{43}, a_{32}. Remove any such cycle from the circuit table. Advance the remaining edges by one row in each of the affected columns. The result is a circuit table because removing a cycle has not affected the fact that the number of edges into a vertex j equals the number of edges out of j.

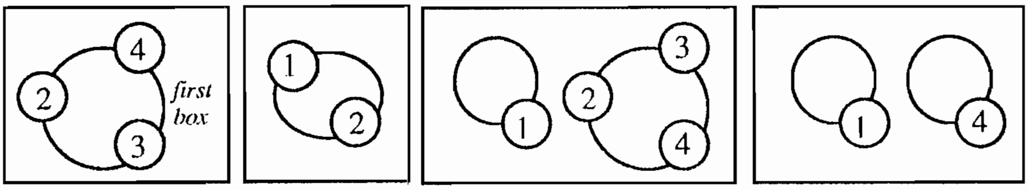
The Horizontal fact allows us to establish a weight preserving correspondence between circuits and stacks of boxes of cycles.

THEOREM 2.2.2 *Let* $\mathbf{N}^n$ *be the set of words on* $1,\ldots,N$ *of length* n. *Then*

$$h_n(\xi_1,\ldots,\xi_N) = \sum_{w\in\mathbf{N}^n} \mathrm{T}_{\mathrm{A}}(w)$$

We know by Theorem 2.1.5 that $h_n(\xi_1,\ldots,\xi_N)$ is a generating function of stacks of boxes of cycles on $1,\ldots,N$ with a total of n places. Our aim is to use the Horizontal fact to establish a weight preserving correspondence between circuit tables and stacks of boxes of cycles. This will prove that $h_n(\xi_1,\ldots,\xi_N) = \sum_{w\in\mathbf{N}^n} \mathrm{T}_{\mathrm{A}}(w)$.

Given a word $w \in \mathbf{N}^n$, let t be the corresponding circuit table with n edges, let b_1 be the box of cycles that consists of all of the cycles formed by the edges in the top row. The Horizontal fact states that this box of cycles is nonempty and removing it results in a circuit table t_1. In general, define b_i to be the box of cycles that consists of all of the cycles formed by the edges in the top row of t_{i-1}, and define t_i to be the circuit table that remains upon removing b_i from t_{i-1}. The Horizontal fact implies that there is a well defined sequence $b_1,\ldots,b_k$ such that b_i is nonempty for all $i \le k$, and empty for all $i > k$. It furthermore implies that every cycle C in b_i overlaps some cycle in b_{i-1} because otherwise C would be in b_{i-1}. This means that $b_1,\ldots,b_k$ defines a stack of boxes of cycles. Below is the stack which corresponds to the circuit that has been serving as our example. Each box in the stack is juxtaposed with the circuit table from which it was removed.



a_{12}	a_{24}	a_{32}	a_{43}	a_{12}	a_{21}	a_{34}	a_{42}	a_{11}	a_{23}	a_{34}	a_{42}	a_{11}			a_{44}
a_{11}	a_{21}	a_{34}	a_{42}	a_{11}	a_{23}		a_{44}	a_{11}			a_{44}				
a_{11}	a_{23}		a_{44}	a_{11}											

This procedure maps every circuit table to a unique stack with the same weight. It is straightforward to recover the circuit table from the stack. In fact, every stack of boxes of cycles gives rise to a different circuit table. Hence there is a weight preserving correspondence between circuit tables on $1,\ldots,N$ with n edges and stacks of boxes of cycles on $1,\ldots,N$ with n places. By Theorem 2.1.5, the stacks are generated by $h_n(\xi_1,\ldots,\xi_N)$, and the theorem follows. QED

As usual, we may examine the consequences of this theorem when $a_{ij}=0$ for all $i\neq j$, and $a_{ii}=x_i$ for all i. Then the word $w\in\mathbf{N}^n$ must have its letters written in weakly increasing order, so that the weight consists of edges a_{ii}. For example, we may have a word 2235 with weight $a_{22}a_{22}a_{33}a_{55}$. Therefore the specialization gives the usual definition of the homogeneous symmetric function $h_n(\xi_1,\ldots,\xi_N)=\sum_{1\le i_1\le\ldots\le i_n\le N} a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n}$ $=\sum_{1\le i_1\le\ldots\le i_n\le N} x_{i_1}x_{i_2}\cdots x_{i_n}$.

As we discussed in Section 1.3, the stacks serve as a canonical description of cycle products. The above correspondence, due to Foata and Cartier [CF], was used by them to show that $1/\det(\mathbf{I}-\mathbf{A})$ is a generating function for cycle products. We may also remark that Theorem 2.2.2 shows that the action in Theorem 2.1.5 gives a proof of the MacMahon Master theorem.

The Vertical fact provides another way of reading information from a circuit table. Once we formalize it, we will be able to use it to prove Theorem 2.2.3, giving a fourth interpretation of $h_n(\xi_1,\dots,\xi_N)$ as a generating function. Given a circuit table, let i_1 be a letter that appears there. There must be an edge $a_{i_1 i_2}$ in the top row. Remove it from the table. Construct a sequence $a_{i_1 i_2}, a_{i_2 i_3},\dots,a_{i_s i_{s+1}},\dots$ in the following way. If the last edge in the sequence is $a_{i_s i_{s+1}}$, then find the edge at the top of the column associated with i_{s+1}, remove it from the column, and add it to the sequence. If $i_{s+1} \neq i_1$, then it is always possible to find this edge. The reason is that among the edges that are left in the table the number of edges that come out of $i_{s+1} \neq i_1$ is one greater than the number going into $i_{s+1} \neq i_1$ (and therefore nonempty), whereas the number of edges that come out of all other $k \neq i_1$ is equal to the number going into k. This means that the sequence can continue indefinitely so long as no edge ever leads us back to the column associated with i_1. But there are only finitely many edges, and therefore the sequence must take us back to the column associated with i_1. Hence the sequence determines a closed walk from i_1 to i_1.

This then, is the Vertical fact: *If a walk along the circuit table starts at the top entry of the ith column, then it eventually returns to the ith column.* Moreover, *a circuit table is left upon removing the walk* .

The following interpretation of $h_n(\xi_1,\dots,\xi_N)$ is a consequence of the Vertical fact.

THEOREM 2.2.3 *Fix $\sigma \in S_N$. Let V_σ be the set of sequences $(w_1, w_2,\dots,w_N)$ of closed walks on $1,\dots,N$ with a total of n places for which $\sigma(j)$ is the letter at the origin of w_j and the letters $\sigma(1),\sigma(2),\dots,\sigma(j-1)$ do not occur in w_j. Then*

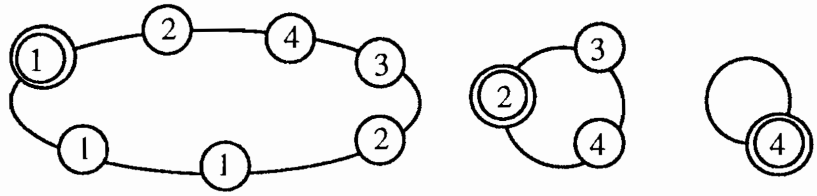
$$h_n(\xi_1,\dots,\xi_N) = \sum_{s \in V_\sigma} \mathrm{W}_\Lambda(s)$$

We present a weight preserving correspondence between circuit tables on $1,\ldots,N$ with n edges and the sequences of walks described above. The theorem will then be a consequence of Theorem 2.2.2.

Given a circuit table t_0, we construct a sequence of circuit tables $t_1,\ldots,t_N$ and a sequence of walks $w_1,\ldots,w_N$. If the letter $\sigma(j)$ does not appear in the circuit table t_{j-1}, then we define w_j to be the empty walk from $\sigma(j)$ to $\sigma(j)$. Otherwise, suppose that the letter $\sigma(j)$ appears in the circuit table t_{j-1}. Then there must be an edge of the form $a_{\sigma(j)s(\sigma(j))}$ in the top row of t_{j-1}. The Vertical fact tells us how to remove a closed walk from $\sigma(j)$ to $\sigma(j)$ from the circuit table t_{j-1}. Continue removing closed walks from $\sigma(j)$ to $\sigma(j)$ until the column corresponding to $\sigma(j)$ has no more edges. Let w_j be the concatenation of these walks. Let t_j be the circuit table that results upon removing w_j. There are no occurrences of $\sigma(j)$ in t_j because the column corresponding to $\sigma(j)$ is empty. We see that the sequence $w_1,\ldots,w_N$ is well defined and that for all j the letters $\sigma(1),\sigma(2),\ldots,\sigma(j-1)$ do not appear in the walk w_j.

This defines a one-to-one weight preserving map from the circuit tables to the sequences of walks. But it is possible to recover the circuit table from the sequence $w_N,\ldots,w_1$ by laying down the edges in each walk in reverse order. In fact, any such sequence of walks gives rise to a different circuit table. Therefore the map defines a weight preserving correspondence and the theorem follows from Theorem 2.2.2. QED

The correspondence in the theorem gives the following closed walks in the example that we have been considering if we let $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=3$, and $\sigma(4)=4$. Each walk is juxtaposed with the circuit table from which it was removed.



a_{12}	a_{24}	a_{32}	a_{43}		a_{23}	a_{34}	a_{42}				a_{44}
a_{11}	a_{21}	a_{34}	a_{42}				a_{44}				
a_{11}	a_{23}		a_{44}								

For purposes of illustration we may also consider what happens when $a_{ij}=0$ for all $i \neq j$, and $a_{ii}=x_i$ for all i. Then the walk that starts and ends at the letter $\sigma(j)$ must be of the form $\left(a_{\sigma(j)\sigma(j)}\right)^{k_j}$. This recovers the usual definition of the homogeneous symmetric function $h_n(\xi_1,\ldots,\xi_N)=\sum_{1\le i_1\le\ldots\le i_n\le N} a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n}=\sum_{1\le i_1\le\ldots\le i_n\le N} x_{i_1}x_{i_2}\cdots x_{i_n}$.

Suppose that $\sigma(i)=i$ for all i. We describe Theorem 2.2.3 in the language of Foata [Lo,197]. The closed walks from i to i that are gotten by the Vertical Fact are called dominated circuits because in each of them the letter i appears only once and all other letters are greater. Theorem 2.2.3 shows that for each circuit there is a unique "dominated circuit factorization". If we let w' be the word gotten by reading the letters of the closed walks $w_N,\ldots,w_1$ in sequence, then it is possible to recover w from w'. The mapping $w \to w'$ is known as the First Fundamental Transformation.

The action that Foata and Cartier use to prove the MacMahon Master theorem is made possible by the Vertical fact. Given a circuit, walk along the first nonempty closed walk w_j. As we walk along, we find ourselves in the same predicament as in our interpretation of the Cayley-Hamilton theorem. We will either make a cycle, which we place in the box of cycles, or we will touch upon a cycle from the box, which we incorporate into the walk. The proof is essentially the same as in Remark 2.1.3.

The combinatorics of closed walks is investigated in the last two sections of this chapter. This will lead to Theorems 2.3.1 and 2.4.3, which are both generalizations of

Theorem 2.2.3. Another generalization occurs in Theorem 3.3.2, which presents an interpretation of the forgotten symmetric functions $f_\lambda(\xi_1,\ldots,\xi_N)$.

SECTION 2.3 THE DETERMINANT OF THE WALK MATRIX

Closed walks underlie much of the combinatorics of symmetric functions of eigenvalues. We have already seen that $p_n(\xi_1,\ldots,\xi_N)$ is the generating function for closed walks of length n. In the next chapter we expand symmetric functions in terms of the basis $\{p_\lambda\}_{\lambda\vdash n}$, and this will thrust closed walks into a prominent role. In this section we become more familiar with the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. We calculate $e_n(\omega_1,\ldots,\omega_N)$ where $\omega_1,\ldots,\omega_N$ are the eigenvalues of $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. In the special case $n=N$, the involution behind this calculation will provide an interpretation for the equation $\sum_{r\geq 0}\sum_{\lambda\succ r} m_\lambda(\xi_1,\ldots,\xi_N)=\sum_{r\geq 0} h_r(\xi_1,\ldots,\xi_N)$ that we discuss in Section 3.4.

Theorem 2.2.3 expressed $h_n(\xi_1,\ldots,\xi_N)$ as the generating function for certain sequences of walks. That there should be a relation between $h_n(\xi_1,\ldots,\xi_N)$ and walks is indicated by the equation

$$\det\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right) = \frac{1}{\det(\mathbf{I}-\mathbf{A})}$$

which follows from the property of determinants that $\det\mathbf{B}^{-1} = 1/\det\mathbf{B}$. The left hand side consists of both positive and negative terms that are products of walks. There is a sign reversing involution on these terms by which they cancel away, leaving behind as fixed points the desired sequences of walks.

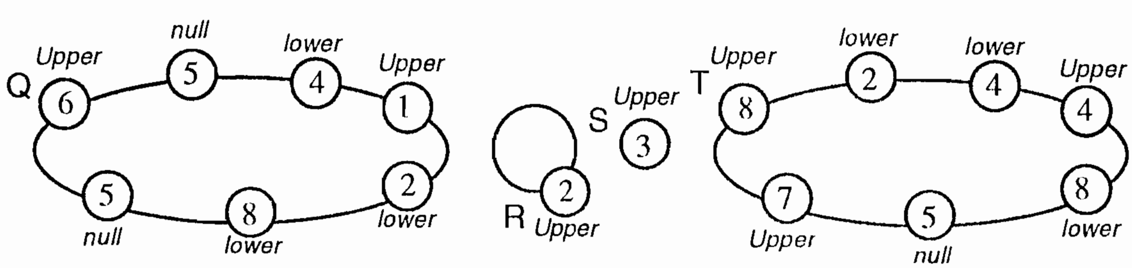
Our labor bears greater fruit if we consider the more general question of calculating the coefficients of the characteristic polynomial of the walk matrix. Ignoring sign, these coefficients are the elementary symmetric functions $e_r(\omega_1,\ldots,\omega_N)$, where $\omega_1=1/(1-\xi_1),\ldots,\ \omega_N=1/(1-\xi_N)$ are the eigenvalues of the walk matrix. If $r=N$, then we have $e_N(\omega_1,\ldots,\omega_N)=\prod_{i=1}^{N}1/(1-\xi_i)=\det(\mathbf{I}/(\mathbf{I}-\mathbf{A}))$.

We know that $e_n(\omega_1,\ldots,\omega_N)$ generates boxes of cycles employing a total of n edges. These edges are taken from the matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$, which means that they are generating functions for walks. In the case of the generating function $(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{ii}$, the walks include the empty walk (walk of length zero) from i to i. We think of each term of $e_n(\omega_1,\ldots,\omega_N)$ as a box of cycles in which the edges have been replaced by walks.

The places in the box of cycles are very important. Assign upper case to every such place. These places are the origins of the walks. The letters at these places are also important. Note that these letters are all distinct. Assign lower case to every place (except the origin of a walk) at which such a letter occurs. Assign null case to any place at which no such letter occurs.

If a walk is empty, then it must occur in a cycle of length one. This cycle has one place and the letter there has upper case. Let the place be undirected, so that it has no weight.

The sign of the term is the same as that of the box of cycles. The result of replacing the edges of a cycle with walks yields a walkalong. The origin of any such walkalong is the place with upper case that has the largest letter. In the following typical term from $e_n(\omega_1,\ldots,\omega_N)$ the origins of the walkalongs are marked by Q, R, S, T, the cycles are (2), (3), (61), (847), and the walks are $1\to2\to8\to5\to6$, $2\to2$, the empty walk from 3 to3, $4\to8\to5\to7$, $6\to5\to4\to1$, $7\to8$, and $8\to2\to4\to4$.



We say that a place of upper or lower case with letter j is a problem place if the letter at the origin of its walkalong is k and $k>j$. In the example above the problem places are $Q+2$, $Q+3$, $Q+4$, $T+1$, $T+2$, $T+3$, $T+6$. In particular, any upper case place is a problem place (excepting the origin of a walkalong). This last fact is the basis for the following result.

THEOREM 2.3.1 *Let $\omega_1,\ldots,\omega_N$ be the eigenvalues of* $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. *Fix $\sigma \in S_N$. Let W_σ be the set of sequences $w_1,\ldots,w_n$ of closed walks on $1,\ldots,N$ for which letters $i_1 < i_2 <\ldots< i_n$ exist such that $\sigma(i_j)$ is the letter at the origin of w_j and the letters $\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_{j-1})$ do not occur in w_j. Then*

$$e_n(\omega_1,\ldots,\omega_N) = \sum_{(w_1,\ldots,w_n)\in W_\sigma} \mathrm{W}_\mathbf{A}(w_1)\cdots \mathrm{W}_\mathbf{A}(w_n)$$

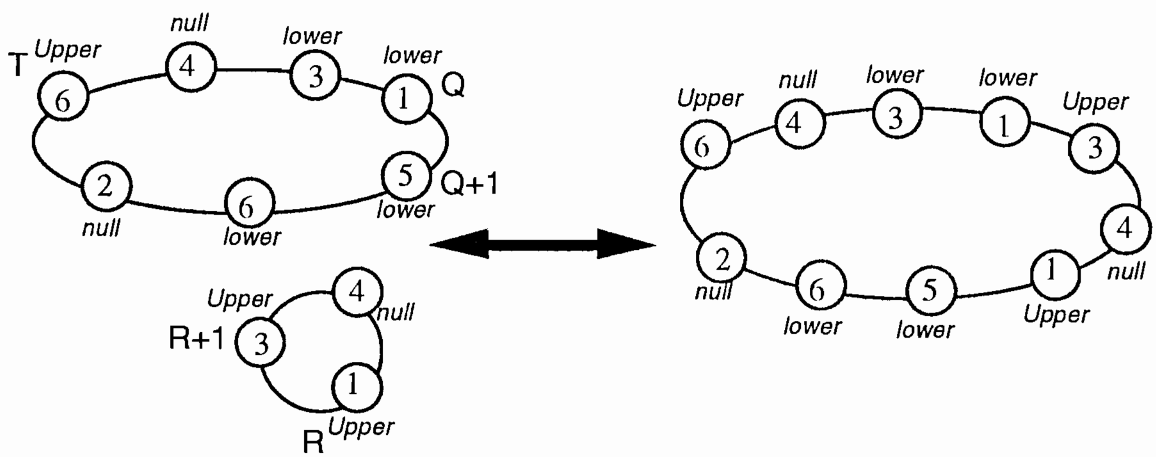
Note that the closed walk w_j may possibly be the empty walk from i_j to i_j. For the purposes of our proof we assume that σ is the identity. The general result will follow by symmetry.

We describe an involution on the sets of walkalongs that we have defined above. Find the smallest j, then the largest k, and then the smallest i such that there is a problem place Q of value j at a place $T+i$ in the walkalong with letter k at origin T.

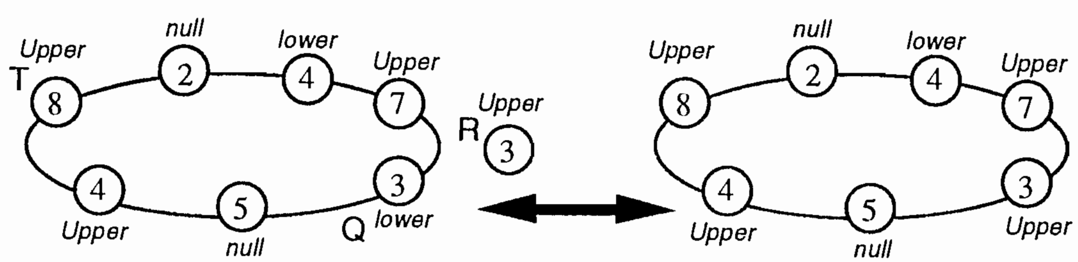
If there are no such j, k, i, then do nothing. Such a term will constitute a fixed point. Otherwise, by our definition of problem place there is a unique occurrence R of value j that has upper case.

i) If R is directed and $R \neq Q$, then redirect Q to $R+1$ and R to $Q+1$
ii) Or if R is undirected and $R \neq Q$, then delete R and let Q have upper case, whereas if R is directed and $R = Q$, then let Q have lower case and introduce an undirected upper case place with value j.

These instructions change the answer to the question, are Q and R in the same circle ? But they preserve the answer to the question, what is the distance from Q to R if they are in the same walkalong, and what is the distance around the walkalong of R, if not ? The instruction ii) addresses the special circumstance in which this distance is zero, so that either $R=Q$ or R is undirected.



Above is an example that illustrates instruction i) and below is an example that illustrates instruction ii).



The map determined by these instructions is well defined. Redirecting Q and R turns one walkalong into two walkalongs or two walkalongs into one walkalong. The instructions preserve the values of the places and their cases: upper, lower, or null.

Therefore the map is weight preserving. In particular, introducing or deleting an undirected place or changing the case of Q does not affect weight.

The map is also sign reversing. The sign of each term is given by the sign of the cycle structure of the upper case letters. The instructions do not affect the number or kind of upper case letters, but they do change the number of cycles by one. This changes the sign.

The map is an involution because Q remains the problem place after the instructions are executed. If instruction i) is executed, then Q and R belong to walkalongs in which no place with upper or lower case has value less than j, and no place with upper case has value greater than k. Also, if Q and R start out in the same cycle, then the coordinate of R may not be less than that of Q. Therefore from the origin of the walkalong to Q no place is redirected. Q continues to be the problem place of highest priority. A second application of instruction i) reverses the effect of the first application. Likewise, if instruction ii) is executed, then no place is redirected. We conclude that the map is a weight preserving, sign reversing involution.

The fixed points of the involution are those sets of walkalongs for which the value at the upper and lower case places of a walkalong attains a minimum at the origin. One consequence of this is that in each walkalong the origin is the unique place with upper case. Therefore the fixed points have positive sign because the corresponding cycles have length one. Note also that some or all of the walkalongs may consist of a single undirected place.

We think of each walkalong as a closed walk with origin at the origin of the walkalong. The letters at the origins are all different and therefore the closed walks may be listed in sequence $w_1,\ldots,w_n$ so that the letters at the origin increase $i_1 < i_2 < \ldots < i_n$. Then for all $k < j$ the word w_j does not contain the letter i_k. However, the letters that do not belong to the set $\{i_1,\ldots,i_n\}$ (and were therefore associated with null case) may occur in any of the closed walks. Finally, some or all of the words may possibly be empty. The fact that $\mathrm{W}(w_j)$ is the weight of the walk brings us to the expression in the statement of the theorem. QED.

In the event that $n = N$, then $e_N(\omega_1,\ldots,\omega_N) = \det(\mathbf{I}/(\mathbf{I}-\mathbf{A}))$ and the theorem generates the sequences found in Theorem 2.2.3, whereas if $n = 1$, then the theorem says that $e_1(\omega_1,\ldots,\omega_N) = \mathrm{tr}(\mathbf{I}/(\mathbf{I}-\mathbf{A}))$ is the generating function for closed walks. Note that the involution used to prove the above theorem did not alter the case of any letter, and therefore it can be applied separately to any $n \times n$ principal minor of the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. We will appeal to this fact when we prove Theorem 2.4.3 at the end of the next section.

SECTION 2.4 CLOSED WALKS AND CYCLE PRODUCTS

In this section we express walks in terms of cycle products, and vice versa. The work of this section and the previous section will allow us to give a combinatorial proof of Theorem 2.4.3 which presents an identity due to Jacobi [Go] concerning the calculation of any principal minor of the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. Given letters $i_1,\ldots,i_r$, consider the $r\times r$ principal minor of $\mathbf{I}/(\mathbf{I}-\mathbf{A})$ that is taken from the rows $i_1,\ldots,i_r$ and the columns $i_1,\ldots,i_r$. We know from the proof of Theorem 2.3.1 that the determinant of this principal minor generates sequences $w_1,\ldots,w_r$ of closed walks such that for all j, the letters $i_1,\ldots,i_{j-1}$ do not appear in w_j, but the letter i_j appears at the origin of w_j. In the proof of Theorem 2.4.3 we shall see that the determinant of this principal minor also generates stacks of boxes of cycles for which the cycles in the first box must each contain at least one letter from among $i_1,\ldots,i_r$. The bijection between these two interpretations is quite involved, and we first consider it in the special case of a 1×1 principal minor, which is the subject of Lemma 2.4.2. In this case the bijection relates closed walks from i to i with stacks of boxes of cycles for which the first box has one cycle and that cycle contains the letter i. In Chapter 4 we refer back to Lemma 2.4.2 in order to arrive at a new quotient formula for the Schur functions $s_\lambda(\xi_1,\ldots,\xi_N)$.

If we think of the walk matrix $\mathbf{I}/(\mathbf{I}-\mathbf{A})$ as the inverse of $\mathbf{I}-\mathbf{A}$, then we know from Cramer's rule that

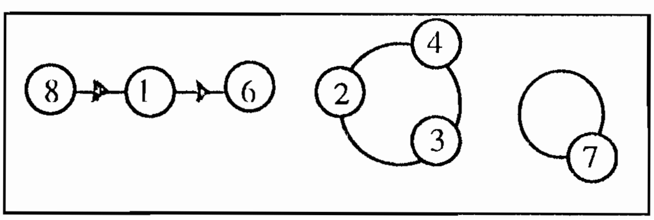
$$\left(\frac{\mathbf{I}}{\mathbf{I}-\mathbf{A}}\right)_{ij}=\frac{\det(\mathbf{I}-\mathbf{A})^{(ij)}}{\det(\mathbf{I}-\mathbf{A})}$$

We showed in the first section of this chapter that the combinatorial approach to the recursion relation $\sum_{r=0}^{N}(-1)^r e_{N-r}(\xi_1,\ldots,\xi_N)p_r(\xi_1,\ldots,\xi_N)=0$ also serves to interpret the above equation if we first cross multiply by $\det(\mathbf{I}-\mathbf{A})$. This time we interpret the above equation without cross multiplying. This will lead us to Theorem 2.4.1 and Lemma 2.4.2 which express walks in terms of cycle products.

On the right hand side of the above equation, $1/\det(\mathbf{I}-\mathbf{A})$ is understood to be the generating function for stacks of boxes of cycles, and $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ is shorthand for the cofactor

$$\det(\mathbf{I}-\mathbf{A})^{(ij)} = \begin{vmatrix} 1-a_{11} & -a_{12} & 0 & & -a_{1N} \\ -a_{21} & 1-a_{22} & 0 & & \\ & & 0 & & \\ 0 & 0 & 1_{ji} & 0 & 0 \\ -a_{N1} & & 0 & & 1-a_{NN} \end{vmatrix}$$

The terms of this cofactor can be gotten from $\det(\mathbf{I}-\mathbf{A})$ by first insisting that in every term there be a cycle with the edge $-a_{ji}$, and then removing this edge. This cycle may therefore be thought of as a path from i to j. Recall that a path from i to j is a walk from i to j for which all of the vertices are distinct. In particular, a path from i to i is understood to be the empty walk from i to i. We think of $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ as generating boxes of cycles for which a path from i to j plays the role of one of the cycles. In the example below, $i=8$, $j=6$, and the path from 8 to 6 takes the role of the cycle (168).

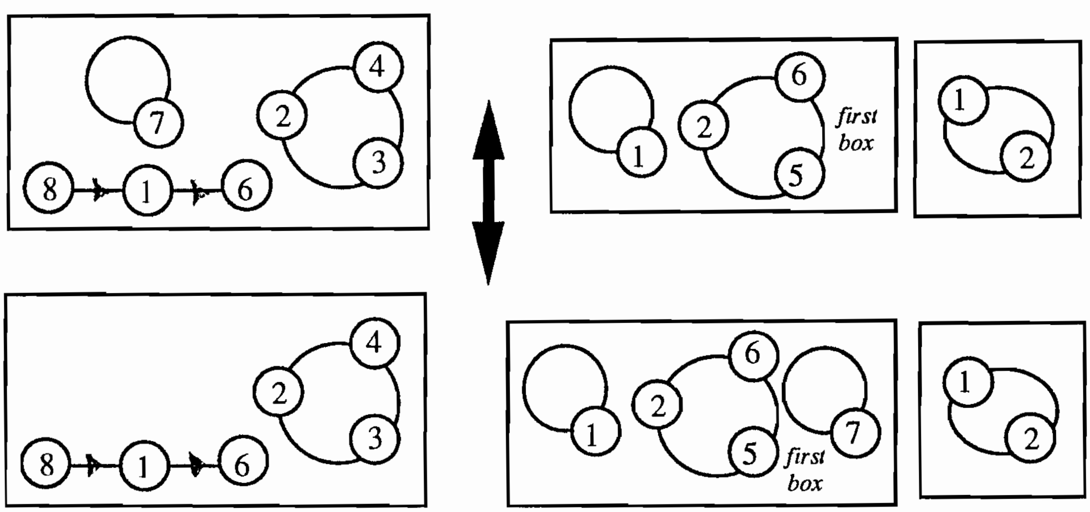


Each box of cycles employs no more than N edges. The edges involved are taken from $-\mathbf{A}$, and therefore, with the exception of the path, each cycle contributes a sign of -1. Therefore adding or deleting a cycle from a term in $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ changes its sign.

The expression $[1/\det(\mathbf{I}-\mathbf{A})][\det(\mathbf{I}-\mathbf{A})^{(ij)}]$ generates pairs of objects, the first of which is a stack of boxes of cycles by Theorem 2.1.5, and the second of which is a signed box of cycles in which a cycle has been replaced by a path from i to j. We now define an involution on these pairs of objects by which most of them will cancel away. Given a pair of objects, say that a cycle is a candidate if it is

> either a cycle from $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ but not the path from i to j
> or a cycle from the stack of boxes of cycles that is disjoint from the
> cycles of $\det(\mathbf{I}-\mathbf{A})^{(ij)}$, including the path from i to j.

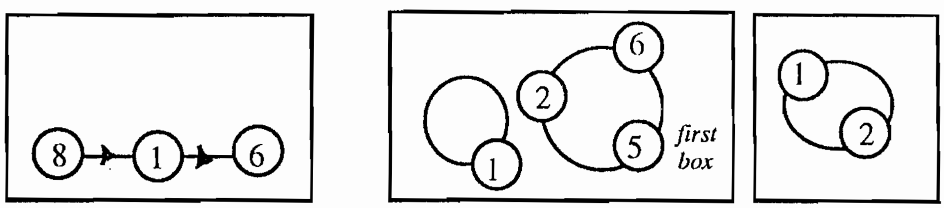
If there is any such candidate, then there is a unique candidate with smallest letter. Move this cycle from the first box to the cofactor, or from the cofactor to the front of the stack (and therefore the first box, upon adjusting the boxes, as in the example below, where the cycle is (7)).



This action changes the total sign because adding or removing a cycle from $\det(\mathbf{I}-\mathbf{A})^{(ij)}$ changes sign, but adding or removing a cycle from $1/\det(\mathbf{I}-\mathbf{A})$ does not.

The action that we have just defined is reversed upon a second application. If the action moved a cycle to the cofactor, then afterwards that cycle has the smallest letter among those in the cofactor. It must also overlap any cycle that subsequently ventured into the first box. This means that the cycle continues to be the candidate of highest priority. Likewise, if the action moved a cycle to the first box, then it also continues to be the candidate of highest priority. The action is reversed upon a second application and therefore it defines a sign reversing weight preserving involution. The involution shows us how to cancel away any pair of objects having a cycle that is a candidate.

The fixed points of the involution are those pairs of objects for which no cycle is a candidate. This means that the only cycle in the cofactor is the path from i to j. Any cycle in the first box of the stack must overlap the path, as in the example below.



We may therefore think of the path as a cycle that is at the front of the stack. Then the newly created stack is such that the first box consists of one cycle that employs the edge a_{ji}, but from which this edge is removed. We restate this result as the following theorem.

THEOREM 2.4.1 *Let K_{ij} be the set of stacks of boxes of cycles on $1,\ldots,N$ with a total of $n+1$ places for which there is one cycle in the first box and that cycle has an edge a_{ji}. Then*

$$\left(\mathbf{A}^n\right)_{ij} = \frac{1}{a_{ji}} \sum_{s \in K_{ij}} \mathrm{W}_{\mathbf{A}}(s)$$

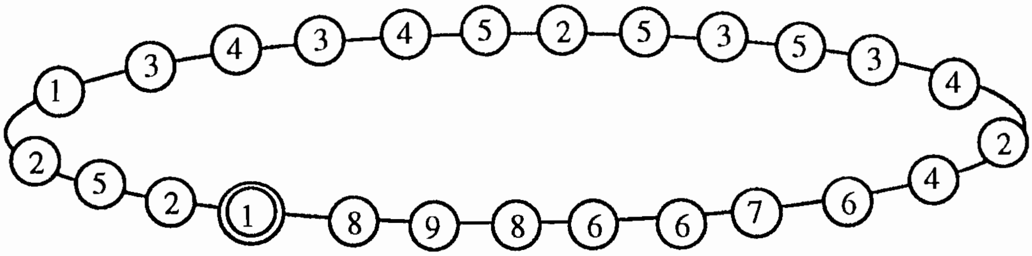
In particular, the theorem holds when $i = j$, in which case the first box consists of the cycle with weight a_{ii}. The second box must then consist of a single cycle that contains the letter i. We may cancel away the a_{ii} in the first box with the factor $1/a_{ii}$. This gives the following interpretation for $\left(\mathbf{A}^n\right)_{ii}$ which proves useful in the chapter on Schur functions.

LEMMA 2.4.2 *Let K_i be the set of stacks of boxes of cycles on $1,\ldots,N$ with a total of n places for which there is one cycle in the first box and that cycle employs the letter i. Then*

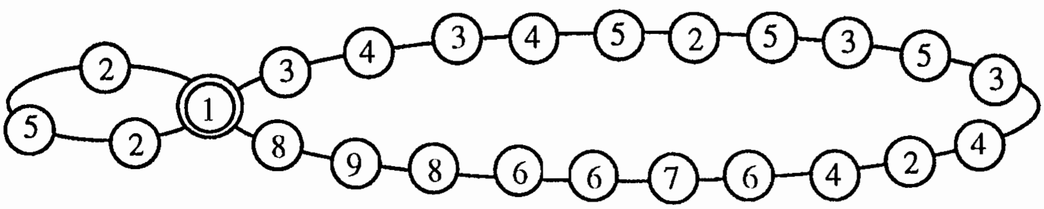
$$\left(\mathbf{A}^n\right)_{ii} = \sum_{s \in K_i} \mathrm{W}_{\mathbf{A}}(s)$$

We conclude our interpretation of the equation $\left(\mathbf{I}/(\mathbf{I}-\mathbf{A})\right)_{ij} =$ $\det(\mathbf{I}-\mathbf{A})^{(ij)}/\det(\mathbf{I}-\mathbf{A})$ by demonstrating a weight preserving correspondence between walks from i to i and the stacks generated in Lemma 2.4.2. We do this by depicting each closed walk as a tree-like structure of cycles, and then identifying these cycles with those of a stack. Afterwards, we modify this weight preserving correspondence to handle the case $i \neq j$. Finally, as promised, we give a combinatorial proof of a more general identity that is due to Jacobi.

Suppose that we are given a closed walk from i to i. The closed walk pictured below will help us illustrate the correspondence that we wish to construct.



If all of the letters of the closed walk are distinct, then we are done. Otherwise, we arrive at a new object by pinning together places in this walk. This is a procedure that does not redirect any places, but simply allows us to visualize the closed walk in a different way. If there are two or more places at which there is the same letter, then we may think of pinning these places together. In the closed walk that we are considering, if we pin together the places at which there is the letter 1, then the result looks like

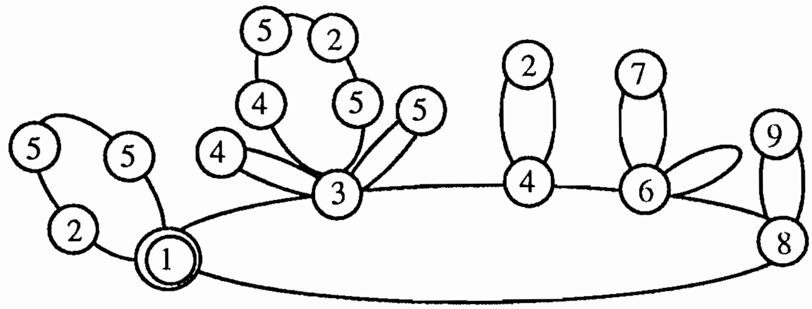


In this way the closed walk from 1 to 1 is thought of as a sequence of closed walks which share an origin. The letter at this origin is 1, and it is the sole appearance of this letter in each of the closed walks. The walks are arranged in the plane from left to right, clockwise around the origin. In creating such a visualization we say that we are pinning together the places at which there is a letter 1. The original closed walk can be immediately recovered by removing the pin which holds these places together.

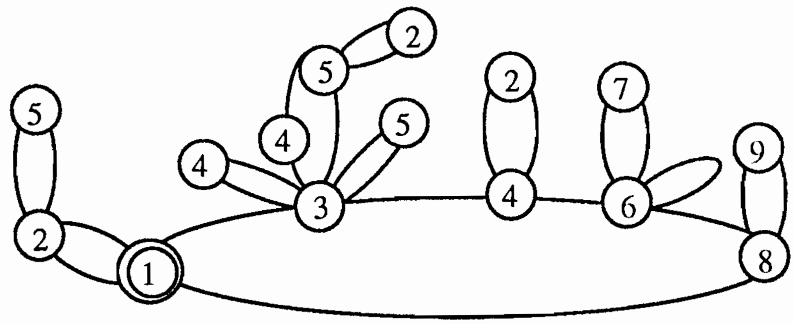
Starting with the original walk, let i be the letter at the origin Q. Let $Q+r_{i1}, Q+r_{i2}, \ldots, Q+r_{is}$ be the places, in order, at which the letter i appears. Pin together these places. This allows us to visualize from left to right a sequence of closed walks from i to i as in the picture above. As we remarked, each of these closed walks has only one occurrence of the letter i, and that is at the origin.

In general, suppose that by pinning together places we created a closed walk W from j to j. Given such a walk, assume that the unique occurrence of j in the walk is at the origin R. We describe a procedure for pinning together places in this walk and for defining a function ψ that maps W into a cycle $\psi(W)$ with origin R. If the walk W is a cycle, then let $\psi(W)$be this cycle with origin R, and do nothing. Otherwise, find the first place at which there is a letter which occurs more than once in this closed walk. If the letter is j_1, then let $R+r_{j_1 1}, R+r_{j_1 2}, \ldots, R+r_{j_1 s_1}$ be the places in this closed walk at which j_1 occurs. Then find the first place after $R+r_{j_1 s_1}$ in which there is a letter which occurs more than once. If the letter is j_2, then let the places be $R+r_{j_2 1}, R+r_{j_2 2}, \ldots, R+r_{j_2 s_2}$. Continue in this way until all of the places of the closed walk are exhausted. Then pin together the places $R+r_{j_1 1}, R+r_{j_1 2}, \ldots, R+r_{j_1 s_1}$, and pin together the places $R+r_{j_2 1}, R+r_{j_2 2}, \ldots, R+r_{j_2 s_2}$, and so on.

For each letter j_k the places $R+r_{j_k 1}, R+r_{j_k 2}, \ldots, R+r_{j_k s_k}$ are pinned together. This means that we visualize, from left to right, a sequence of closed walks that share an origin at which there is a letter j_k. Note that each of these walks has exactly one occurrence of j_k, and that is at the origin. If we were to equate places that are pinned together, then we would see that the sequences of closed walks are organized, from left to right, around a cycle $\psi(W)$. This cycle contains the letters $j_1, j_2, \ldots, j_k, \ldots$, among others, and we say that it has origin R. In the example below there is such a cycle (12) for which the letter at the origin is 1 and $j_1 = 2$, and another such cycle (13468) for which the letter at the origin is 1 and $j_1 = 3$, $j_2 = 4$, $j_3 = 6$, and $j_4 = 9$.



In the example above, all of the resulting closed walks are cycles except for $3 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 5 \rightarrow 3$, and applying the procedure to this cycle gives



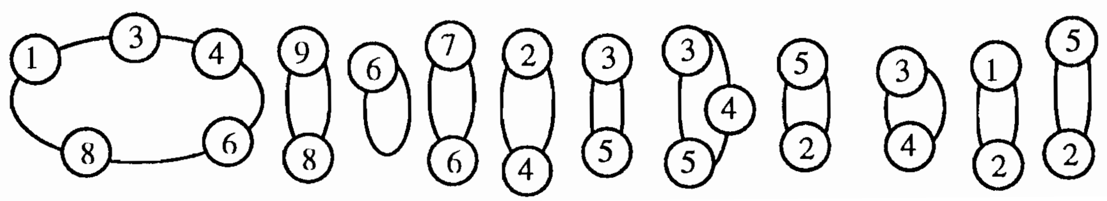
In general, if we keep applying the same procedure to the closed walks that result, then ultimately there is no walk in which a letter occurs more than once. If the places that are pinned together are not yet equated, then we can recover the original closed walk without difficulty, which can done by simply removing the pins. If we equate places that are pinned together, then we are left with a tree-like structure of cycles, which we call a cycle tree. There are two restrictions on the cycles of a cycle tree that are a consequence of the construction. These restrictions are best understood in terms of subtrees. Each cycle arose as a cycle C uniquely identified with a closed walk $\psi^{-1}(C)$, and may therefore be identified with a cycle subtree T_C that was constructed by pinning together places from that walk. The first condition on the cycles of the cycle tree is that if r is the letter at the origin R of a cycle C, then this is the unique occurrence of r in the corresponding subtree T_C. The second condition is that if the letter r occurs at a place R_1, and there is a cycle in which R_1 precedes R_2, then r does not occur in any subtree whose origin is R_2.

As we already remarked, introducing pins does not redirect any places and does not affect their weight. Therefore the closed walk may be immediately recovered by removing all of the pins.

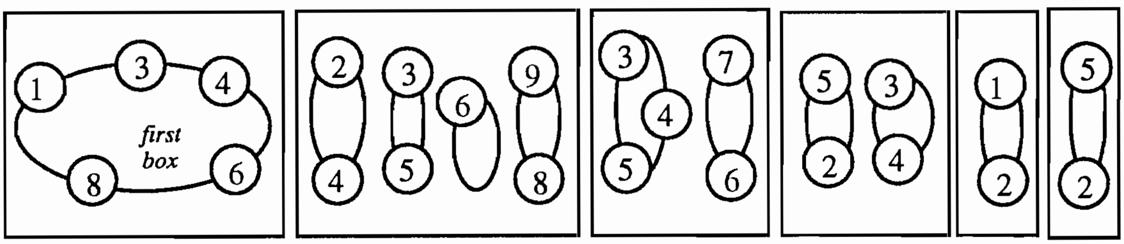
The object that we have constructed from the closed walk has the same weight that it would have if we broke it apart into its cycles. We now show how to list out these cycles in such a way that they form the desired stack. First of all, equate places that are pinned together, so that the cycles are well defined. Then write out the cycles as a sequence, from left to right, observing the following rules.

i) write C to the left of all other cycles in the corresponding subtree T_C.
ii) if the origins of T_C and $T_{C'}$ belong to the same cycle, and T_C appears the left of $T_{C'}$, then write C to the right of C'.

These rules determine a total order on the cycles that allows us to determine, given any two cycles, which we must write to the left. It is a total order because exactly one of the rules must apply, and the rules do not interfere with each other, but both are transitive. Therefore the rules determine a well defined way of writing out the cycles in a sequence. With respect to the example we have been considering, this gives the following sequence of cycles.



As we have seen in Section 1.3, there is a canonical way of presenting any such sequence of cycles as a stack of boxes of cycles. Introduction of boxes does not affect the fact that a cycle is written to the left of all cycles located outwards from it in the cycle tree because any two such cycles C and C' must be separated by a sequence of cycles $C = D_1, D_2, \ldots, D_r = C'$ for which D_i and D_{i+1} always overlap. In particular, C and C' must be in different boxes. Recall the closed walk from i to i and origin Q that corresponds to the entire cycle tree. Then the cycle from the cycle tree that is the rightmost among those with letter i is the unique cycle in the first box.

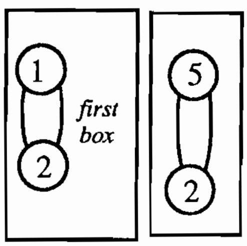


This gives us a stack of boxes of cycles for which the first box consists of a single cycle that includes the letter i. This is the kind of stack described in Lemma 2.4.2, and we see that every cycle tree yields a different stack. We now show that we can recover the cycle tree from the stack, and that every such stack yields a different cycle tree. In order to do this, we first define a new way of removing cycles from the stack.

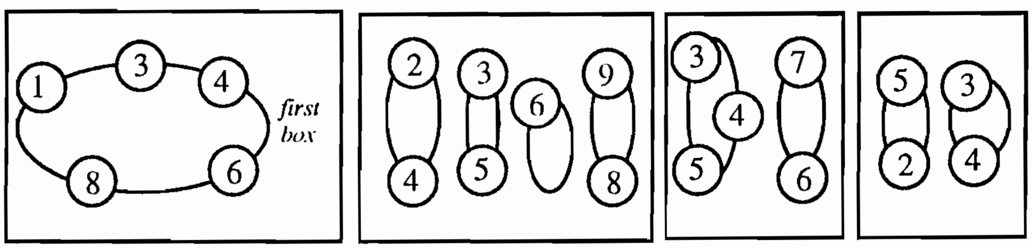
Given a cycle C from a sequence, we say that to *push back* C is to remove from the sequence C and all of the cycles C' that are separated from C by a sequence of cycles $C = D_1, D_2, \ldots, D_r = C'$ for which D_i and D_{i+1} always overlap. In particular, we may push back several cycles $C_1, \ldots, C_s$, and regardless of the order in which we do this, we find that ultimately the same sequence of cycles remains.

In order to reconstruct the tree, we assume that we know the letter i. We first recover the order of the cycles which contain this letter. If they are located from $C_1, \ldots, C_r$

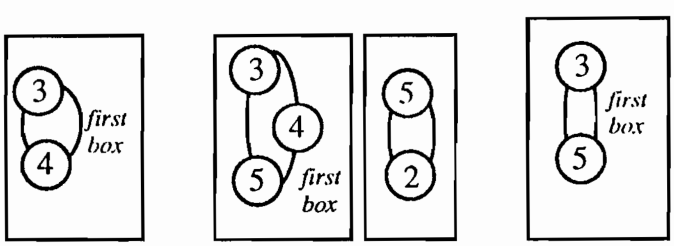
left to right in the sequence, then we know that they should be positioned from right to left in the cycle subtree. We want to find out which cycles belong to each of the corresponding subtrees. Before we consider the general case, we first consider the example that we have been working with. The cycle with 1 that is rightmost in the sequence depicted above is (12). If we push this cycle to the right, we get the stacks



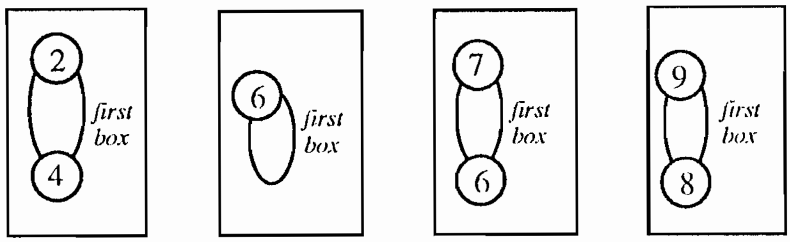
and



The cycles that belong to the stacks with (12) and (13468) in their first boxes are the cycles that belong to the corresponding subtrees. The cycle (25) is in the same subtree as (12) and must lie outward of it. As for the stack that starts with (13468), find the rightmost cycle that contains the letter 3, and push it back. Then find the next rightmost cycle with the letter 3, and push it back, and so on, until all of the 3's are exhausted.



Then find the rightmost cycle that contains the letter 4, and push it back, and then the next rightmost, if any, and so on. Proceeding in this way, we separate the cycles of the stack into the following stacks with the following order.



In this way we can tell which cycles belong to which cycle subtrees and the order in which these subtrees appear in the cycle tree.

In general, in order to recover a cycle tree from a stack, we show, given a cycle from the stack, which cycles belong to the corresponding subtree. This information is sufficient to reconstruct the cycle tree. Assume that we know which cycles in the stack belong to the subtree T_D that corresponds to D. These cycles form a subsequence w_D of the stack which we may also think of as a stack. Let D have origin R and letters $j_0,\ldots,j_{\ell-1}$ at $R, R+1,\ldots,R+\ell-1$, respectively. In order to recover the subtrees with origin at $R+1$, find the cycles in w_D that have the letter j_1. Push back the rightmost of these cycles, and the next rightmost, and so on. The resulting stacks correspond to the subtrees associated with each of these cycles. In the stack that remains, find the cycles that have the letter j_2, and proceed in a similar way, and so on with all of the letters $j_3,\ldots,j_{\ell-1}$. This recovers the order of the subtrees, from left to right, that come out of the cycle D, and the cycles which belong to them. In general, this means that our procedure yields a cycle tree, and one with the same weight as the stack. Furthermore, this recovers the cycle tree that we want because given a subtree T_D any cycle in it must have been removed from the stack any time D was pushed back. In particular, any such cycle must have been to the right of D in the stack. As this is so for all D, then we know that we are reversing the effects of the algorithm and recovering the original cycle tree.

This shows that there is a weight preserving correspondence between the stacks of Lemma 2.4.2 and cycle trees. Likewise, we have shown that every cycle tree corresponds to a closed walk. Taken together these two correspondences determine a correspondence between the desired stacks and closed walks in the case when $i = j$.

When $i \neq j$, we remark that the same algorithm establishes a weight preserving correspondence between walks from i to j and the stacks described in Theorem 2.4.1. Adding an edge a_{ji} to the end of such a walk yields a closed walk from i to i. The algorithm described above folds together this closed walk in such a way that the edge a_{ji} belongs to the rightmost cycle with the letter i. The algorithm then takes this cycle and makes it the unique cycle in the first box of a stack. This cycle has the edge a_{ji} and therefore the stack is of the kind described in Theorem 2.4.1. These instructions may be reversed, and this recovers the walk from i to j. This modification of the algorithm that interprets the equation in Lemma 2.4.2 therefore yields a combinatorial interpretation of the equation in Theorem 2.4.1.

We have shown how to express walks in terms of cycle products. We want to generalize this result, as promised, to prove our next result, Theorem 2.4.3. But first, we show how to express cycle products in terms of closed walks. For this we make use of the notion of pushing back a cycle C from a stack w, which we defined above.

Suppose that we are given a stack w. We use induction to express w as a sequence of closed walks $v_{i_k}, v_{i_{k-1}}, \ldots, v_{i_1}$. Let i_1 be the smallest letter in w and let C_{i_1} be the first cycle in w with that letter. Push back C_{i_1} in w. This describes w as a sequence $w = w_1 u_{i_1}$ where u_{i_1} is the sequence of cycles that were pushed back along with C_{i_1}, and w_1 is the sequence of cycles that remains. Note that C_{i_1} is by definition the only cycle in the first box of u_{i_1}. Let v_{i_1} be the closed walk from i_1 to i_1 that corresponds to u_{i_1}. In general, suppose that we have expressed w as the product of stacks $w = w_{j-1}u_{i_{j-1}}\cdots u_{i_1}$, where the letters $i_1,\ldots,i_{j-1}$ do not appear in w_{j-1}, and each u_{i_k} corresponds to a closed walk v_{i_k} in which none of the letters $i_1,\ldots,i_{k-1}$ appear. Let i_j be the smallest letter in w_{j-1} and let C_{i_j} be the first cycle in w_{j-1} with that letter. Push back C_{i_j} in w_{j-1}. This describes w_{j-1} as a sequence $w_{j-1} = w_j u_{i_j}$ where u_{i_j} is the unique sequence of cycles that were pushed back along with C_{i_j}, and w_j is the unique sequence of cycles that remains. By definition C_{i_j} is the only cycle in the first box of u_{i_j}, and therefore there is a unique closed walk v_{i_j} from i_j to i_j that corresponds to u_{i_j}. This walk does not employ any of the letters $i_1,\ldots,i_{j-1}$, and the stack w_j does not employ any of the letters $i_1,\ldots,i_j$. By induction we conclude that this procedure expresses w as a sequence of closed walks $v_{i_k}, v_{i_{k-1}},\ldots,v_{i_1}$ in a unique way. The walk v_{i_j} does not contain any of the letters $i_1,\ldots,i_{j-1}$, and therefore the sequence of closed walks is the same as described in Theorem 2.2.3 and Theorem 2.3.1 (when $n = N$). The stack w can be recovered from the walks $v_{i_k}, v_{i_{k-1}},\ldots,v_{i_1}$ by concatenating the sequences $u_{i_k}, u_{i_{k-1}},\ldots,u_{i_1}$.

For example, the stack $w=$ || (12)(3)(45) || (267)(13)(48) || (12)(34) || gives rise to stacks $u_1=$ || (12) || (13)(267) || (12)(34) ||, $u_3=$ || (3) ||, $u_4=$ || (45) || (48) ||, that by Lemma 2.4.2 correspond to walks from 1 to 1, 3 to 3, and 4 to 4, respectively.

Another correspondence between stacks and the sequences of closed walks referred to above can be defined by expressing each in terms of a table as we did in Theorem 2.2.2 and Theorem 2.2.3. We do not know, however, what is the relationship between this correspondence and the one described above.

Finally, we state and prove the generalization due to Jacobi [Go], that we mentioned at the start of this section.

THEOREM 2.4.3 *Divide the letters* $1,\ldots,N$ *into two disjoint sequences* $i_1<\ldots<i_r$ *and* $j_1<\ldots<j_{N-r}$. *Let* $(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{i_1,\ldots,i_r}$ *be the principal minor that corresponds to the rows*

$i_1,\dots,i_r$ and columns $i_1,\dots,i_r$ of $\mathbf{I}/(\mathbf{I}-\mathbf{A})$. Let $(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}$ be the principal minor that corresponds to the rows $j_1,\dots,j_{N-r}$ and columns $j_1,\dots,j_{N-r}$ of $(\mathbf{I}-\mathbf{A})$. Then

$$\det(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{i_1,\dots,i_r} = \frac{\det(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}}{\det(\mathbf{I}-\mathbf{A})}$$

We give a bijective proof. In order to do this, we first identify the right hand side of the above equation with pairs of weighted objects generated by $1/\det(\mathbf{I}-\mathbf{A})$ and $\det(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}$. The first object is a stack of boxes of cycles, and the second object is a box of cycles with edges from $-\mathbf{A}$ that does not employ any of the letters $i_1,\dots,i_r$. We perform an involution on these pairs that is in the same spirit as the ones that we used to prove Theorem 2.1.5 and Lemma 2.4.2. We say that a cycle is a candidate if it is

either a cycle from $\det(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}$
or a cycle from the stack of boxes of cycles that is disjoint from the
cycles of $\det(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}$ and does not contain any of the letters $i_1,\dots,i_r$.

If there is any such candidate, then there is a unique candidate with smallest letter. Move this cycle from the first box to the cofactor, or from the cofactor to the front of the stack. The fixed points are those pairs for which there are no cycles in the box of cycles that corresponds to $\det(\mathbf{I}-\mathbf{A})_{j_1,\dots,j_{N-r}}$, and any cycle in the first box of the stack must contain one of the letters $i_1,\dots,i_r$.

We remarked at the end of Theorem 2.3.1 that the proof of that theorem shows that $\det(\mathbf{I}/(\mathbf{I}-\mathbf{A}))_{i_1,\dots,i_r}$ is a generating function for sequences of closed walks $w_{i_1},\dots,w_{i_r}$ on the letters $1,\dots,N$ such that for all j, w_{i_j} is a closed walk from i_j to i_j with no occurrence of any of the letters $i_1,\dots,i_{j-1}$. It therefore remains for us to present a weight preserving correspondence between these sequences of walks and the stacks that are the fixed points of the involution defined above. But this correspondence is practically the same as the one described just before the statement of this theorem, by which a stack may be expressed as a sequence of closed walks. Given such a stack s, find the first, if any, cycle C_{i_1} with the letter i_1, and push it back. This describes s as a sequence $s = s_1 u_{i_1}$ where u_{i_1} is the sequence of cycles that were pushed back along with C_{i_1}, and s_1 is the sequence of cycles that remains. Note that C_{i_1} is by definition the only cycle in the first box of u_{i_1}. Let v_{i_1} be the closed walk from i_1 to i_1 that corresponds to u_{i_1}. In general, suppose that we have expressed s as the product of stacks $s = s_{j-1} u_{i_{j-1}} \cdots u_{i_1}$, where the letters $i_1, \ldots, i_{j-1}$ do not appear in s_{j-1}, and each u_{i_k} corresponds to a closed walk v_{i_k} in which none of the letters $i_1, \ldots, i_{k-1}$ appear. Let C_{i_j} be the first cycle in s_{j-1} with the letter i_j. Push back C_{i_j} in s_{j-1}. This describes s_{j-1} as a sequence $s_{j-1} = s_j u_{i_j}$ where u_{i_j} is the unique sequence of cycles that were pushed back along with C_{i_j}, and s_j is the unique sequence of cycles that remains. By definition C_{i_j} is the only cycle in the first box of u_{i_j}, and therefore there is a unique closed walk v_{i_j} from i_j to i_j that corresponds to u_{i_j}. This walk does not employ any of the letters $i_1, \ldots, i_{j-1}$, and the stack s_j does not employ any of the letters $i_1, \ldots, i_j$. By induction we conclude that this procedure expresses s as a sequence of closed walks $v_{i_k}, v_{i_{k-1}}, \ldots, v_{i_1}$ in a unique way. The walk v_{i_j} does not contain any of the letters $i_1, \ldots, i_{j-1}$, and therefore the sequence of closed walks is the same as described in Theorem 2.2.3 and Theorem 2.3.1 (when $n = N$). As in the discussion before this theorem, the stack s can be recovered from the walks $v_{i_k}, v_{i_{k-1}}, \ldots, v_{i_1}$ by concatenating the sequences $u_{i_k}, u_{i_{k-1}}, \ldots, u_{i_1}$. This demonstrates the correspondence and proves the theorem. QED

CHAPTER 3

FORGOTTEN AND MONOMIAL SYMMETRIC FUNCTIONS

Our object in this chapter is to evaluate forgotten symmetric functions f_λ and monomial symmetric functions m_λ at the eigenvalues $\xi_1,\ldots,\xi_N$ of an arbitrary $N\times N$ matrix $\mathbf{A}$. In the previous chapter we did this for the bases of power, elementary, and homogeneous symmetric functions. In principle it is possible to evaluate any symmetric function $b(\xi_1,\ldots,\xi_N)$ of degree n by expressing it in terms of one of these bases. For example, we may write $b=\sum_{\lambda\vdash n} V_\lambda(b)\cdot p_\lambda$ and evaluate the power symmetric functions at $\xi_1,\ldots,\xi_N$. Moreover, this last equation may be compared with the expression $b=\frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda B(\lambda)p_\lambda$ from Section 1.1, where B is the character associated with $b=\mathrm{ch}(B)$ by the Frobenius characteristic map, and $B(\lambda)$ is its value at the conjugacy class of type λ and size C_λ. It becomes apparent that $V_\lambda(b)=\frac{C_\lambda}{n!}B(\lambda)$. We may rightly say, from an algebraic point of view, that our work is done. From a combinatorial point of view, however, it has just begun. In this chapter we use interpretations of the character tables $\left(\mathrm{ch}^{-1}(f_\mu)(\lambda)\right)_{\mu,\lambda\vdash n}$ and $\left(\mathrm{ch}^{-1}(m_\mu)(\lambda)\right)_{\mu,\lambda\vdash n}$ to evaluate the forgotten symmetric functions and the monomial symmetric functions. This brings out unexpected but satisfying descriptions of $f_\mu(\xi_1,\ldots,\xi_N)$ in terms of Lyndon words and of $m_\mu(\xi_1,\ldots,\xi_N)$ in terms of the determinant of the walk matrix. These descriptions also reaffirm the central location of the power symmetric functions (and closed walks) within the combinatorics of the symmetric functions. Indeed, we devote the first section of this chapter to showing that Littlewood's formula for expressing $b(\xi_1,\ldots,\xi_N)$ in terms of immanants is equivalent to evaluating the equation $b=\frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda B(\lambda)p_\lambda$ at $\xi_1,\ldots,\xi_N$.

SECTION 3.1 LITTLEWOOD'S FORMULA

One of the themes of Littlewood's book The Theory of Group Characters is that of the immanant of a matrix [L, 81-121]. In the context of group characters this seems to be a natural generalization of the concept of the determinant of a matrix. Of interest to us is a formula from his book by which a symmetric function $b(\xi_1,\ldots,\xi_N)$ may be expressed as the sum of immanants. We present a weight preserving bijection between the terms of this formula and those of the expression $b(\xi_1,\xi_2,\ldots\xi_N) = \frac{1}{N!}\sum_{\sigma\in S_N} B(\sigma)p_\sigma(\xi_1,\xi_2,\ldots\xi_N)$, thereby showing that the two are equivalent. We also discuss the difficulty of working with the immanant of a matrix. It is fair to say that the immanant has yet to compare with the determinant in mathematical importance, even though the latter is but a special case of the former.

The determinant of a matrix is certainly among the most useful of mathematical concepts. We have seen the special role that it plays in combinatorics. The determinant lists out all permutations with their sign. But is it meaningful to replace the sign function with some other function ? As noted in Section 1.1, the sign function is an irreducible character of the symmetric group. With this in mind, given an $N \times N$ matrix $\mathbf{A}$ we may consider generalizing

$$\det \mathbf{A} = \sum_{\sigma\in S_N} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}$$

by replacing $\chi^{1^N}(\sigma) = \operatorname{sgn}(\sigma)$ with any irreducible character $\chi^\lambda(\sigma)$, $\lambda \succ N$. In this way we arrive at the immanant

$$\operatorname{Imm}_{\chi^\lambda} \mathbf{A} = \sum_{\sigma\in S_N} \chi^\lambda(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}.$$

We shall see, however, that in the results that we consider the combinatorics of immanants has nothing to do with the irreducible characters, and therefore we allow immanants to be defined most generally as $\mathrm{Imm}_B \mathbf{A} = \sum_{\sigma \in S_N} B(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}$, where $B(\sigma)$ is any character or any function constant on conjugacy classes.

Littlewood used the immanant as a unifying concept. By considering the case when the matrix $\mathbf{A}$ is defined to be

$$\mathbf{Z}_N = \begin{pmatrix} p_1 & 1 & & & & \\ p_2 & p_1 & 2 & & & \\ p_3 & p_2 & p_1 & 3 & & \\ \vdots & & & & \ddots & \\ \vdots & & & & & N-1 \\ p_N & p_{N-1} & p_{N-2} & \cdots & \cdots & p_1 \end{pmatrix}$$

he arrived at the formulation $N! s_\lambda = \mathrm{Imm}_{\chi^\lambda} \mathbf{Z}_N$. This nevertheless is just a device for writing $s_\lambda = \frac{1}{N!} \sum_{\sigma \in S_N} \chi^\lambda(\sigma) p_\sigma$. More interesting are his results about "immanants of complementary coaxial minors", especially those that allow for "repetitions of rows or columns" [L, 118-121]. We relate his observations, but we recast them in our own language.

Given a word $w = x_1 \cdots x_n$, define the word matrix $\mathbf{A}_w$ to be $\left(a_{x_i x_j}\right)_{1 \le i,j \le n}$. For example, if the word is 3552, then

$$\mathbf{A}_{3552} = \begin{pmatrix} a_{33} & a_{35} & a_{35} & a_{32} \\ a_{53} & a_{55} & a_{55} & a_{52} \\ a_{53} & a_{55} & a_{55} & a_{52} \\ a_{23} & a_{25} & a_{25} & a_{22} \end{pmatrix}$$

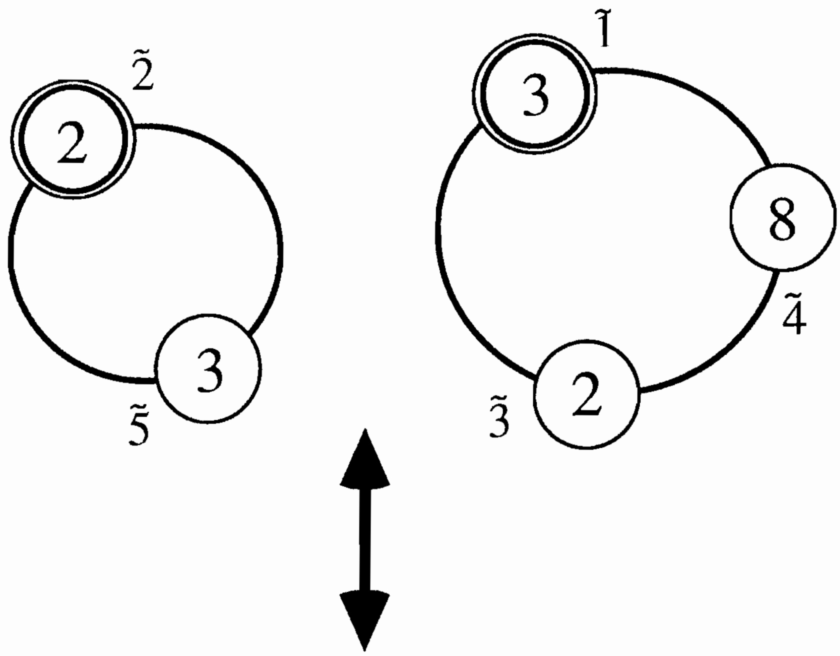
In particular, note that if the word is $12 \cdots n$, then the word matrix is $\mathbf{A}_{12\cdots n} = \left(a_{ij}\right)_{1 \le i,j \le n}$.

Let the symmetric function b and the character B be associated by the Frobenius characteristic map $b = \mathrm{ch}(B)$. Recall from Section 1.1 the equation $b = \frac{1}{n!}\sum_{\sigma\in S_n} B(\sigma)p_\sigma$, and the convention that the power symmetric function p_σ is indexed by the cycle structure of the permutation σ. From this equation we derive the following formula which constitutes Theorem III of three theorems that Littlewood [L, 118-121] gives concerning immanants. He states the formula for the case when $b = s_\lambda$ and $B = \chi^\lambda$. The proof of the general case is no more demanding, but of interest to us.

THEOREM 3.1.1 *Let $b = \mathrm{ch}(B)$ be a symmetric function of degree n. Then*

$$b(\xi_1, \xi_2, \ldots \xi_N) = \frac{1}{n!}\sum_{w\in \mathbf{N}^n} \mathrm{Imm}_B\, \mathbf{A}_w .$$

To prove this theorem we show that $\sum_{w\in\mathbf{N}^n} \mathrm{Imm}_B\, \mathbf{A}_w$ and $\sum_{\sigma\in S^n} B(\sigma)p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$ generate the same terms (the same objects with the same weights). Each term in $\sum_{\sigma\in S^n} B(\sigma)p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$ can be gotten by fixing σ and a term from $p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$ and associating with them the number $B(\sigma)$. Align the cycles of σ with the closed walks from the term in $p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$ so that the smallest letter in each cycle is aligned with the origin of the respective closed walk. Each edge of a cycle corresponds to an edge of a walk and vice versa. The edge $i \to \sigma(i)$ of the cycles may be associated with the $(i,\sigma(i))$th position in a permutation matrix. Place in these positions the corresponding edges from the walks, as in Figure 2. Note that there is always but one nonzero entry $a_{k_i l_j}$ in the ith column of the matrix and that $k_1k_2\cdots k_n = l_1l_2\cdots l_n$. If we let $k_1k_2\cdots k_n$ be the word w, then the matrix represents the term from $\mathrm{Imm}_B\, \mathbf{A}_w$ that depends on σ (and w) and is associated with $B(\sigma)$. Conversely, there is one term in $\sum_{w\in\mathbf{N}^n} \mathrm{Imm}_B\, \mathbf{A}_w$ for every σ and w and it is associated with $B(\sigma)$. From σ and w we can recover the term in



$$\begin{pmatrix} a_{33} & a_{32} & a_{32} & a_{38} & a_{33} \\ a_{23} & a_{22} & a_{22} & a_{28} & a_{23} \\ a_{23} & a_{22} & a_{22} & a_{28} & a_{23} \\ a_{83} & a_{82} & a_{82} & a_{88} & a_{83} \\ a_{33} & a_{32} & a_{32} & a_{38} & a_{33} \end{pmatrix}$$

Figure 2: A bijection equating $\sum_{\sigma\in S_n} B(\sigma)\mathrm{p}_\sigma(\xi_1,\ldots,\xi_N)$ and $\sum_{w\in \mathrm{N}^n} \mathrm{Imm}_\mathrm{B}\mathbf{A}_w$.

At the top is a term from $\sum B(\sigma)\mathrm{p}_\sigma(\xi_1,\ldots,\xi_N)$ consisting of closed walks $3 \to 8 \to 2 \to 3$ and $2 \to 3 \to 2$ that are labelled by the permutation σ with cycles $\tilde{1} \to \tilde{4} \to \tilde{3} \to \tilde{1}$ and $\tilde{2} \to \tilde{5} \to \tilde{2}$. Below is the corresponding term from $\sum \mathrm{Imm}_\mathrm{B}\mathbf{A}_w$, likewise indexed by σ. Asssociated to both terms is $B(\sigma)$.

$p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$. Therefore $\sum_{w\in\mathbf{N}^n}\mathrm{Imm}_B\,\mathbf{A}_w = \sum_{\sigma\in S^n} B(\sigma)p_\sigma(\xi_1,\xi_2,\ldots,\xi_N)$. The fact that $b(\xi_1,\xi_2,\ldots\xi_n) = \frac{1}{n!}\sum_{\sigma\in S_n} B(\sigma)p_\sigma(\xi_1,\xi_2,\ldots\xi_n)$ proves the theorem. QED.

We may use Littlewood's formula to calculate some examples. When $b = e_n$, then $B(\sigma) = \chi^{1^n}(\sigma) = \mathrm{sgn}(\sigma)$ and therefore the immanant $\mathrm{Imm}_{\chi^{1^n}}\mathbf{A}_w$ is just the determinant $\det\mathbf{A}_w$. If two columns of a matrix are the same, then the determinant is zero. Therefore $\det\mathbf{A}_{w_0} \neq 0$ only when $w_0 = x_1x_2\cdots x_n$ consists of letters that are all distinct. There are $n!$ words w of length n that consist of the letters $x_1,x_2,\ldots,x_n$. Note that for any such word w the rows and columns of $\mathbf{A}_w$ can be permuted to get $\mathbf{A}_{w_0}$. The number of rows and the number of columns permuted is the same so that $\det\mathbf{A}_w = \det\mathbf{A}_{w_0}$. Therefore the sum of the immanants of all $n!$ word matrices equals $n!\cdot\det\mathbf{A}_{w_0}$ where $\det\mathbf{A}_{w_0}$ is the generating function of boxes of cycles that employ the letters $x_1,x_2,\ldots,x_n$. And so in general $e_n(\xi_1,\ldots,\xi_N) = \frac{1}{n!}\sum_{w\in\mathbf{N}^n}\mathrm{Imm}_{\chi^{1^n}}\mathbf{A}_w$ generates boxes of cycles that employ n letters taken from $1,\ldots,N$.

Another example to consider is when b is the homogeneous symmetric function $h_n(\xi_1,\ldots,\xi_N)$, in which case $B(\sigma) = \chi^n(\sigma) = 1$. Then $\mathrm{per}\,\mathbf{A} = \mathrm{Imm}_{\chi^n}\mathbf{A} =$ $\sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}$ is known as the permanent of $\mathbf{A}$ and is a generating function for permutations $\sigma\in S_n$. If two words w and w' are rearrangements of each other, then permuting rows and columns shows that $\mathrm{per}\,\mathbf{A}_w = \mathrm{per}\,\mathbf{A}_{w'}$. There are $n!/m_1!m_2!\cdots m_N!$ ways to rearrange a word $w = 1^{m_1}2^{m_2}\cdots N^{m_N}$ of length n. This gives an identity which was the subject of a paper by Vere-Jones in 1984 [VJ][BR, 313] (upon replacing $\sum_{n=0}^{\infty} h_n(\xi_1,\ldots,\xi_N)$ with $1/\det(\mathbf{I}-\mathbf{A})$ and $(a_{ij})_{1\le i,j\le N}$ with $(a_{ij}y_j)_{1\le i,j\le N}$).

THEOREM 3.1.2 $$h_n(\xi_1,\ldots,\xi_N) = \sum_{\substack{w=1^{m_1}2^{m_2}\ldots N^{m_N}\\ m_1+\ldots+m_N=n}} \frac{1}{m_1!m_2!\cdots m_N!}\cdot\mathrm{per}\,\mathbf{A}_w.$$

As we know from Theorem 2.2.2, $h_n(\xi_1,\ldots,\xi_N)$ is a generating function of circuits, and we now show how this interpretation unfolds from the above identity. Let $w = 1^{m_1}2^{m_2}\cdots N^{m_n} = x_1x_2\cdots x_n$. If σ and τ are permutations, then $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} =$ $a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)}$ if and only if the letter $x_{\sigma(i)}$ always equals the letter $x_{\tau(i)}$. There are exactly $m_1!m_2!\cdots m_N!$ permutations that give the same term $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ as σ. Therefore $1/m_1!m_2!\cdots m_N!\cdot \operatorname{per} \mathbf{A}_w$ generates one term $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ for each distinct rearrangement $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ of $x_1x_2\cdots x_n$. Hence $h_n(\xi_1,\ldots,\xi_N)$ is the generating function of circuits.

In general, if words w and w' are rearrangements of each other, then permuting rows and columns shows that $\operatorname{Imm}_B \mathbf{A}_w = \operatorname{Imm}_B \mathbf{A}_{w'}$. The fact that a word $w = 1^{m_1}2^{m_2}\cdots N^{m_N}$ of length n can be rearranged in $n!/m_1!m_2!\cdots m_N!$ ways means that

$$b(\xi_1,\ldots,\xi_N) = \sum_{\substack{w=1^{m_1}2^{m_2}\cdots N^{m_N} \\ m_1+\cdots+m_N=n}} \frac{1}{m_1!m_2!\cdots m_N!}\cdot \operatorname{Imm}_B \mathbf{A}_w .$$

Any algebraic relation that relates Schur functions (and therefore symmetric functions in general) is satisfied by the corresponding sequences of immanants. Of course, if the symmetric functions are all in the same variables $\xi_1,\ldots,\xi_N$, then the same matrix $\mathbf{A}$ must underlie all of the sequences. We may then think of the relation as being satisfied by the immanants themselves if we contrive the appropriate rule for their multiplication. This is Littlewood's Theorem II concerning immanants. It may also be stated in the following way.

THEOREM 3.1.3 *Fix an* $N\times N$ *matrix* $\mathbf{A}$. *Fix an element* $F(y_1,\ldots,y_r)$ *of the field* $\mathbb{C}(y_1,\ldots,y_r)$. *For all* i, $1\le i\le r$, *let* $b_i(x_1,\ldots,x_N)$ *be a symmetric function of degree* $d_i \le N$, *and let* $b_i = \operatorname{ch}(B_i)$. *Then* $F(b_1,\ldots,b_r) = 0$ *if and only if*

$$F\left(\sum_{\substack{w=1^{m_1}2^{m_2}\cdots N^{m_N}\\ m_1+\ldots+m_N=d_1}} \frac{1}{m_1!m_2!\cdots m_N!}\cdot \mathrm{Imm}_{B_1}\mathbf{A}_w,\ \ldots\ ,\ \sum_{\substack{w=1^{m_1}2^{m_2}\cdots N^{m_N}\\ m_1+\ldots+m_N=d_r}} \frac{1}{m_1!m_2!\cdots m_N!}\cdot \mathrm{Imm}_{B_r}\mathbf{A}_w\right)=0.$$

Goulden and Jackson [GJ] have observed that applying this theorem to the relation $\sum_{r=0}^{n}(-1)^r e_r h_{n-r}=0$ yields the MacMahon Master theorem.

Rather than consider the entire sequence, we may consider the "leading terms", that is, those immanants for which all $m_i!=1$. We write this as $\sum_{w\in\mathbf{N}^n_<}\mathrm{Imm}_B\mathbf{A}_w$, where $\mathbf{N}^n_<$ is the set of all words $w\in\mathbf{N}^n$ whose letters are strictly increasing. Suppose that $b_1b_2\cdots b_\ell=c$ are symmetric functions in the variables $\xi_1,\ldots,\xi_N$, that the b_i have degree n_i and c has degree n, and that $b_i=\mathrm{ch}(B_i)$, $c=\mathrm{ch}(C)$. Then

$$\prod_{\substack{w_1w_2\cdots w_\ell\in\mathbf{N}^n_{\neq}\\ w_i\in\mathbf{N}^{n_i}_<}}\mathrm{Imm}_{B_i}\mathbf{A}_{w_i}=\sum_{w\in\mathbf{N}^n_<}\mathrm{Imm}_C\mathbf{A}_w$$

and we may define the multiplication

$$\sum_{w\in\mathbf{N}^{n_1}_<}\mathrm{Imm}_{B_1}\mathbf{A}_w\times\cdots\times\sum_{w\in\mathbf{N}^{n_\ell}_<}\mathrm{Imm}_{B_\ell}\mathbf{A}_w \equiv \prod_{\substack{w_1w_2\cdots w_\ell\in\mathbf{N}^n_{\neq}\\ w_i\in\mathbf{N}^{n_i}_<}}\mathrm{Imm}_{B_i}\mathbf{A}_{w_i}=\sum_{w\in\mathbf{N}^n_<}\mathrm{Imm}_C\mathbf{A}_w$$

accordingly. Here $\mathbf{N}^n_{\neq}$ is the set of words $w\in\mathbf{N}^n$ with all letters distinct, and therefore $w_1w_2\cdots w_\ell\in\mathbf{N}^n_{\neq}$ and $w_i\in\mathbf{N}^{n_i}_<$ indicate that the w_i do not share any letters. With multiplication defined in this way and addition defined in the usual way the sums $\sum_{w\in\mathbf{N}^n_<}\mathrm{Imm}_B\mathbf{A}_w$ satisfy the same relations as the corresponding symmetric functions b. This is Littlewood's Theorem I concerning immanants [L, 118-121].

THEOREM 3.1.4 *Fix an* $N \times N$ *matrix* $\mathbf{A}$. *Fix an element* $F(y_1,\ldots,y_r)$ *of the field* $\mathbb{C}(y_1,\ldots,y_r)$. *For all* i, $1 \le i \le r$, *let* $b_i(x_1,\ldots,x_N)$ *be a symmetric function of degree* $d_i \le N$, *and let* $b_i = \mathrm{ch}(B_i)$. *Then* $F(b_1,\ldots,b_r) = 0$ *if and only if*

$$F\left(\sum_{w\in\mathbf{N}_<^{d_1}} \mathrm{Imm}_{B_1} \mathbf{A}_w,\ldots, \sum_{w\in\mathbf{N}_<^{d_r}} \mathrm{Imm}_{B_r} \mathbf{A}_w\right) = 0$$

with multiplication in the latter relation defined as described above.

Littlewood himself shows that applying this theorem to the equation $\sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$ yields a recursion relation of Muir's that dates back to 1897. In our notation the recursion relation has the form

$$\sum_{r=0}^{n}(-1)^r\left(\sum_{w\in\mathbf{N}_<^{r}} \det \mathbf{A}_w \times \sum_{w\in\mathbf{N}_<^{n-r}} \mathrm{per}\,\mathbf{A}_w\right) = 0$$

More recently, Chu [C][BR, 314] uses the inclusion-exclusion principle to prove an identity by which $\mathrm{per}\,\mathbf{A}$ can be expanded in terms of determinants of the principal submatrices of $\mathbf{A}$. But his formula may be gotten by applying Littlewood's theorem to the identity $h_n = \det\left(e_{1-i+j}\right)_{1\le i,j\le n}$, and in our notation it looks like

$$\sum_{w\in\mathbf{N}_<^{n}} \mathrm{per}\,\mathbf{A}_w = \begin{vmatrix} \sum_{w\in\mathbf{N}_<^{1}} \det \mathbf{A}_w & \sum_{w\in\mathbf{N}_<^{2}} \det \mathbf{A}_w & & \sum_{w\in\mathbf{N}_<^{n}} \det \mathbf{A}_w \\ 1 & \sum_{w\in\mathbf{N}_<^{1}} \det \mathbf{A}_w & \ddots & \\ & \ddots & \ddots & \sum_{w\in\mathbf{N}_<^{2}} \det \mathbf{A}_w \\ & & 1 & \sum_{w\in\mathbf{N}_<^{1}} \det \mathbf{A}_w \end{vmatrix}$$

Likewise, he proves an identity by which $\det \mathbf{A}$ can be expanded in terms of permanents of the principal submatrices of $\mathbf{A}$. This may be gotten by applying Littlewood's theorem to the equation $e_n = \det\left(h_{1-i+j}\right)_{1\le i,j\le n}$. In our notation it looks like

$$\sum_{w\in \mathbf{N}_<^n} \det \mathbf{A}_w = \begin{vmatrix} \sum\limits_{w\in \mathbf{N}_<^1} \operatorname{per} \mathbf{A}_w & \sum\limits_{w\in \mathbf{N}_<^2} \operatorname{per} \mathbf{A}_w & & \sum\limits_{w\in \mathbf{N}_<^n} \operatorname{per} \mathbf{A}_w \\ 1 & \sum\limits_{w\in \mathbf{N}_<^1} \operatorname{per} \mathbf{A}_w & \ddots & \\ & \ddots & \ddots & \sum\limits_{w\in \mathbf{N}_<^2} \operatorname{per} \mathbf{A}_w \\ & & 1 & \sum\limits_{w\in \mathbf{N}_<^1} \operatorname{per} \mathbf{A}_w \end{vmatrix}$$

These are both special cases of a recent theorem of Goulden and Jackson's [GJ], which is equal to applying Littlewood's theorem to the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i-i+j}\right)_{1\le i,j\le n}$, as they themselves submit. This gives

THEOREM 3.1.5 $$\sum_{w\in \mathbf{N}_<^n} \operatorname{Imm}_{\chi^\lambda} \mathbf{A}_w = \det\left(\sum_{w\in \mathbf{N}_<^{\lambda_i-i+j}} \operatorname{per} \mathbf{A}_w \right)_{1\le i,j\le n}.$$

The dual identity $s_{\lambda'} = \det\left(e_{\lambda_i'-i+j}\right)_{1\le i,j\le n}$ yields an analogous equation in terms of determinants [GJ].

These several examples attest to the power of Theorem 3.1.1. But in a sense the theorem suggests that immanants are not interesting mathematical objects. If the multiplication of immanants parallels that of the Schur functions, then the question is, are the immanants easier to multiply, and are they interesting in their own right ? Multiplication of Schur functions is best understood by using the well documented Littlewood-Richardson's rule to expand the product itself in terms of Schur functions. If the Schur functions are thought of as generating functions for column strict tableaux, then this rule has a natural combinatorial interpretation. It would be very difficult to arrive at this rule by considering immanants because they depend on the irreducible characters. There are problems with the various ways of interpreting these characters, and this makes for difficulties in trying to multiply them directly. For these reasons formulas relating immanants (with the multiplication as defined above) are best understood as formulas relating Schur functions.

Are immanants interesting in their own right ? In sharp contrast with the determinant, they are peripheral to almost every branch of mathematics. Brualdi and Ryser's book [BR] contains a chapter on the combinatorics of the permanent. They do show that the relation between the problem of calculating $\operatorname{per}\mathbf{M}$, $m_{ij} = 1$ or 0, and of deciding whether $\operatorname{per}\mathbf{M} \neq 0$ is an important one in complexity theory. But ultimately, the most interesting question is the original one: why is the determinant such a central concept in mathematics?

Indeed, determinants and permanents are the only immanants in the several identities that we have just considered, with the exception of that given by Goulden and Jackson. At the end of Section 4.3 we present a bijective proof of the latter identity. We will see that this proof is essentially the same as that of the character equation $\chi^{\lambda}(\alpha) = \sum_{\mu \succ n} \eta^{\mu}(\alpha) K^{-1}_{\mu,\lambda}$ where $h_{\mu} = \mathrm{ch}(\eta^{\mu})$, and that the matrix $\mathbf{A}$ hardly figures.

What makes the combinatorics of the symmetric functions of $\xi_1, \ldots, \xi_N$ interesting as opposed to that of the immanants of $\mathbf{A}$ is the difference between Littlewood's Theorems I and II. That is, the symmetric functions of $\xi_1, \ldots, \xi_N$ capture the combinatorics of not only the matrix $\mathbf{A}$, but every word matrix $\mathbf{A}_w$ as well. This is made relevant by the four different interpretations for $h_n(\xi_1, \ldots, \xi_N)$ that we presented in the previous chapter. If we restrict our attention to the terms in which no letters are repeated, then all four interpretations are trivialized (much as if we had set $a_{ij} = 0$, $i \neq j$), and we are left with $\operatorname{per}\mathbf{A}$. For example, recall that $h_n(\xi_1, \ldots, \xi_N)$ is a generating function for multisets of Lyndon words. If we only consider terms in which no letters are repeated, then the Lyndon words are cycles and the multisets are sets of disjoint cycles. The combinatorics has disappeared, even though from an algebraic point of view $\operatorname{per}\mathbf{A}$ is always more general than $\operatorname{per}\mathbf{A}_w$.

Littlewood's formula allows us to evaluate a symmetric function at its eigenvalues, provided that we know the values taken by the corresponding character. By this approach we are able to find original and satisfying interpretations for the forgotten symmetric functions $f_\lambda(\xi_1,\ldots,\xi_N)$ and the monomial symmetric functions $m_\lambda(\xi_1,\ldots,\xi_N)$. However, we choose to work with the equation $b = \frac{1}{n!}\sum_{\sigma\in S_n} B(\sigma)p_\sigma$ instead of with Littlewood's formula. The two are combinatorially equivalent, but we find the former easier to work with.

SECTION 3.2 CHARACTER TABLES

We have seen that the power symmetric basis is a sensible basis in which to expand a symmetric function $b = \mathrm{ch}(B)$. It may be argued that the values $B(\sigma)$ in the expansion $b = \frac{1}{n!}\sum_{\sigma \in S_n} B(\sigma) p_\sigma$ capture the combinatorics inherent in some action of which the character B is the trace. However, B need not be a character in the true sense of the word. Indeed, B is a character if and only if b can be expressed as a nonnegative linear combination of Schur functions. Otherwise B is said to be a virtual character and does not correspond to any representation, as we saw in Section 1.1. This is the situation when b is f_λ or m_λ. As we shall see, it is still useful to express b as $\frac{1}{n!}\sum_{\sigma \in S_n} B(\sigma) p_\sigma$, and in the two sections to follow we do this for $f_\lambda(\xi_1,\dots,\xi_N)$ and $m_\lambda(\xi_1,\dots,\xi_N)$. However, in order to calculate the values $B(\sigma)$ we exploit the combinatorics of the ring of symmetric functions. In this section we present an interpretation of $(\mathrm{ch}^{-1} f_\lambda)(\mu)$ and $(\mathrm{ch}^{-1} m_\lambda)(\mu)$ that is due to Eğecioğlu and Remmel [ER2] and then modify it for our purposes.

We first prove a helpful lemma which expresses character tables as transition matrices. Let $\{B^\lambda\}_{\lambda \vdash n}$ be a complete set of linearly independent class functions, let $\{b_\lambda\}_{\lambda \vdash n}$ be the corresponding basis $b_\lambda = \mathrm{ch}(B^\lambda)$ and let $\{\tilde{b}_\lambda\}_{\lambda \vdash n}$ be the dual basis of $\{b_\lambda\}_{\lambda \vdash n}$ with respect to the inner product of Λ_n, that is $\langle b_\lambda, \tilde{b}_\mu \rangle = \delta_{\lambda\mu}$.

LEMMA 3.2.1 $\quad B^\mu(\lambda) = M(p,\tilde{b})_{\lambda,\mu}$ *where* $p_\lambda = \sum_\mu M(p,\tilde{b})_{\lambda,\mu} \tilde{b}_\mu$.

In Section 1.1 we defined the inner product of Λ_n by setting $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. We remarked that $\tilde{p}_\lambda = \frac{C_\lambda}{n!} p_\lambda$ where C_λ is the size of the conjugacy class indexed by the partition λ. It follows that

$$\begin{aligned} M(p,\tilde{b})_{\lambda,\mu} &= \left\langle \sum_{\mu \succ n} M(p,\tilde{b})_{\lambda,\mu} \tilde{b}_\mu,\ b_\mu \right\rangle \\ &= \langle p_\lambda, b_\mu \rangle \\ &= \left\langle p_\lambda, \frac{1}{n!} \sum_{\sigma \in S_n} B^\mu(\sigma) p_\sigma \right\rangle \\ &= \left\langle p_\lambda, \frac{1}{n!} \sum_{\lambda \succ n} B^\mu(\lambda) C_\lambda p_\lambda \right\rangle \\ &= \left\langle p_\lambda, \sum_{\lambda \succ n} B^\mu(\lambda) \tilde{p}_\lambda \right\rangle \quad = B^\mu(\lambda) \qquad \text{QED.} \end{aligned}$$

Recall that $\{f_\lambda\}_{\lambda \succ n}$ and $\{e_\lambda\}_{\lambda \succ n}$ are dual bases, as are $\{m_\lambda\}_{\lambda \succ n}$ and $\{h_\lambda\}_{\lambda \succ n}$. Therefore $(\mathrm{ch}^{-1} f_\lambda)(\mu) = M(p,e)_{\lambda,\mu}$ and $(\mathrm{ch}^{-1} m_\lambda)(\mu) = M(p,h)_{\lambda,\mu}$.

We start with an interpretation of $(\mathrm{ch}^{-1} f_\lambda)(\mu)$ that is due to Eğecioğlu in Remmel [ER2]. They proved that their interpretation is correct by showing that it satisfies the recursion relation $ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}$. Their construction mirrors the determinantal formula

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & \ddots & \vdots \\ 3e_3 & e_2 & e_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{vmatrix} = \sum_{\nu \succ n} M(p,e)_{n,\nu} e_\nu$$

This formula is gotten by writing out the relations $ie_i = \sum_{r=1}^{i} (-1)^{r-1} p_r e_{i-r}$ for all $1 \le i \le n$, and then solving them for p_n. It may be compared with the determinantal formula that we referred to in Section 1.1 in constructing a row of a brick tabloid. The factor ie_i in the first column may be attributed to the leftmost brick in that row. We may think of the leftmost brick (or as Eğecioğlu and Remmel do, the rightmost) as having a distinguished square. As in the case of brick tabloids, consider a sequence of rows of lengths

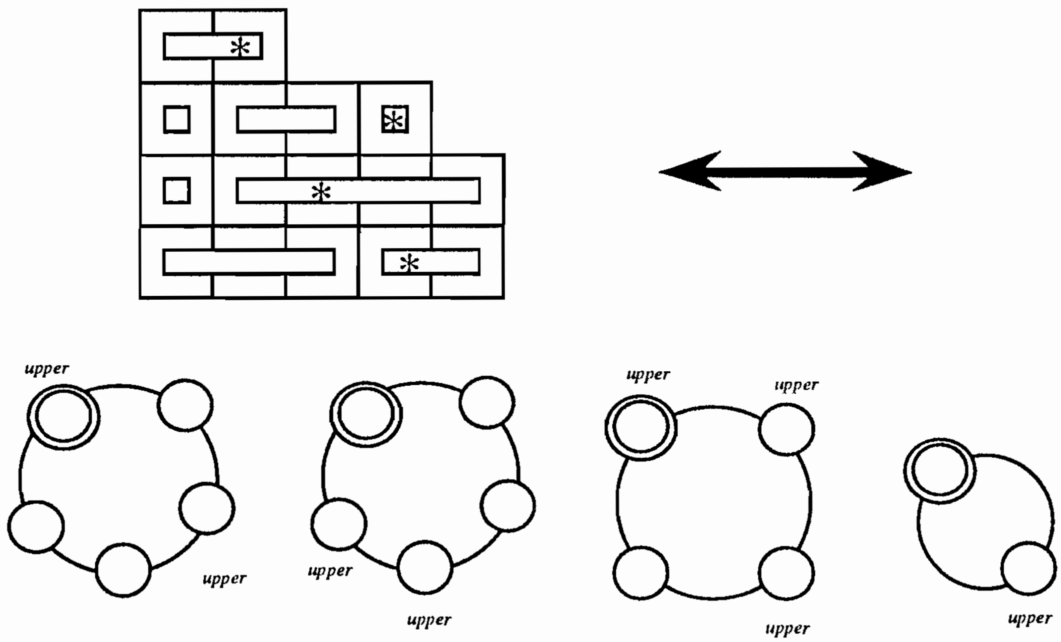
$\mu_1,\ldots,\mu_{\ell(\mu)}$ filled with bricks of lengths $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$, but let the rightmost brick in each row have a distinguished square. The resulting objects are weighted brick tabloids of shape μ and type λ. Let $C(B_{\lambda,\mu})$ be the number of such tabloids. In the expansion of the determinant each brick corresponds to a cycle and has the sign of a cycle. Therefore the total sign is $\operatorname{sgn}(\lambda)$. We conclude that $\operatorname{ch}^{-1}(f_\lambda)(\sigma)=\operatorname{sgn}(\lambda)C(B_{\lambda,\sigma})$ (here σ signifies both a permutation and its cycle structure). [ER2]

Likewise, an interpretation for the character table of $\operatorname{ch}^{-1} m_\lambda$ can be gotten by examining the determinant

$$p_n = (-1)^{n-1}\begin{vmatrix} h_1 & 1 & 0 & \ldots & 0 \\ 2h_2 & h_1 & 1 & \ddots & \vdots \\ 3h_3 & h_2 & h_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ nh_n & h_{n-1} & h_{n-2} & \ldots & h_1 \end{vmatrix} = \sum_{\nu\succ n} M(p,h)_{n,\nu} h_\nu$$

Suppose $n=\mu_i$. The sign $(-1)^{\mu_i-1}$ is that which a row of length μ_i would have if it were a cycle. The total sign is $\operatorname{sgn}(\lambda)\operatorname{sgn}(\mu)$ and equals $(-1)^r$ where r is the number of gaps between bricks. The value of the character $\operatorname{ch}^{-1} m_\lambda$ at σ is $\operatorname{sgn}(\lambda)\operatorname{sgn}(\sigma)C(B_{\lambda,\sigma})$. [ER2]

We now recast the definition of weighted brick tabloid to suit our purposes in the sections to follow. Let there be a sequence of walkalongs $C_1,\ldots,C_{\ell(\mu)}$ on a single letter O, a letter which we never draw because it occurs at every place. For all i, $1\le i\le \ell(\mu)$, let C_i have an origin and length μ_i. In the walkalongs, let every place have either upper or lower case. Consider all distinct ways of ascribing case to the places such that the circular distances separating the upper case places, as defined in Section 1.3, are given by $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$. We call the resulting objects circular brick tabloids (this resembles a construction used by Stembridge) [St2]. If we think of the places with upper case as starts of bricks, then we see that there is a correspondence between circular brick tabloids and weighted brick tabloids. Each circle becomes a row, and the origin of each walkalong becomes the distinguished square in the row. The order of the squares of the row from left to right matches the usual clockwise order of the places of the walkalong, as in the example below, where $\mu = 5542$ and $\lambda = 43222111$. We see that $C(B_{\lambda,\mu})$ gives the number of circular brick tabloids of shape μ and type λ.



We do not actually draw bricks in the circular brick tabloids because we want to focus our attention on the places at the start of each brick, that is, the places with upper case. It is helpful, however, to keep in mind that weighted brick tabloids can be thought of as the circular analogue of brick tabloids. It is typical to identify the rows of a weighted brick tabloid with cycles of permutations, in which case the distinguished square is identified with the smallest letter of the corresponding cycle. Indeed, this is the case in the sections that follow.

SECTION 3.3 FORGOTTEN SYMMETRIC FUNCTIONS

This section presents an interpretation of the forgotten symmetric function $f_\lambda(\xi_1,\xi_2,\dots\xi_N)$. The equation $f_\lambda=\frac{1}{n!}\sum_{\sigma\in S_n}(\mathrm{ch}^{-1}f_\lambda)(\sigma)p_\sigma$ and the combinatorial result $(\mathrm{ch}^{-1}f_\lambda)(\sigma)=\mathrm{sgn}(\lambda)\mathrm{C}(B_{\lambda,\sigma})$ together yield a formula $f_\lambda=\mathrm{sgn}(\lambda)\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{C}(B_{\lambda,\sigma})p_\sigma$ which allows us to express $f_\lambda(\xi_1,\xi_2,\dots\xi_N)$ in terms of multisets of Lyndon words. This result is one of the two main results of this thesis. We then arrive at a second interpretation of $f_\lambda(\xi_1,\xi_2,\dots\xi_N)$ by relating multisets of Lyndon words with sequences of closed walks.

Recall that at the end of Section 1.3 we introduced the concept of case. In order to express $f_\lambda(\xi_1,\xi_2,\dots\xi_N)$ in terms of Lyndon words we make use of both upper case and lower case. If a letter i appears at a place with upper case, then we may think of it as an upper case letter, and denote it $\bar{i}$, whereas if a letter i appears at a place with lower case, then we may think of it as a lower case letter, and denote it i, as in Figure 3. With this in mind we denote by $\pm\mathbf{N}$ the alphabet made up of upper case and lower case letters $\bar{1}<\cdots<\bar{N}<1<\cdots<N$. We also use the concept of circular distance that we introduced in Section 1.3.

THEOREM 3.3.1 *Define* $a_{ij}=a_{\bar{i}j}=a_{i\bar{j}}=a_{\bar{i}\bar{j}}$ *for all letters* $i,\ j,\ \bar{i},\ \bar{j}\in\pm\mathbf{N}$. *Let* M *be the multiset of Lyndon words on* $\bar{1}<\cdots<\bar{N}<1<\cdots<N$ *with* n *places such that every Lyndon word has at least one upper case letter, and the upper case letters are separated by the circular distances* $\lambda_1,\lambda_2,\dots,\lambda_{\ell(\lambda)}$. *Then*

$$f_\lambda(\xi_1,\dots,\xi_N)=\mathrm{sgn}(\lambda)\sum_{m\in M}\mathrm{W}(m).$$

This interpretation arises from the equation $f_\lambda(\xi_1,\ldots,\xi_N) = \operatorname{sgn}(\lambda)\frac{1}{n!}\sum_{\sigma\in S_n} C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$. The function $\sum_{\sigma\in S_n} C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ generates triplets $(\sigma, B_{\lambda,\sigma}, P_\sigma)$ where σ is a permutation, $B_{\lambda,\sigma}$ is a circular brick tabloid of shape σ and type λ, and P_σ is a term from p_σ. Each member of such a triplet may be expressed as a sequence. Identify σ with the sequence of its cycles of lengths $\sigma_1,\ldots,\sigma_{\ell(\sigma)}$. Let the cycles of equal length be ordered so that the later a cycle is listed, the greater its smallest label. Identify $B_{\lambda,\sigma}$ with the sequence of its walkalongs of lengths $\sigma_1,\ldots,\sigma_{\ell(\sigma)}$. Identify P_σ with the sequence of closed walks of lengths $\sigma_1,\ldots,\sigma_{\ell(\sigma)}$. Then for all i, $1 \le i \le \ell(\sigma)$, the ith cycle, the ith walkalong, and the ith closed walk all have the same length σ_i.

Furthermore, the ith cycle has a smallest label, the ith walkalong has an origin , and the ith closed walk has an origin. For all i, $1 \le i \le \ell(\sigma)$, match the smallest label with the origin of the walkalong and with the origin of the walk. This determines an alignment of the ith cycle, the ith walkalong, and the ith closed walk. It associates each label of the cycle with a place in the walkalong and a place in the closed walk. Once this is done it is no longer necessary to distinguish the origin of the walkalong or of the walk because this information can be recovered from the smallest letter. Consider the example in Figure 3, where σ has cycle type 6433 and $\lambda = 4222111111$. In what follows the closed walk is to be thought of as a circular walk.

Introduce the alphabet $\bar{1} < \cdots < \bar{N} < 1 < \cdots < N$ and let $a_{ij} = a_{\bar{i}j} = a_{i\bar{j}} = a_{\bar{i}\bar{j}}$ for all $1 \le i,j \le n$. Each place has upper or lower case as dictated by the circular brick tabloid. If a vertex k of the circular walk is at a place with upper case, then replace this k with $\bar{k}$. This does not affect the weight of the circular walk. The circular brick tabloid may then be disregarded and we may speak of places of upper and lower case in the circular walk. The circular distances separating the upper case places are given by the lengths $\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)}$. There is a unique way of expressing the resulting circular walk as a power of a Lyndon word on the new alphabet.

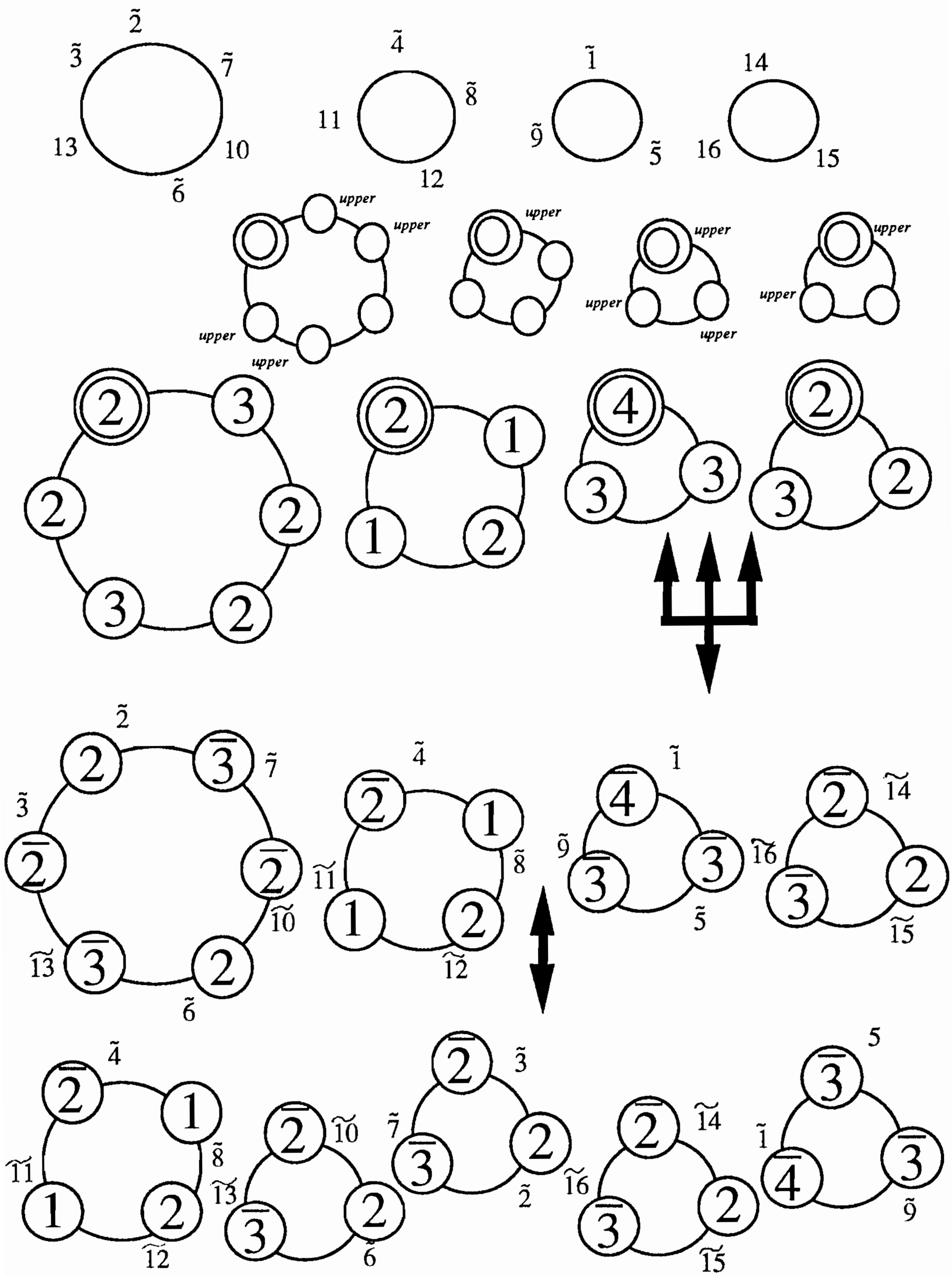


Figure 3: A bijection expressing $f_\lambda(\xi_1,\ldots,\xi_N)$ in terms of multisets of Lyndon words.

Order the circular walks so that their respective Lyndon words increase from left to right with respect to the lexicographic order defined in Section 1.3. Order circular walks that are powers of the same Lyndon word so that the smallest label of the walk increases from left to right. Given a circular walk, break it apart into the Lyndon words of which it is a power, as in Lemma 1.3.4. Note that the weight of the circular walk equals the weight of the product of the resulting Lyndon words. Also, the circular distances that separate the upper case letters of the circular walk remain unaffected. Order the resulting Lyndon words from right to left so that the rightmost is the one with the smallest label, and the others are listed in the order that they appear in the circular walk. Throughout all of this each place of the walk continues to be identified with a label of the permutation σ.

This results in a sequence of weakly increasing Lyndon words for which the places are labeled with labels $\sigma(1),\ldots,\sigma(n)$. Each Lyndon word is built from the alphabet $\overline{1}<\cdots<\overline{N}<1<\cdots<N$ and must contain at least one upper case letter. The circular distances between the upper case letters are given by $\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)}$. The weight of the sequence is the product of the weights of the Lyndon words with the additional requirement that $a_{ij}=a_{\bar{i}j}=a_{i\bar{j}}=a_{\bar{i}\bar{j}}$ for all i, j, $1\le i,j\le N$.

We claim that the above map defines a weight preserving correspondence between all such sequences and the triplets that are generated by $\sum_{\sigma\in S_n}\mathrm{C}(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$. The algorithm is reversible and therefore the inverse of the map exists and the map is one-to-one. For suppose that a sequence of Lyndon words is given as above. Within each Lyndon word ℓ_j find the place c_j that has the smallest label k_j. Mark those places with labels k_i for which $k_i<k_j$ whenever $i<j$ and $\ell_i=\ell_j$. Suppose that the marked places are $c_{i_1},c_{i_2},\ldots,c_{i_r}$. Starting with any such place c_{i_s}, walk along the places of the Lyndon word until the word is traversed. Then walk through the Lyndon words $\ell_{i_s-1},\ell_{i_s-2},\ldots,\ell_{i_{s-1}+1}$, successively, always starting and ending at the place

that corresponds to c_{i_r}. This converts these Lyndon words into a circular walk adorned with labels. The smallest of these labels is k_{i_r} at place c_{i_r}. Then using the places $c_{i_1}, c_{i_2}, \ldots, c_{i_r}$ as reference points it is possible to read off the sequence of cycles σ, the sequence of walkalongs $B_{\lambda,\sigma}$, and the sequence of walks P_σ. This is possible once the case of the places in every circle of $B_{\lambda,\sigma}$ is determined from the case of the corresponding places in the circular walk.

In summary, two weight preserving maps have been defined. The first maps the triplets to labeled multisets of Lyndon words with certain additional properties. The second maps these multisets back to the triplets. Each map reverses the other and therefore they define a weight preserving correspondence.

Finally, we remark that given a multiset of Lyndon words, its places may be labeled in exactly $n!$ ways, and the labeling has no bearing on the weight of the multiset. Removing the labels adds a factor of $n!$ and completes the theorem. QED

Having completed our proof of the theorem, we consider what it has to say about the limiting cases $\lambda = 1^n$ and $\lambda = n$. If $\lambda = 1^n$, then all of the letters are upper case and the function generates all multisets of Lyndon words with n places, and we have $h_n(\xi_1, \ldots, \xi_N)$, as in Theorem 2.1.4. If $\lambda = n$, then there is one upper case letter and a single Lyndon word. The unique letter is taken to be the first and last point of a closed walk. A walk with such a unique upper case letter can never have rotational symmetry. Therefore the function generates all closed walks of length n and is $\mathrm{sgn}(n) p_n(\xi_1, \ldots, \xi_N)$.

If we set $a_{ij} = 0$ for all $i \neq j$, then our expression for $f_\lambda(\xi_1, \xi_2, \ldots \xi_N)$ should yield the description of the forgotten symmetric function in terms of brick tabloids that we saw in Section 1.1. We postpone a demonstration of this, however, until we arrive at a second interpretation of $f_\lambda(\xi_1, \xi_2, \ldots \xi_N)$, from which it will be more evident. First we consider some of the consequences of the relation between $f_\lambda(\xi_1, \xi_2, \ldots \xi_N)$ and Lyndon words.

For all λ, the terms of the function $\hat{f}_\lambda(\xi_1,\xi_2,\ldots\xi_N)=\operatorname{sgn}(\lambda)f_\lambda(\xi_1,\xi_2,\ldots\xi_N)$ are all positive. We relate these functions to $h_n(\xi_1,\ldots,\xi_N)$, whose terms are also all positive. Consider multisets of Lyndon words on the alphabet $\pm\mathbf{N}$. Any such multiset can be divided into a pair of multisets, the first multiset consisting of Lyndon words that each have at least one upper case letter, and the second multiset consisting of Lyndon words that have no upper case letter. Therefore the generating function of the multisets is

$$\left(\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\operatorname{sgn}(\lambda)\cdot f_\lambda(\xi_1,\ldots,\xi_N)\right)\left(\sum_{n=0}^{\infty}h_n(\xi_1,\ldots,\xi_N)\right)$$
$$=\left(\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\operatorname{sgn}(\lambda)\cdot f_\lambda(\xi_1,\ldots,\xi_N)\right)\frac{1}{\det(\mathbf{I}-\mathbf{A})}$$

On the other hand the multisets are generated by the homogeneous symmetric function evaluated at the eigenvalues of the $2N\times 2N$ matrix $(a_{ij})_{i,j\in\pm\mathbf{N}}$. The fact that $a_{ij}=a_{\bar{i}j}=a_{i\bar{j}}=a_{\bar{i}\bar{j}}$ for all letters $i,\ j,\ \bar{i},\ \bar{j}\in\pm\mathbf{N}$ means that $(a_{ij})_{i,j\in\pm\mathbf{N}}$ equals the tensor product $\mathbf{A}\otimes\mathbf{M}$ where

$$\mathbf{A}\otimes\mathbf{M}=\begin{pmatrix}\mathbf{A} & \mathbf{A}\\ \mathbf{A} & \mathbf{A}\end{pmatrix}\quad\text{and}\quad\mathbf{M}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

The eigenvalues of $\mathbf{M}$ are 2 and 0 and therefore the eigenvalues of $\mathbf{A}\otimes\mathbf{M}$ are $2\xi_1,\ldots,2\xi_N,0\xi_1,\ldots,0\xi_N$. We note that $h_n(2\xi_1,\ldots,2\xi_N,0,\ldots,0)=h_n(2\xi_1,\ldots,2\xi_N)$. Therefore the generating function is $\sum_{n=0}^{\infty}h_n(2\xi_1,\ldots,2\xi_N)=\prod_{i=1}^{N}\frac{1}{1-2\xi_i}=\frac{1}{\det(\mathbf{I}-2\mathbf{A})}$.

Combining the two interpretations gives the formula

$$\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\hat{f}_\lambda(\xi_1,\ldots,\xi_N)=\frac{\det(\mathbf{I}-\mathbf{A})}{\det(\mathbf{I}-2\mathbf{A})}$$

Also, applying the involution ω gives the formula

$$\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\operatorname{sgn}(\lambda)\cdot m_{\lambda}(\xi_1,\ldots,\xi_N)=\frac{\det(\mathbf{I}-2\mathbf{A})}{\det(\mathbf{I}-\mathbf{A})}$$

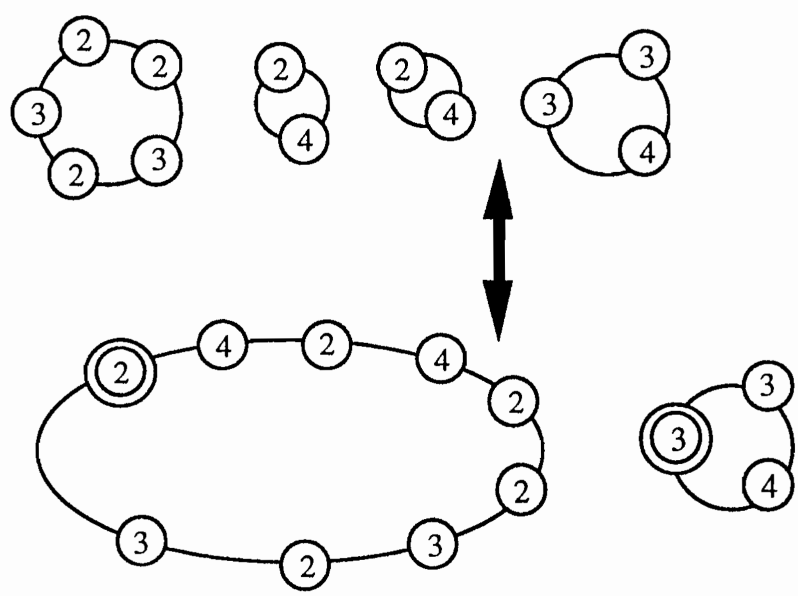
Finally, setting $\xi_1=x_1,\ldots,\xi_N=x_N$ gives

$$\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\hat{f}_{\lambda}(x_1,\ldots,x_N)=\prod_{i=1}^{N}\frac{1-x_i}{1-2x_i}$$
$$\sum_{n=0}^{\infty}\sum_{\lambda\succ n}\operatorname{sgn}(\lambda)m_{\lambda}(x_1,\ldots,x_N)=\prod_{i=1}^{N}\frac{1-2x_i}{1-x_i}$$

This expression of words on the alphabet $\pm\mathbf{N}$ is a factorization, specifically, a bisection of each word into two words, where the first is a product of Lyndon words, each having at least one upper case letter, and the second is a product of Lyndon words, each having no upper case letter. By a theorem from chapter 5, "Factorization of Free Monoids", of [Lo], it is known that to any bisection of a free monoid is associated a decomposition of the corresponding free Lie algebra into a direct sum of two submodules.

A second interpretation for $f_{\lambda}(\xi_1,\xi_2,\ldots\xi_N)$ is gotten in terms of sequences of closed walks. It follows from an observation about $h_n(\xi_1,\ldots,\xi_N)$, whose terms may be described by multisets of Lyndon words, or alternatively, by certain sequences of walks. We present a bijection that relates the terms resulting from each of these two descriptions. This then leads us to a second interpretation of the forgotten symmetric functions $f_{\lambda}(\xi_1,\xi_2,\ldots\xi_N)$.

Given a multiset of Lyndon words, group them by the first letter in the word, which is necessarily the smallest letter in the word. In each group, list the Lyndon words in reverse order, so that the one with the smallest prefix is listed last. But write the letters of each Lyndon word in the usual order, so that the word starts with its smallest prefix. Stringing the Lyndon words together in each group gives a closed walk for each group because each Lyndon word in the group starts with the same letter. This closed walk does not include any letters less than the letter at the origin. Conversely, suppose that we are given a sequence of n closed walks, possibly empty, such that no letter in the walk is smaller than the letter at the origin. Consider the nonempty walks. Write out the letters of each walk in the usual order. Given the walk that starts with the letter k, factor the resulting word u into Lyndon words. That is, let $u = u_1 \cdots u_\ell$ where u_ℓ is the right factor of u with smallest prefix, and in general, u_j is the right factor of $u_1 \cdots u_j$ of smallest prefix. As we noted in Section 1.3, this factorization is unique, and we see that it recovers the Lyndon words that start with the letter k. The following example illustrates our bijection.



Note that this bijection is weight preserving because the words that are strung together all have the same letter at the origin, as do the words that are removed from the closed walk. We now apply this action to the multisets of Lyndon words that describe the terms of the forgotten symmetric functions. This gives rise to a second combinatorial description of these functions. Again we make use of the concept of circular distance from Section 1.3. In the theorem below some of the closed walks may be empty, in which case we do not associate any circular distance with the letter at the origin.

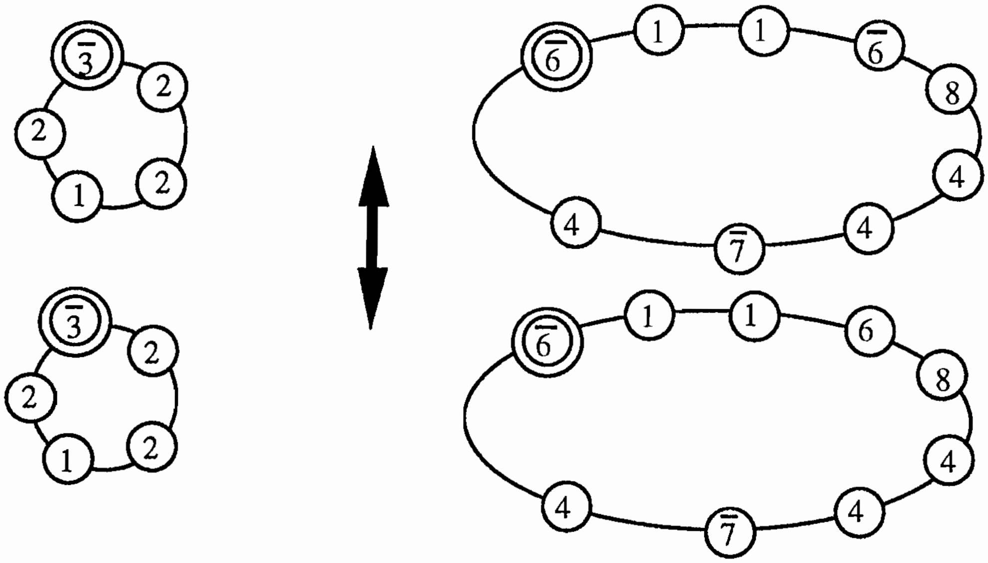
THEOREM 3.3.2 *Define* $a_{ij} = a_{\bar{i}j} = a_{i\bar{j}} = a_{\bar{i}\bar{j}}$ *for all letters* $i, j, \bar{i}, \bar{j} \in \pm\mathbf{N}$. *Let* S *be the set of sequences* $(w_1, w_2, \ldots, w_N)$ *of closed walks on* $\bar{1} < \cdots < \bar{N} < 1 < \cdots < N$ *with a total of* n *places for which* $\bar{i}$ *is the letter at the origin of the possibly empty closed walk* w_i, *the letters* $\bar{1}, \ldots, \overline{i-1}$ *do not occur in* w_i, *and the upper case letters of the sequence are separated by the circular distances* $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$. *Then*

$$f_\lambda(\xi_1, \ldots, \xi_N) = \operatorname{sgn}(\lambda) \sum_{s \in S} \mathrm{W}(s).$$

This result is a consequence of reworking the bijection described above. Suppose that we start with a multiset of Lyndon words on the alphabet $\pm\mathbf{N}$ for which each Lyndon word has at least one upper case letter, so that each Lyndon word starts with an upper case letter. In stringing together Lyndon words that start with the same letter, the bijection does not affect the circular distances separating the upper case letters. Therefore we get walks as before, but those for which there is an upper case letter at the origin of each walk, and for which the circular distances are given by the partition λ. QED

As usual, we consider what the theorem has to say about the cases $\lambda = 1^n$ and $\lambda = n$. If $\lambda = 1^n$, then every letter is upper case and the function generates the description of $h_n(\xi_1, \ldots, \xi_N)$ in terms of sequences of closed walks that was given in Theorem 2.2.3. If $\lambda = n$, then there is only one closed walk, and the unique upper case letter occurs at the origin. This generates all closed walks of length n and gives $\operatorname{sgn}(n) p_n(\xi_1, \ldots, \xi_N)$. Finally, we may ask, what does $f_\lambda(\xi_1, \xi_2, \ldots \xi_N)$ generate if $a_{ij} = 0$ for all $i \neq j$? List the walks by length, and list walks of equal length by the letter at the origin so that it increases. If the length of the jth closed walk is μ_j and the letter at the origin is i_j, then the walk has weight $\left(a_{i_j i_j}\right)^{\mu_j}$. The places in each walk may have upper or lower case. If the places with upper case mark the first squares of a brick, then we see that the function generates brick tabloids of type λ for which a different letter i_j is assigned to each row, and in rows with the same length the letters appear in order, and each row of length μ_j has weight $\left(a_{i_j i_j}\right)^{\mu_j}$. This is the interpretation of f_λ in terms of brick tabloids that we saw in Section 1.1.

Our second expression for $f_\lambda(\xi_1,\xi_2,\ldots\xi_N)$ may be used to provide a combinatorial interpretation of the equation $e_n = \sum_\mu f_\mu$. Given a term in $f_\mu(\xi_1,\xi_2,\ldots\xi_N)$, find the least i for which there exists a walk with letter $\bar{i}$ at the origin Q and there exists an r such that there is a letter j or $\bar{j}$ at $Q+r \neq Q$ for which $j \geq i$. If i exists, then find the least such r and change the case of the letter at $Q+r$. This action changes sign and is reversible because it does not affect the origin of the walk. Consider the example below, pairing terms of opposite sign from $f_{5432}(\xi_1,\ldots,\xi_9)$ and $f_{752}(\xi_1,\ldots,\xi_9)$. Here $\bar{i} = \bar{6}$ and $r = 3$.

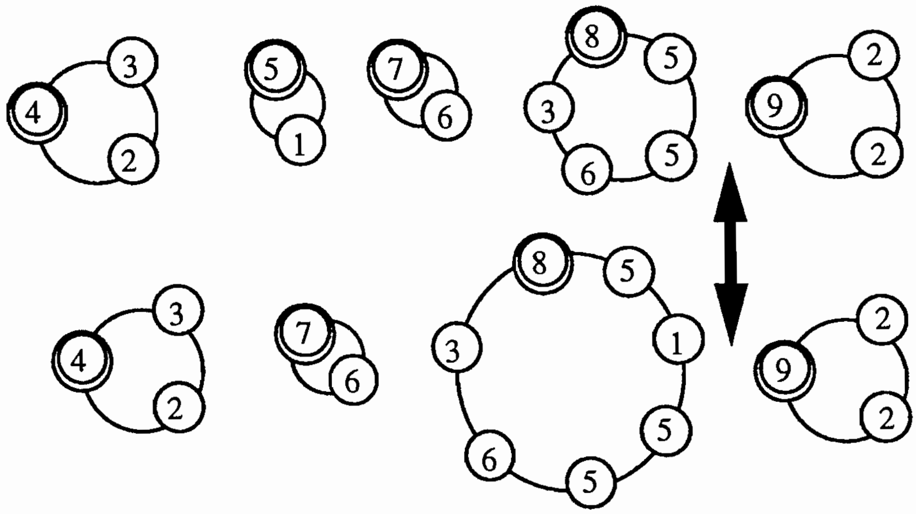


The fixed points that we are left with are those for which in each walk the unique place with upper case is the origin, and if the case of the origin is changed to lower case, then the letter at the origin is strictly greater than the letter at any other place in the walk. This allows us to ignore case from henceforth, and to describe $e_n(\xi_1,\xi_2,\ldots\xi_N)$ as the generating function of sequences $(w_1,w_2,\ldots,w_N)$ of closed walks with a total of n places, with sign $\mathrm{sgn}(\lambda)$, where λ is given by the lengths of the walks, and such that for all i, $1 \le i \le N$, the letter i appears at the origin of the walk w_i but at every other place of the walk w_i the letters are strictly less than i.

We now perform a second involution. Find the least i such that w_i is a nonempty walk which is not a cycle or w_i is not disjoint from some walk w_k, $k<i$. Given w_i, note that the walks $w_1,\ldots,w_{i-1}$ are disjoint cycles. The letter at the origin P of w_i is i. Find the least r such that there is a letter j at $P+r$ and

either j is at a place $P+s$, $0<s<r$,
or j occurs at a place R in a cycle w_k, $k<i$.

If j is chosen by virtue of the first event, then redirect $P+r-1$ to $P+s$ and $P+s-1$ to $P+r$, creating a cycle. Define the origin of this cycle to be the place with largest letter $k<i$, and denote it as w_k, noting that previously the walk w_k must have been empty, as otherwise w_k would have been chosen by virtue of the second event. If j is chosen by virtue of the second event, then note that w_k is unique because the cycles $w_1,\ldots,w_{i-1}$ are disjoint. Incorporate the cycle w_k into the closed walk by redirecting $R-1$ to $P+r$ and $P+r-1$ to R. The two events are reciprocal, as is demonstrated in the example below, where $i=8$ and $k=5$.



The action of adding or removing a cycle changes sign. It is reversible because if a cycle is removed, then it is denoted w_k with $k < i$ and is disjoint from $w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_{i-1}$. Therefore if the action is performed again, then we again choose i. What fixed points remain ? Those for which every closed walk is either empty or a disjoint cycle. Each cycle has the usual sign, and so we are left with signed boxes of cycles. This completes our interpretation of $e_n = \sum_\mu f_\mu$.

SECTION 3.4 MONOMIAL SYMMETRIC FUNCTIONS

In this section we find a combinatorial interpretation for the monomial symmetric function $m_\lambda(\xi_1,\ldots,\xi_N)$. The method used is similar to that used to prove Theorem 3.3.1, but the result attained appears very different. We start with the equation $m_\lambda = \frac{1}{n!}\sum_{\sigma\in S_n}(\mathrm{ch}^{-1} m_\lambda)(\sigma)p_\sigma$ and substitute $(\mathrm{ch}^{-1} m_\lambda)(\sigma) = \mathrm{sgn}(\lambda)\mathrm{sgn}(\sigma)\mathrm{C}(B_{\lambda,\sigma})$ to get $m_\lambda = \mathrm{sgn}(\lambda)\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\mathrm{C}(B_{\lambda,\sigma})p_\sigma$. We work with triplets $(\sigma, B_{\lambda,\sigma}, P_\sigma)$ as before, but this time we perform an involution that takes into account the sign of σ. In the objects that survive this involution, each upper case letters appears at most once. These letters are linked by walks the lengths of which are given by $\lambda_1,\lambda_2,\ldots,\lambda_N$. We show as in the last section that the factor $1/n!$ is eliminated upon removing the labels given by σ. The resulting objects are readily understood as terms selected from $\det(\mathbf{I}/\mathbf{I}-\mathbf{A})$, the determinant of the walk matrix.

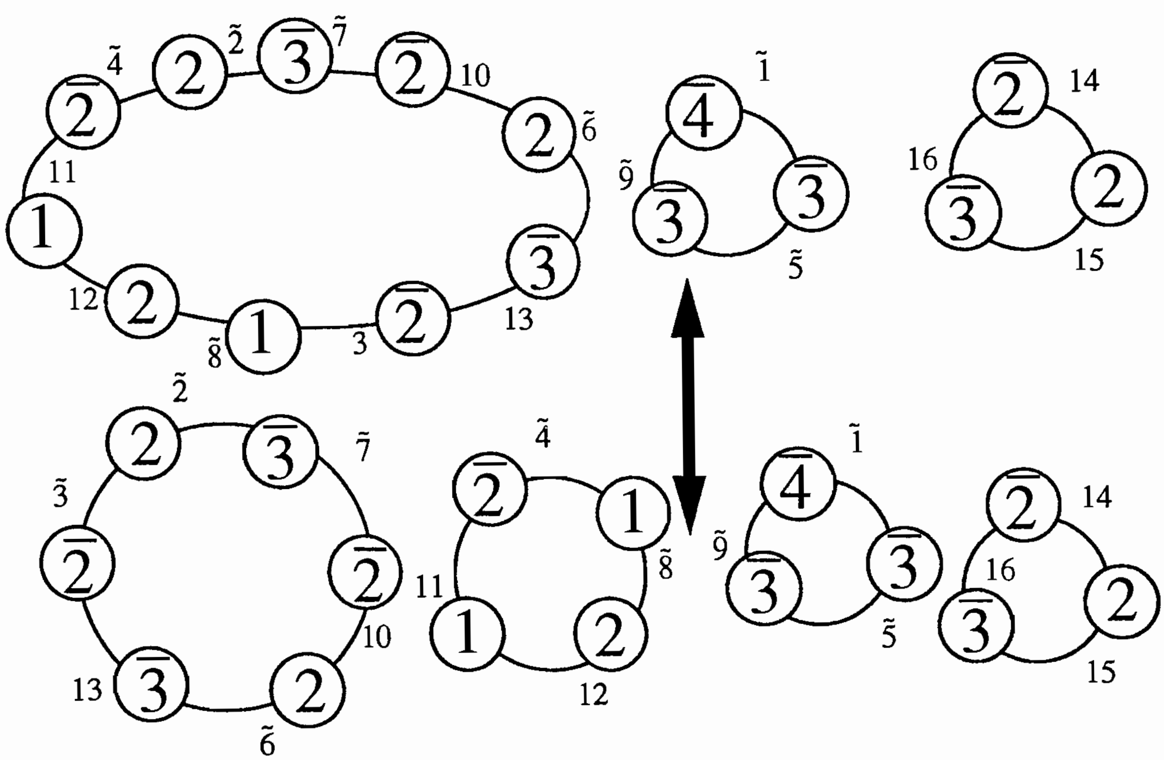
If $\lambda \succ n$, then we declare the convention that $\lambda_i = 0$ for $i > \ell(\lambda)$. Define $\mathbf{R}[\lambda_1\lambda_2\cdots\lambda_N]$ to be the set of distinct rearrangements of the word $\lambda_1\lambda_2\cdots\lambda_N$. Recall that $(\mathbf{A}^{\lambda_k})_{ij}$ is the generating function for walks of length λ_k from i to j.

THEOREM 3.4.1 *Define $a_{ij} = a_{i\bar{j}} = a_{\bar{i}j} = a_{\bar{i}\bar{j}}$ for all letters $i,\ j,\ \bar{i},\ \bar{j} \in \pm\mathbf{N}$. If $N < \ell(\lambda)$, then $m_\lambda(\xi_1,\ldots,\xi_N) = 0$. If $N \geq \ell(\lambda)$, then*

$$m_\lambda(\xi_1,\ldots,\xi_N) = \sum_{\substack{\tau\in S_N \\ x_1x_2\cdots x_N \in \mathbf{R}[\lambda_1\lambda_2\cdots\lambda_N]}} \mathrm{sgn}(\tau)(\mathbf{A}^{x_1})_{1\tau(1)}(\mathbf{A}^{x_2})_{2\tau(2)}\cdots(\mathbf{A}^{x_N})_{N\tau(N)}.$$

We start with $m_\lambda(\xi_1,\ldots,\xi_N) = \mathrm{sgn}(\lambda)\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\mathrm{C}(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ and proceed as in the proof of Theorem 3.3.1. The function $\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ generates triplets $(\sigma, B_{\lambda,\sigma}, P_\sigma)$, as in that proof, but with sign $\mathrm{sgn}(\sigma)$. In the beginning of that proof each triplet was identified with a pairing of the cycles of σ with circular walks. The circular walks had vertices taken from the alphabet $\pm\mathbf{N}$, and the upper case vertices were separated by the circular lengths $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$. We now remark that this identification of triplets with pairings preserves not only weight, but also sign. Therefore $\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ may be understood as the generating function of the pairings, with sign.

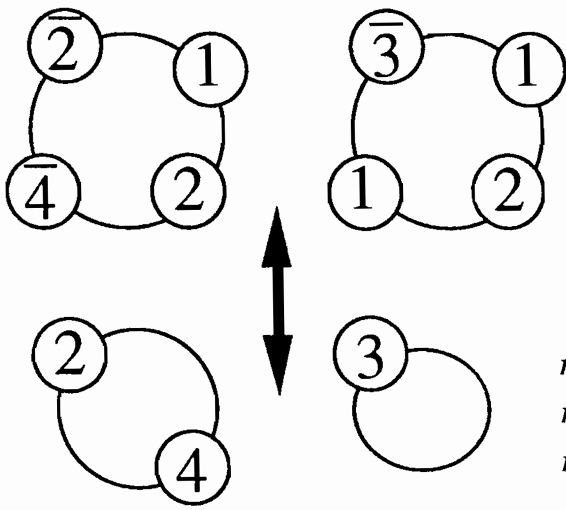
At this point we define an involution on the pairings which exploits their sign. Find the smallest label of σ, and then the next smallest label of σ, for which the letters at the corresponding places P and Q are equal and have upper case. Redirect place P to $Q+1$ and place Q to $P+1$. This action creates two walkalongs from one or one from two, as in Lemma 1.3.1. It is weight preserving, but sign reversing.



As can be seen in the example above, the action reverses itself upon a second application because it redirects places, but never affects their letters or labels. Therefore it defines an involution by which terms are shown to cancel away.

The fixed points of the involution are those for which each letter with upper case appears at most one place. In particular, this means that if $N < \ell(\lambda)$, then $m_\lambda(\xi_1,\ldots,\xi_N) = 0$. In general, it means that the circular walks, which are powers of Lyndon words, must themselves be Lyndon words. Furthermore, the resulting Lyndon words must all be distinct. The surviving objects are sets of Lyndon words with a total of n vertices from $\pm\mathbf{N}$ for which the upper case letters are all distinct and separated by circular distances $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$, and for which places are labelled with the labels $\tilde{1},\ldots,\tilde{N}$. The function $\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ generates all such objects. Each set of Lyndon words is labelled in $n!$ ways because each place is distinguishable as the words are all different and they are all Lyndon words. Therefore the function $\frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)C(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ generates the same objects, but unlabeled.

We now show how to do away with the alphabet $\pm\mathbf{N}$ and think of these last objects in terms of the alphabet $\mathbf{N}$. Note that in any object the letters at the upper case places are all distinct. Suppose they are $\bar{i}_1,\ldots,\bar{i}_{\ell(\lambda)}$. Then their positions in the walkalongs determine a cycle structure on the letters $\bar{i}_1,\ldots,\bar{i}_{\ell(\lambda)}$ upon ignoring the lower case vertices. Note also that $\operatorname{sgn}(\sigma)\operatorname{sgn}(\lambda)$ is equal to the sign of this cycle structure. In this light the object may be thought of as a box of cycles on the letters $i_1,\ldots,i_{\ell(\lambda)}$, with sign, in which the $\ell(\lambda)$ edges have been replaced by walks of length $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$.



replacing $2 \to 4$ *with* $2 \to 1 \to 2 \to 4$
replacing $3 \to 3$ *with* $3 \to 1 \to 2 \to 1 \to 3$
replacing $4 \to 2$ *with* $4 \to 2$

We conclude that $m_\lambda(\xi_1,\ldots,\xi_N) = \operatorname{sgn}(\lambda)\frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\mathrm{C}(B_{\lambda,\sigma})p_\sigma(\xi_1,\ldots,\xi_N)$ generates all objects consisting of a box of cycles on $\ell(\lambda)$ places, with sign, and with the edges a_{ij} replaced by walks from i to j of length $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$. For given such an object, we can recover the upper case letters, and then the Lyndon words. We can introduce labels, think of the Lyndon words as circular walks, and then as closed walks, and finally find for our object a match among those original triplets that survived the involution. We have first constructed a map from the these triplets to our objects, and now demonstrated a map from the objects to the triplets. The two maps are inverses, and consequently determine a weight and sign preserving correspondence, which proves the result. For purposes of notation, we may assume that walks of length zero are attributed to the letters that do not appear in the box of cycles. The result is then the same as in the statement of the theorem. QED

The combinatorial objects that we have constructed may be thought of as terms in the determinant $\det(\mathbf{I}/\mathbf{I}-\mathbf{A})$ of the walk matrix. Let W_{ij} be the generating function for nonempty walks from i to j, and let I_{ii} be the generating function for the empty walk from i to i. The determinant of the walk matrix is given by

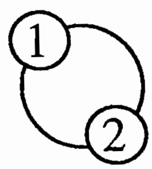
$$\det\begin{pmatrix} I_{11}+W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & I_{22}+W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N1} & W_{N2} & \cdots & I_{NN}+W_{NN} \end{pmatrix}.$$

For any term in this determinant there is some $\lambda \succ n$ and some n such that the term is made up of $\ell(\lambda)$ nonempty walks of lengths $\lambda_1,\ldots,\lambda_{\ell(\lambda)}$ that are placed into a cycle structure. Such a term corresponds to a unique term in $\sum_{n=0}^{\infty}\sum_{\lambda\succ n} m_\lambda(\xi_1,\ldots,\xi_N)$, as we have described it, and the two terms have the same sign. Likewise, every term in $\sum_{n=0}^{\infty}\sum_{\lambda\succ n} m_\lambda(\xi_1,\ldots,\xi_N)$, as we have described it, corresponds to a unique term in $\det(\mathbf{I}/\mathbf{I}-\mathbf{A})$. This correspondence yields a striking interpretation of the equation $h_n = \sum_{\lambda\succ n} m_\lambda$, if we recall the equation $\det(\mathbf{I}/\mathbf{I}-\mathbf{A}) = \sum_{n\geq 0} h_n(\xi_1,\ldots,\xi_N)$ that we interpreted in Section 2.3.

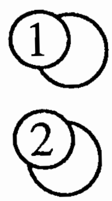
If $\lambda = 1^n$, then every walk is of length one and the function generates all boxes of cycles with n places, with sign, and we have $e_n(\xi_1,\ldots,\xi_N)$. If $\lambda = n$, then there is a single place that is replaced by a single walk of length n. This gives all closed walks of length n, and we have $p_n(\xi_1,\ldots,\xi_N)$. Also, if we set $a_{ij}=0$ for all $i\neq j$, then each cycle must be a loop, and each walk must be of the form $(a_{ii})^{\lambda_k}$. This recovers the usual definition of the monomial symmetric function m_λ, generating all distinct terms of the form $(a_{i_1i_1})^{\lambda_1}(a_{i_2i_2})^{\lambda_2}\cdots(a_{i_{\ell(\lambda)}i_{\ell(\lambda)}})^{\lambda_{\ell(\lambda)}}$, where the letters $i_1,\ldots,i_{\ell(\lambda)}$ are all distinct.

For purposes of comparison we consider the elementary symmetric functions $e_\lambda(\xi_1,\xi_2,\ldots\xi_N)$. The elementary symmetric function $e_r(\xi_1,\xi_2,\ldots\xi_N)$ is generated by the function $\det(\mathbf{I}+x\mathbf{A})$, where we take the coefficient of x^r. The generating function for $e_\lambda(\xi_1,\xi_2,\ldots\xi_N)$ is $\prod_{k=1}^{N}\det(\mathbf{I}+x_k\mathbf{A}) = \det\left(\prod_{k=1}^{N}(\mathbf{I}+x_k\mathbf{A})\right)$, where we take the coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_n^{\lambda_n}$. Note that $\left(\prod_{k=1}^{N}(\mathbf{I}+x_k\mathbf{A})\right)_{ij}$ generates walks from i to j that have been paired with a subset of the labels $\tilde{1},\ldots,\tilde{N}$ of size equal to the length of the walk. Therefore $e_\lambda(\xi_1,\xi_2,\ldots\xi_N)$ may be understood as the generating function of terms gotten by taking a box of cycles of length λ_k, with sign, and replacing the edges with walks, making sure that each walk of length k is associated with a subset of $\tilde{1},\ldots,\tilde{N}$ of size k, and that taken together the subsets form the multiset $1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$.

The main disadvantage of our expression for $m_\lambda(\xi_1,\ldots,\xi_N)$ is that many of the terms that it expresses cancel. It is in fact possible to insist that no two walks have the same letter in the same position. Even so, further cancellation does take place, as implied by the example below, where the objects correspond to terms of $m_{42}(\xi_1,\ldots,\xi_N)$ of equal weight, but opposite sign.



replacing $1 \to 2$ *with* $1 \to 2 \to 2$
replacing $2 \to 1$ *with* $2 \to 3 \to 4 \to 2 \to 1$



replacing $1 \to 1$ *with* $1 \to 2 \to 1$
replacing $2 \to 2$ *with* $2 \to 3 \to 4 \to 2 \to 2$

It would be interesting to discover a combinatorial interpretation of $m_\lambda(\xi_1,\ldots,\xi_N)$ in which no further cancellation occurs.

Of the many features of our expression for $m_\lambda(\xi_1,\ldots,\xi_N)$ that do make it attractive, the most intriguing is the way in which the closed walks $\left(\mathrm{A}^{\lambda_k}\right)_{ii}$ slide into the role of the products $\xi_i^{\lambda_k}$ when we set $a_{ij} = 0$ for all $i \neq j$. The connection is mysterious, but one that has been with us from the very beginning, upon our evaluation of $p_{\lambda_k}(\xi_1,\xi_2,\ldots\xi_N) = \sum_{i=1}^{N} \xi_i^{\lambda_k} = \sum_{i=1}^{N} \left(\mathrm{A}^{\lambda_k}\right)_{ii}$. Awaiting us is another such example, a quotient formula for Schur functions which is the first result of the next chapter.

CHAPTER 4

SCHUR FUNCTIONS

The object of this chapter is to evaluate the Schur functions s_λ at the eigenvalues $\xi_1,\ldots,\xi_N$ of an arbitrary $N\times N$ matrix $\mathbf{A}$. We collect together results from three different approaches. First, in Section 4.1 we consider the Schur function as the quotient of alternants $s_\lambda(x_1,\ldots,x_N) = \det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N} \Big/ \det\left(x_j^{\delta_i}\right)_{1\le i,j\le N}$. It is possible to evaluate the alternant $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ at the eigenvalues $\xi_1,\ldots,\xi_N$, but only as the square root of a symmetric function. We find a more interesting approach, which is to express the Schur function as a quotient $s_\lambda(\xi_1,\ldots,\xi_N) = \det\left(\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}\right)_{1\le i,j\le N} \Big/ \det\left(\left(\mathbf{A}^{\delta_i}\right)_{jj}\right)_{1\le i,j\le N}$ of determinants which are not equal to the alternants, but play an analogous role.

Next, Theorem 4.3.1 presents an interpretation of $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux which generalizes the fact that s_λ generates the column strict tableaux of shape λ. This generalization is one of the two main results of this thesis. Its proof makes use of a combinatorial interpretation of the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i-i+j}\right)_{1\le i,j\le N}$ in terms of special rim hook tabloids that is due to Eğecioğlu and Remmel [ER1]. The relation between the rim hook tableaux and the special rim hook tabloids is worked out in Section 4.2.

Finally, in Section 4.4 we address the fact that $s_\lambda(\xi_1,\ldots,\xi_N)$ is the trace of the irreducible representation of the general linear group that is associated with the partition $\lambda \succ n$. We do not arrive at any new results, but simply record a combinatorial interpretation of $s_\lambda(\xi_1,\ldots,\xi_N)$ that can be gotten by taking the trace of the irreducible representations described by Littlewood in his book [L]. This interpretation is encouraging when λ is a hook shape, but rather involved in the general case.

SECTION 4.1 A NEW QUOTIENT FORMULA

A natural way to define the Schur functions is as the quotient of the monomial antisymmetric functions $a_{\lambda+\delta}=\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ and $\alpha_\delta = \det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$, as we did in Section 1.1. We are able to evaluate these functions at the eigenvalues $\xi_1,\ldots,\xi_N$, but this does not produce anything of combinatorial interest. We then start over with a slightly different approach that best captures the spirit in which we have been working. Machinery from a proof of the Jacobi-Trudi identity is reworked in terms of edges a_{ij} instead of vertices x_i. A result about walks from Section 2.4 makes it possible to express the Schur function as a quotient of determinants $s_\lambda(\xi_1,\ldots,\xi_N) =$ $\det\left(\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}\right)_{1\le i,j\le N} \Big/ \det\left(\left(\mathbf{A}^{\delta_i}\right)_{jj}\right)_{1\le i,j\le N}$ that brings to mind the familiar quotient of alternants.

How can an antisymmetric function $a(\xi_1,\ldots,\xi_N)$ be expressed in terms of the entries of $\mathbf{A}$? This cannot be done in any straightforward way because the eigenvalues themselves cannot be calculated in terms of the entries of $\mathbf{A}$. We therefore try to express $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ in terms of symmetric functions of $\xi_1,\ldots,\xi_N$. Indeed, this can be done by taking the square of $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$. Let $\alpha_j = \lambda_j + \delta_j$. Then

$$a_\alpha^2 = \det\begin{pmatrix} x_1^{\alpha_1} & \cdots & x_1^{\alpha_N} \\ x_2^{\alpha_1} & \cdots & x_2^{\alpha_N} \\ \vdots & & \vdots \\ x_N^{\alpha_1} & \cdots & x_N^{\alpha_N} \end{pmatrix} \cdot \det\begin{pmatrix} x_1^{\alpha_1} & \cdots & x_1^{\alpha_N} \\ x_2^{\alpha_1} & \cdots & x_2^{\alpha_N} \\ \vdots & & \vdots \\ x_N^{\alpha_1} & \cdots & x_N^{\alpha_N} \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} x_1^{\alpha_1} & \cdots & x_N^{\alpha_1} \\ x_1^{\alpha_2} & \cdots & x_N^{\alpha_2} \\ \vdots & & \vdots \\ x_1^{\alpha_N} & \cdots & x_N^{\alpha_N} \end{pmatrix}\begin{pmatrix} x_1^{\alpha_1} & \cdots & x_1^{\alpha_N} \\ x_2^{\alpha_1} & \cdots & x_2^{\alpha_N} \\ \vdots & & \vdots \\ x_N^{\alpha_1} & \cdots & x_N^{\alpha_N} \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} p_{\alpha_1+\alpha_1} & \cdots & p_{\alpha_1+\alpha_N} \\ p_{\alpha_2+\alpha_1} & \cdots & p_{\alpha_2+\alpha_N} \\ \vdots & & \vdots \\ p_{\alpha_N+\alpha_1} & \cdots & p_{\alpha_N+\alpha_N} \end{pmatrix}$$

and consequently $a_\alpha = \pm\sqrt{\det\left(p_{\alpha_i+\alpha_j}\right)_{1\le i,j\le N}}$. The Schur function is therefore the following quotient of square roots.

$$s_\lambda = \sqrt{\det \frac{\left(p_{\alpha_i+\alpha_j}\right)_{1\le i,j\le N}}{\left(p_{\delta_i+\delta_j}\right)_{1\le i,j\le N}}}$$

It is not clear what to do with this square root. Another formula that may be relevant in this regard is $s_\lambda = \sqrt{\det\left(h_{\lambda_i+\lambda_{N-i}-i+j}\right)_{1\le i,j\le N}}$, which can be found in Macdonald's book [M, 30]. Either of these formulas may be evaluated at the eigenvalues $\xi_1,\ldots,\xi_N$, but this does not appear to add any combinatorial significance.

There is another way of expressing $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ as a combination of symmetric functions, and this leads us to one of the two main results of this thesis. It is impossible to express $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ as a rational function of symmetric functions in the variables $\xi_1,\ldots,\xi_N$ because then $\det\left(x_j^{\lambda_i+\delta_i}\right)_{1\le i,j\le N}$ would itself be a symmetric function. However, it is possible to come very close, as will be apparent in the proof of the next theorem.

THEOREM 4.1.1 *Let $\ell(\lambda)\le N$ and $\delta_i = N-i$. Then*

$$s_\lambda(\xi_1,\xi_2,\ldots,\xi_N) = \frac{\det\left(\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}\right)_{1\le i,j\le N}}{\det\left(\left(\mathbf{A}^{\delta_i}\right)_{jj}\right)_{1\le i,j\le N}}$$

The inspiration for this formula is a proof of the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le N}$ taken from Macdonald's book [M, 25]. That proof involves functions $e_r^{(j)}$ that are modifications of the elementary symmetric functions $e_r(x_1, \ldots, x_N)$ in that the variable x_j is omitted so that $e_r^{(j)} = e_r(x_1, \ldots, x_N)\big|_{x_j = 0}$. Equations $\sum_{k=1}^{N} h_{\lambda_i + \delta_i - N + k}(-1)^{N-k} e_{N-k}^{(j)} = x_j^{\lambda_i + \delta_i}$ are derived and placed in matrix form (let $h_r = 0$ when $r < 0$ and note that $\delta_i = N - i$):

$$\begin{pmatrix} h_{\lambda_1} & h_{\lambda_1 + 1} & \cdots & h_{\lambda_1 + (N-1)} \\ h_{\lambda_2 - 1} & h_{\lambda_2} & \cdots & h_{\lambda_2 + (N-2)} \\ \vdots & \vdots & & \vdots \\ h_{\lambda_N - (N-1)} & h_{\lambda_N - (N-2)} & \cdots & h_{\lambda_N} \end{pmatrix} \cdot \begin{pmatrix} e_{N-1}^{(1)} & e_{N-1}^{(2)} & \cdots & e_{N-1}^{(N)} \\ e_{N-2}^{(1)} & e_{N-2}^{(2)} & \cdots & e_{N-2}^{(N)} \\ \vdots & \vdots & & \vdots \\ e_0^{(1)} & e_0^{(2)} & \cdots & e_0^{(N)} \end{pmatrix}$$
$$= \begin{pmatrix} x_1^{\lambda_1 + \delta_1} & x_2^{\lambda_1 + \delta_1} & \cdots & x_N^{\lambda_1 + \delta_1} \\ x_1^{\lambda_2 + \delta_2} & x_2^{\lambda_2 + \delta_2} & \cdots & x_N^{\lambda_2 + \delta_2} \\ \vdots & \vdots & & \vdots \\ x_1^{\lambda_N + \delta_N} & x_2^{\lambda_N + \delta_N} & \cdots & x_N^{\lambda_N + \delta_N} \end{pmatrix}$$

Taking determinants of this matrix equation $H_\lambda M = A_{\lambda+\delta}$ gives the formula $\det(A_{\lambda+\delta}) = \det(H_\lambda)\det(M)$ where $\det(A_{\lambda+\delta})$ is the monomial antisymmetric function $a_{\lambda+\delta}$. When $\lambda = 0$, then $\det(H_0) = 1$, so that $\det(M) = \det(A_\delta) = a_\delta$. Therefore $\det(H_\lambda) = a_{\lambda+\delta}/a_\delta = s_\lambda$.

The equation $\det(A_{\lambda+\delta}) = \det(H_{\lambda+\delta})\det(M)$ does express $\det(A_{\lambda+\delta})$ as a polynomial of symmetric functions. However, the functions $e_{N-k}^{(j)}$ are not symmetric functions in the variables $x_1, \ldots, x_N$ because they are missing the variable x_j. If it were possible to evaluate the functions $e_{N-k}^{(j)}$ at the eigenvalues and express them in terms of the entries of $\mathbf{A}$, then it would be possible to calculate $\xi_j^{\lambda_i + \delta_i}$, which is a contradiction. Indeed, given $\mathbf{A}$ it is impossible to construct the entries of a matrix $\mathbf{A}'$ with one less eigenvalue.

However, the proof of the Jacobi-Trudi identity can be employed if $e_r^{(j)}$ is placed with a variant: Given $\mathbf{A}=\left(a_{ij}\right)_{1\le i,j\le N}$, let $\tilde{e}_r^{(j)}$ be the generating function for boxes of cycles, with sign, that employ r edges but do not employ the letter j. Then consider the expression $\sum_{k=1}^{N} h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)(-1)^{N-k}\tilde{e}_{N-k}^{(j)}$. It is the same as $\sum_{k=N-(\lambda_i+\delta)}^{N} h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)(-1)^{N-k}\tilde{e}_{N-k}^{(j)}$ because $\tilde{e}_{N-k}^{(j)}=0$ when $k<1$. In this latter expression, think of $(-1)^{N-k}$ as adding sign -1 to each edge of every box of cycles generated by $\tilde{e}_{N-k}^{(j)}$. And think of $h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)$ as the generating function for stacks of boxes of cycles that employ $\lambda_i+\delta_i-N+k$ edges. The pairs of objects generated by the expression $\sum_{k=N-(\lambda_i+\delta)}^{N} h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)(-1)^{N-k}\tilde{e}_{N-k}^{(j)}$ resemble those generated in Theorem 2.1.4 in the interpretation of $\sum_{k=0}^{n}(-1)^{n-k}h_k e_{n-k}=0$ (note especially the remark concerning the case $N<n$). Apply the involution from that theorem, but modify it so that a cycle in the first box of a stack generated by $h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)$ cannot be a candidate if it has the letter j. Adding this extra condition does not affect the reversibility of the involution and its fixed points are those in which no cycle is a candidate. Any cycle in the box generated by $\tilde{e}_{N-k}^{(j)}$ could have been a candidate, and therefore this box must be empty. A fixed point therefore has no sign and the stack of boxes of cycles must be generated by $h_{\lambda_i+\delta_i}(\xi_1,\ldots,\xi_N)$ and employ all $\lambda_i+\delta_i$ edges. The fixed points are those stacks of boxes of cycles with $\lambda_i+\delta_i$ edges for which the first box has exactly one cycle, and that cycle has the letter j.

By Lemma 2.4.2 the generating function for these fixed points is the same as that for walks from j to j of length $\lambda_i+\delta_i$. Therefore $\sum_{k=1}^{N} h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)(-1)^{N-k}\tilde{e}_{N-k}^{(j)}$ $=\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}$. These equations lead as before to a matrix equation

$$\begin{pmatrix} h_{\lambda_1}(\xi_1,\ldots,\xi_N) & \cdots & h_{\lambda_1+(N-1)}(\xi_1,\ldots,\xi_N) \\ h_{\lambda_2-1}(\xi_1,\ldots,\xi_N) & \cdots & h_{\lambda_2+(N-2)}(\xi_1,\ldots,\xi_N) \\ \vdots & & \vdots \\ h_{\lambda_N-(N-1)}(\xi_1,\ldots,\xi_N) & \cdots & h_{\lambda_N}(\xi_1,\ldots,\xi_N) \end{pmatrix} \cdot \begin{pmatrix} \tilde{e}_{N-1}^{(1)} & \tilde{e}_{N-1}^{(2)} & \cdots & \tilde{e}_{N-1}^{(N)} \\ \tilde{e}_{N-2}^{(1)} & \tilde{e}_{N-2}^{(2)} & \cdots & \tilde{e}_{N-2}^{(N)} \\ \vdots & \vdots & & \vdots \\ \tilde{e}_{0}^{(1)} & \tilde{e}_{0}^{(2)} & \cdots & \tilde{e}_{0}^{(N)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\mathbf{A}^{\lambda_1+\delta_1}\right)_{11} & \left(\mathbf{A}^{\lambda_1+\delta_1}\right)_{22} & \cdots & \left(\mathbf{A}^{\lambda_1+\delta_1}\right)_{NN} \\ \left(\mathbf{A}^{\lambda_2+\delta_2}\right)_{11} & \left(\mathbf{A}^{\lambda_2+\delta_2}\right)_{22} & \cdots & \left(\mathbf{A}^{\lambda_2+\delta_2}\right)_{NN} \\ \vdots & \vdots & & \vdots \\ \left(\mathbf{A}^{\lambda_N+\delta_N}\right)_{11} & \left(\mathbf{A}^{\lambda_N+\delta_N}\right)_{22} & \cdots & \left(\mathbf{A}^{\lambda_N+\delta_N}\right)_{NN} \end{pmatrix}$$

If this matrix equation is $H_\lambda \tilde{M} = \tilde{A}_{\lambda+\delta}$, then upon taking determinants we have $\det\left(\tilde{A}_{\lambda+\delta}\right) = \det\left(H_\lambda\right)\det\left(\tilde{M}\right)$ where $s_\lambda(\xi_1,\ldots,\xi_N) = \det\left(H_\lambda\right)$. If $\lambda = 0$, then $\det\left(H_0\right) = 1$ which shows that $\det\left(\tilde{M}\right) = \det\left(\tilde{A}_\delta\right)$. Therefore

$$s_\lambda(\xi_1,\xi_2,\ldots\xi_N) = \frac{\det\left(\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}\right)_{1\le i,j\le N}}{\det\left(\left(\mathbf{A}^{\delta_i}\right)_{jj}\right)_{1\le i,j\le N}} \qquad \text{QED}$$

The argument above hinges on the equation $\sum_{k=1}^{N} h_{\lambda_i+\delta_i-N+k}(\xi_1,\ldots,\xi_N)(-1)^{N-k}\tilde{e}_{N-k}^{(j)}$ $= \left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}$. We arrived at this equation by combinatorial means, but it is possible to derive it algebraically by applying Cramer's rule. This gives

$$\begin{aligned} \left(\mathbf{A}^s\right)_{jj} = \left(\frac{\mathbf{I}}{\mathbf{I}-x\mathbf{A}}\right)_{jj}\Bigg|_{x^s} &= \frac{\det(\mathbf{I}-x\mathbf{A})^{(jj)}}{\det(\mathbf{I}-x\mathbf{A})}\Bigg|_{x^s} \\ &= \sum_{k=0}^{s}\left(\frac{1}{\det(\mathbf{I}-x\mathbf{A})}\right)\Bigg|_{x^{s-k}} \det(\mathbf{I}-x\mathbf{A})^{(jj)}\Big|_{x^k} \\ &= \sum_{k=0}^{N}\left(\frac{1}{\det(\mathbf{I}-x\mathbf{A})}\right)\Bigg|_{x^{s-(N-k)}} \det(\mathbf{I}-x\mathbf{A})^{(jj)}\Big|_{x^{N-k}} \\ &= \sum_{k=0}^{N} h_{s-(N-k)}(\xi_1,\ldots,\xi_N)\tilde{e}_{N-k}^{(j)}(\xi_1,\ldots,\xi_N)(-1)^{N-k} \end{aligned}$$

where the last equality holds because $\tilde{e}_{N-k}^{(j)}(\xi_1,\ldots,\xi_N)(-1)^{N-k} = \det(\mathbf{I}-x\mathbf{A})^{(jj)}\Big|_{x^{N-k}}$.

As noted in the proof, $\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj}$ is the generating function for closed walks from j to j of length $\lambda_i + \delta_i$ It plays a role akin to that of $x_j^{\lambda_i+\delta_i}$. Setting $a_{ij} = 0$, $i \ne j$, and

$a_{ii} = x_i$ makes $\left(\mathbf{A}^{\lambda_i+\delta_i}\right)_{jj} = \left(a_{jj}\right)^{\lambda_i+\delta_i}$ and gives the usual quotient of alternants $s_\lambda\left(x_1,\ldots,x_N\right) = a_{\lambda+\delta}/a_\delta$.

SECTION 4.2 THE GEOMETRY OF RIM HOOKS

One of the two main results of this thesis is Theorem 4.3.1, which presents a combinatorial interpretation of $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux. This interpretation is attractive because setting $a_{ij}=0$, $i \neq j$, recovers the well known fact that s_λ generates the column strict tableaux of shape λ. The purpose of this section is to develop the facts about rim hooks that we will refer to in establishing Theorem 4.3.1. Our point of departure is the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le N}$. In our proof of Theorem 4.3.1 this identity takes the form $s_\lambda = \sum_{\nu \succ n} K^{-1}_{\nu,\lambda} h_\nu$. As we mentioned in Section 1.2, the integers $K^{-1}_{\nu,\lambda}$ have a combinatorial interpretation due to Eğecioğlu and Remmel [ER1] in terms of special rim hook tabloids. Two facts about these tabloids are of special importance, and we record them as Lemma 4.2.3 and Lemma 4.2.5. Given a special rim hook tabloid, Lemma 4.2.3 shows us ways of constructing a tabloid of opposite sign. This is essential to the sign reversing involution that figures in the first half of our proof of Theorem 4.3.1. Lemma 4.2.5 relates special rim hook tabloids with rim hook tableaux. This relationship makes possible the bijection that figures in the second half of our proof of Theorem 4.3.1.

In our search for a combinatorial interpretation of $s_\lambda(\xi_1,\ldots,\xi_N)$, our first observation is that the limiting cases $s_n(\xi_1,\ldots,\xi_N) = h_n(\xi_1,\ldots,\xi_N)$ and $s_{1^n}(\xi_1,\ldots,\xi_N) = e_n(\xi_1,\ldots,\xi_N)$ can both be expressed in terms of cycles, as was stated in Theorems 2.1.2 and 2.1.5. Perhaps it is possible to express $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of cycles ? The answer is yes, if we use the Jacobi-Trudi identity $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le N}$ or its dual $s_\lambda = \det\left(e_{\lambda'_i - i + j}\right)_{1 \le i,j \le N}$. The latter identity in fact provides a very helpful means of actually calculating $s_\lambda(\xi_1,\ldots,\xi_N)$. We find this approach somewhat disappointing, however, because setting $a_{ij}=0$ for $i \neq j$ does not give rise to column strict tableaux, as we would like, but simply recovers the Jacobi-Trudi identity. It is true that there exists an involution due to Gessel and Viennot [S, 82] which shows how the terms of the Jacobi-Trudi identity give rise to column strict tableaux. However, we would like the column strict tableaux to appear in a direct way. Our purpose is not merely to interpret $s_\lambda(\xi_1,\ldots,\xi_N)$, but to find an interpretation that generalizes s_λ as the generating function of column strict tableaux of shape λ.

In our proof of Theorem 4.3.1, we do take the Jacobi-Trudi identity $\sum_{\nu\succ n} K^{-1}_{\nu,\lambda} h_\nu$ as our starting point. We use special rim hook tabloids to express $K^{-1}_{\nu,\lambda}$, but we express $h_{\lambda_i-i+j}(\xi_1,\ldots,\xi_N)$ in terms of Lyndon words instead of cycles. Of interest to us is a sign reversing involution that Eğecioğlu and Remmel [ER1] describe on the terms of $\sum_{\nu\succ n} K^{-1}_{\nu,\lambda} h_\nu$. In our proof of Theorem 4.3.1, we perform a similar involution on the terms of $\sum_{\lambda\succ n} K^{-1}_{\nu,\lambda} h_\nu(\xi_1,\ldots,\xi_N)$ for which the fixed points are rim hook tableaux, instead of column strict tableaux. The purpose of this section is to describe the properties of rim hook tableaux and special rim hook tabloids that our proof will depend on.

We recall from Section 1.2 that a rim hook tableau of shape λ is an object that consists of a sequence of shapes $\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots \subset \lambda^{(r)} = \lambda$ such that for all i, $1 \le i \le r$, the skew shape $\lambda^{(i)} - \lambda^{(i-1)}$ is a rim hook. The sign of a rim hook that starts at a square $(E,N)_{E\times N}$ and ends at a square $(E',N')_{E\times N}$ is defined to be $(-1)^{N'-N}$, and the sign of a rim hook tableau is the product of the sign of its rim hooks. The Murnaghan-Nakayama rule for calculating the irreducible character $\chi^\lambda(\sigma)$ states that if we fix a rearrangement $r_1,\ldots,r_{\ell(\sigma)}$ of the lengths $\sigma_1 \ge \cdots \ge \sigma_{\ell(\sigma)}$ of the cycles of a permutation $\sigma \in S_n$, then

$$\chi^\lambda(\sigma) = \sum_{T\in\Pi} \operatorname{sgn}(T),$$

where Π is the set of rim hook tableaux of shape λ with rim hooks of length $r_1,\ldots,r_{\ell(\sigma)}$. For example, if $\lambda = 543$, $\sigma_1 = 4$, $\sigma_2 = 3$, $\sigma_3 = 3$, $\sigma_4 = 2$, and we fix $r_1 = 4$, $r_2 = 3$, $r_3 = 2$, $r_4 = 3$, then Π consists of the following tableaux.

1	3	4		
1	3	4	4	
1	1	2	2	2

1	3	3		
1	2	2	4	
1	1	2	4	4

2	2	3		
1	2	3	4	
1	1	1	4	4

2	2	4		
1	2	4	4	
1	1	1	3	3

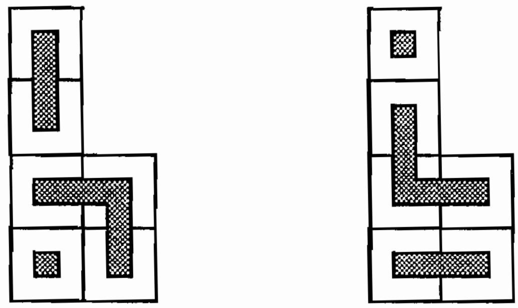
The first three tableaux have positive sign, and the fourth one has negative sign. It follows from the Murnaghan-Nakayama rule that the value of χ^{543} on elements of the conjugacy class 4332 is $\chi^{543}(4332) = 3 - 1 = 2$. If we choose a different order $r_1 = 3$, $r_2 = 3$, $r_3 = 2$, $r_4 = 4$, then the number of tableaux is different,

1	4	4		
1	2	4	4	
1	2	2	3	3

2	4	4		
2	2	4	4	
1	1	1	3	3

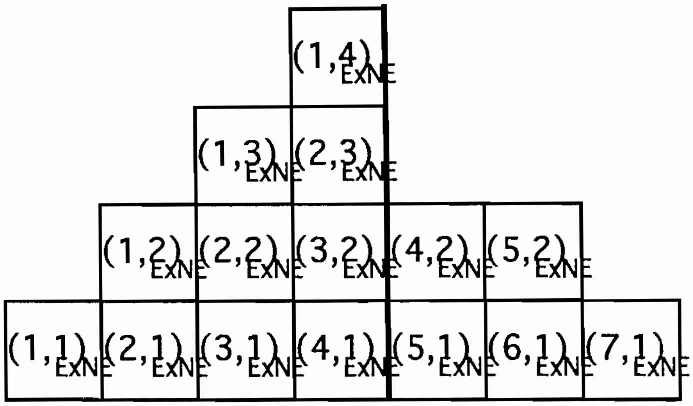
but the final answer is still the same: $\chi^{543}(4332) = 2 - 0 = 2$.

A special rim hook tabloid T of shape λ is a rim hook tableau of shape λ for which every rim hook is a special rim hook, that is, every rim hook starts in the leftmost column of the shape λ. The ith special rim hook of T is the one, if any, which starts at $(1,i)_{E\times N}$. If no special rim hook of T starts in the ith row, then we say that the ith special rim hook of T is empty and has length zero. The type $\nu \succ n$ of T is the partition that consists of the nonzero lengths of the special rim hooks of T. The sign of T is the same as if it were a rim hook tableau. We remarked in Section 1.2 that $K^{-1}_{\nu,\lambda} = \sum_{T\in\Phi} \mathrm{sgn}(T)$ where Φ is the set of special rim hook tabloids of shape λ and type ν. In drawing a special rim hook tabloid the special rim hooks are indicated by solid lines. The examples below show that $K^{-1}_{321,2211} = 1 - 1 = 0$.



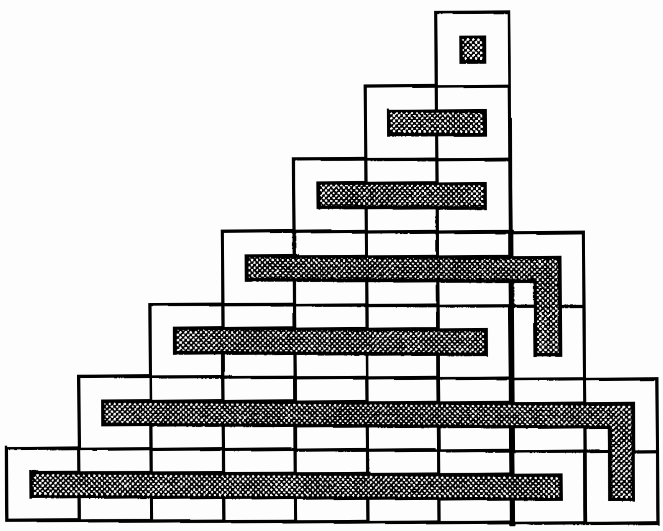
One way to relate rim hook tableaux and special rim hook tabloids is to apply the map ch^{-1}, described in Section 1.1, to the Jacobi-Trudi identity $s_\lambda = \sum_{\nu \succ n} K^{-1}_{\nu,\lambda} h_\nu$. Recall that $(\mathrm{ch}^{-1} s_\lambda)(\sigma) = \chi^\lambda(\sigma)$ and $(\mathrm{ch}^{-1} h_\nu)(\sigma) = \eta^\nu(\sigma)$. This yields an identity $\chi^\lambda(\sigma) = \sum_{\nu \succ n} K^{-1}_{\nu,\lambda} \eta^\nu(\sigma)$. There exists a combinatorial interpretation of the character $\eta^\nu(\sigma)$ in terms of ordered brick tabloids, as we remarked in Section 1.2. The involution that we use in Theorem 4.3.1 and the bijection that we use in Lemma 4.2.5 are very similar to those that would be used to give a combinatorial proof of $\chi^\lambda(\sigma) = \sum_{\nu \succ n} K^{-1}_{\nu,\lambda} \eta^\nu(\sigma)$. This point will be made more explicit at the conclusion of Section 4.3.

The fact that rim hooks have one square per northeasterly line makes it very sensible to work in an east by northeast coordinate system. But such a coordinate system works especially well if, given a shape $\lambda \succ n$, we fix $H \geq \ell(\lambda)$ and work within a larger set $\delta^H \cup \lambda$ of squares which for all j, $1 \leq j \leq H$, extends the jth row of λ to the west by $\delta^H_j = H + 1 - j$ squares. We establish a coordinate system $E \times NE$ on the squares of $\delta^H \cup \lambda$ by defining $(i + \delta^H_j, j)_{E \times NE} = (i, j)_{E \times N}$, as illustrated below, when $H = 4$ and $\lambda = 32$.



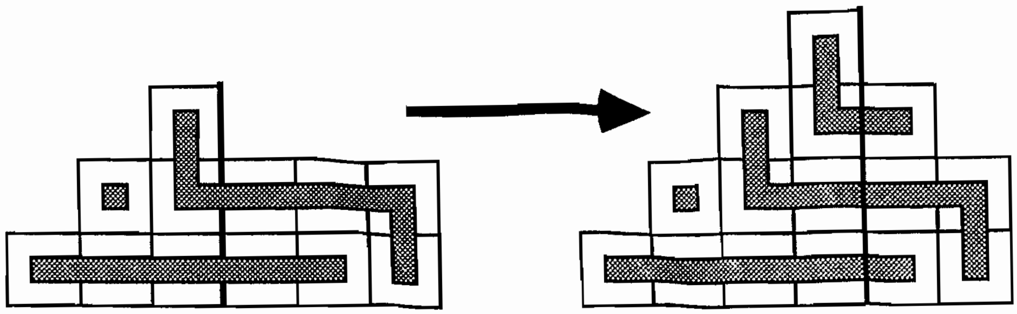
We do not abandon the coordinate system $E \times N$ altogether, but we will use the coordinate system $E \times NE$ whenever we work with northeasterly lines. In particular, we denote by $(i,\bullet)_{E\times NE}$ the northeasterly line which contains the square $(i,1)_{E\times NE}$.

A particularly helpful way of exploring the properties of a special rim hook tabloid T of shape $\lambda \succ n$ is to extend the special rim hooks to the shape $\delta^H \cup \lambda$. For all $i,\ j,\ 1 \le i \le \delta_j^H,\ 1 \le j \le H$, let the square $(i,j)_{E\times NE}$ belong to the jth special rim hook, as shown below.



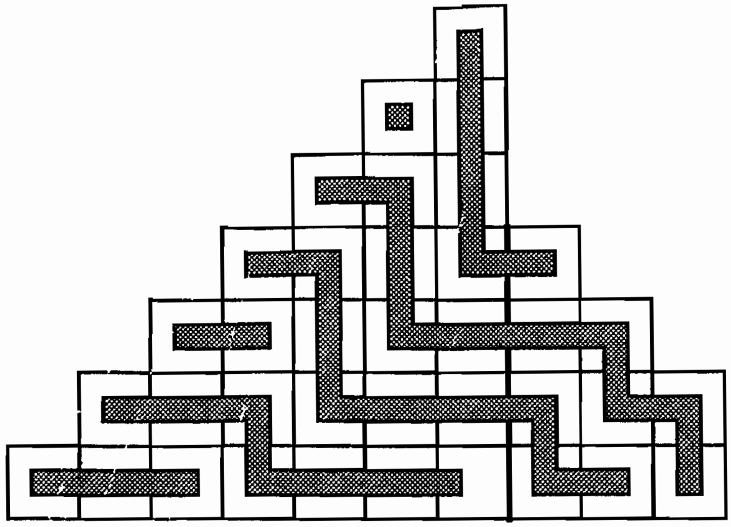
The resulting object is said to be the augmented special rim hook tabloid $\psi_{\delta^H}(T)$ of shape $\delta^H \cup \lambda$ that corresponds to T. It clarifies the properties of T. For example, from the illustration of $\psi_{\delta^H}(T)$ above it becomes reasonable to claim that the 3rd and 6th special rim hooks of T end at $(0,3)_{E\times N}$ and $(0,6)_{E\times N}$, respectively, even though we say that they start at $(1,3)_{E\times N}$ and $(1,6)_{E\times N}$, respectively. This point of view helps eliminate the difference between those special rim hooks of T of length zero and those special rim hooks of length greater than zero. It becomes sensible to declare that if the jth special rim hook of T has length zero, then it starts at $\left(1+\delta_j^H, j\right)_{E\times NE} = (1,j)_{E\times N}$ and ends at $\left(\delta_j^H, j\right)_{E\times NE} = (0,j)_{E\times N}$. This convention makes sense even when $j \geq \ell(\lambda)$.

In general, we define an augmented special rim hook tabloid U of shape $\delta^H \cup \lambda$ to be any decomposition of the squares of $\delta^H \cup \lambda$ into a disjoint union of rim hooks that start at $(1,j)_{E\times NE}$, $1 \leq j \leq H$. We emphasize that the special rim hook tabloid $\psi_{\delta^H}^{-1}(U)$ need not exist. Given a shape $\delta^H \cup \lambda$, note that the northeasterly lines which pass through the rightmost square of some row of $\delta^H \cup \lambda$ are precisely $\left(\lambda_j + \delta_j^H, \cdot\right)_{E\times NE}$, $1 \leq j \leq H$. Hence the number of squares in the intersection of $\delta^H \cup \lambda$ with $\left(\lambda_j + \delta_j^H, \cdot\right)_{E\times NE}$ is one greater than the number of squares in the intersection of $\delta^H \cup \lambda$ with $\left(\lambda_j + \delta_j^H + 1, \cdot\right)_{E\times NE}$. A consequence of this fact is that if an augmented special rim hook tabloid U has shape $\delta^H \cup \lambda$, then exactly one rim hook of U ends on each of the northeasterly lines $\left(\lambda_j + \delta_j^H, \cdot\right)_{E\times NE}$, $1 \leq j \leq H$. Furthermore, observe that the addition to U of an $H+1$th rim hook with start at $(1, H+1)_{E\times NE}$ gives rise to an augmented special rim hook tabloid (with $H+1$ rim hooks, and moved one column over) if and only if this $H+1$th rim hook does not end on any of the lines $\left(\lambda_j + \delta_j^H, \cdot\right)_{E\times NE}$, $1 \leq j \leq H$, as illustrated below , where $H+1$ is 4.



By induction on H, it follows that given a permutation $\tau \in S_H$ it is possible to construct an augmented special rim hook tabloid of shape $\delta^H \cup \lambda$ such that for all j, $1 \le j \le H$, the rim hook that starts at $(1,j)_{E\times NE}$ has length $\lambda_{\tau(j)} + \delta^H_{\tau(j)}$. Hence there are exactly $H!$ augmented special rim hook tabloids of shape $\delta^H \cup \lambda$, and we may index them by $\tau \in S_H$, so that U_τ denotes the augmented special rim hook tabloid for which the rim hook that starts at $(1,j)_{E\times NE}$ has length $\lambda_{\tau(j)} + \delta^H_{\tau(j)}$, $1 \le j \le H$. Define the sign of U_τ to be the product of the signs of its rim hooks. Then it can be shown that $\text{sgn}(U_\tau) = \text{sgn}(\tau)$ by considering the case when τ is a transposition $(j-1, j)$, $2 \le j \le H$. Therefore the $H!$ terms of the determinant $\det\left(\lambda_i + \delta^H_i\right)_{1\le i,j\le H}$ provide all of the information about augmented special rim hook tabloids of shape $\delta^H \cup \lambda$.

We compare below an augmented special rim hook tabloid U_τ and the corresponding term from $\det\left(\lambda_i + \delta^H_i\right)_{1\le i,j\le H}$ when $\lambda = 3321$ and $\tau = (15)(247632)$.



$$\begin{vmatrix} 10 & 10 & 10 & 10 & [10] & 10 & 10 \\ 9 & 9 & 9 & [9] & 9 & 9 & 9 \\ 7 & [7] & 7 & 7 & 7 & 7 & 7 \\ 5 & 5 & 5 & 5 & 5 & 5 & [5] \\ [3] & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & [2] & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & [1] & 1 \end{vmatrix}$$

The choice of τ is such that $\psi^{-1}_{\delta^H}(U_\tau)$ does not exist. The following criterion tells us when $\psi^{-1}_{\delta^H}(U_\tau)$ does exist.

LEMMA 4.2.1 *Let U_τ be an augmented special rim hook tabloid of shape $\delta^H \cup \lambda$ such that for all j, $1 \le j \le H$, the rim hook that starts at $(1,j)_{E\times NE}$ has length $\lambda_{\tau(j)} + \delta^H_{\tau(j)}$. Then $\psi^{-1}_{\delta^H}(U_\tau)$ exists if and only $\lambda_{\tau(j)} + \delta^H_{\tau(j)} \ge \delta^H_j$ for all i, $1 \le i \le H$.*

If $\psi^{-1}_{\delta^H}(U_\tau)$ exists, then for all j, $1 \le j \le H$, the squares $(1,j)_{E\times NE},\ldots,(\delta^H_j, j)_{E\times NE}$ belong to the same rim hook of U_τ, and therefore $\lambda_{\tau(j)} + \delta^H_{\tau(j)} \ge \delta^H_j$ for all j, $1 \le j \le H$. Conversely, if $\lambda_{\tau(j)} + \delta^H_{\tau(j)} \ge \delta^H_j$ for all j, $1 \le j \le H$, then find the smallest j for which the squares $(1,j)_{E\times NE},\ldots,\left(\delta_j^H,j\right)_{E\times NE}$ do not all belong to the same rim hook of U_τ. If j exists, then $\lambda_{\tau(j)}+\delta_{\tau(j)}^H<\delta_j^H$, with a contradiction. Therefore j does not exist and $\psi_{\delta^H}^{-1}(U_\tau)$ does exist. QED

We now consider what an augmented special rim hook tabloid $\psi_{\delta^H}(T)$ can tell us about a special rim hook tabloid T.

LEMMA 4.2.2 *Let T be a special rim hook tabloid. Then no two of its special rim hooks end on the same northeasterly line.*

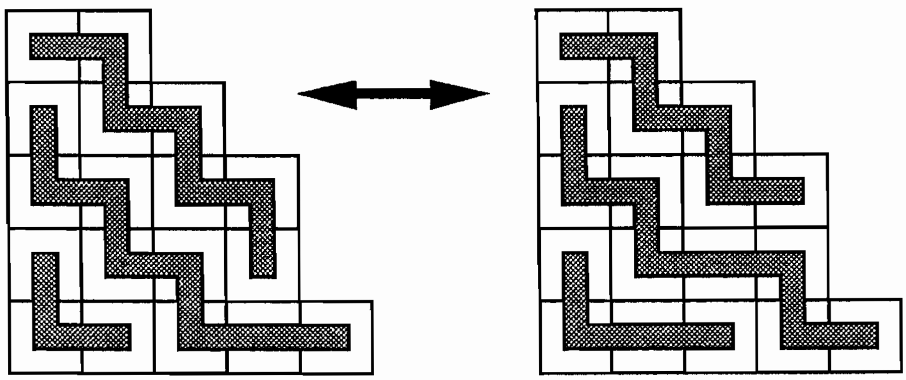
This follows from the fact that the rim hooks of $\psi_{\delta^H}(T)$ end on the lines $\left(\lambda_j+\delta_j^H,\bullet\right)_{E\times NE}$, $1\le j\le H$, which are all different. QED

LEMMA 4.2.3 *Let T be a special rim hook tabloid of shape λ such that for all i, $1\le i\le \ell(\lambda)$, the ith special rim hook of T ends at $(h_i,\bullet)_{E\times NE}$. Fix $j\ne k$, $1\le j,k\le \ell(\lambda)$, such that $(h_j,\bullet)_{E\times NE}$ passes through the kth special rim hook. Then there exists a special rim hook tabloid $\tilde{T}$ of shape λ such that the jth special rim hook ends at $(h_k,\bullet)_{E\times NE}$, the kth ends at $(h_j,\bullet)_{E\times NE}$, and for all i, $i\ne j$, $i\ne k, 1\le i\le \ell(\lambda)$, the ith ends at $(h_i,\bullet)_{E\times NE}$. Moreover,* $\operatorname{sgn}(T)=-\operatorname{sgn}\left(\tilde{T}\right)$.

Consider an augmented special rim hook tabloid $\psi_{\delta^H}(T)$, $\ell(\lambda)\le H$. There exists a permutation $\sigma\in S_H$ such that $\psi_{\delta^H}(T)=U_\sigma$. Consider the augmented special rim hook tabloid $U_{\sigma\cdot(j,k)}$. Its kth rim hook ends on $(h_j,\bullet)_{E\times NE}$, its jth ends on $(h_k,\bullet)_{E\times NE}$, and for all i, $i\ne j$, $i\ne k$, its ith ends on $(h_i,\bullet)_{E\times NE}$. Furthermore, $\operatorname{sgn}\left(U_{\sigma\cdot(j,k)}\right)=-\operatorname{sgn}(U_\sigma)$. The fact that $(h_j,\bullet)_{E\times NE}$ passes through the kth special rim hook of T implies that the length $\lambda_{\sigma(k)}+\delta_{\sigma(k)}^H-\delta_k^H$ of this special rim hook is greater than or equal to

$h_k - h_j = \lambda_{\sigma(k)} + \delta^H_{\sigma(k)} - \lambda_{\sigma(j)} - \delta^H_{\sigma(j)}$ and therefore $\lambda_{\sigma\cdot(j,k)(k)} + \delta^H_{\sigma\cdot(j,k)(k)} = \lambda_{\sigma(j)} + \delta^H_{\sigma(j)} \geq \delta^H_k$. It is also true that $\lambda_{\sigma\cdot(j,k)(j)} + \delta^H_{\sigma\cdot(j,k)(j)} \geq \delta^H_j$ and so it follows from Lemma 4.2.2 that $\psi^{-1}_{\delta^H}\left(U_{\sigma\cdot(j,k)}\right)$ exists. Setting $\tilde{T} = \psi^{-1}_{\delta^H}\left(U_{\sigma\cdot(j,k)}\right)$ completes the proof. QED

We say that Lemma 4.2.3 has the effect of switching the northeasterly lines at which the jth and kth special rim hooks of T end. Note that this switch changes the lengths of these special rim hooks. The example below illustrates an application of Lemma 4.2.3 with $j = 5$ and $k = 2$



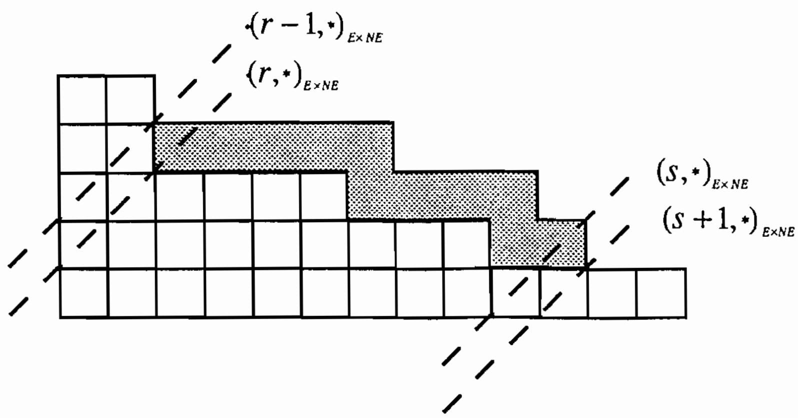
If $k = j+1$ or $j = k+1$, then Lemma 4.2.3 describes the switching employed by Eğecioğlu and Remmel in their combinatorial proof of $s_{\lambda/\mu} = \det\left(h_{\lambda_j - \mu_i + j - i}\right)_{1 \leq i,j \leq N}$. We will use Lemma 4.2.3 in order to create a sign reversing involution as part of our proof of Theorem 4.3.1.

Another important element in our proof of Theorem 4.3.1 is a bijection that relates special rim hook tabloids and rim hook tableaux. The following lemma explains the lengthening or shortening of a special rim hook tabloid T in terms of the adding or removing of a rim hook tableaux from the shape of T.

LEMMA 4.2.4 *Fix* $\lambda \succ n$ *and* $H \geq \ell(\lambda)$. *Let* T *be a special rim hook tabloid of shape* $\lambda \succ n$, *and for all* i, $1 \leq i \leq H$, *let its* i*th special rim hook have length* r_i. *Let* $\mu \succ (n+x)$. *Then* $\mu - \lambda$ *is a rim hook of length* x *if and only if there is a unique* k, $1 \leq k \leq H$, *for which there exists a special rim hook tabloid* T_+ *of shape* μ *having a* k*th special rim hook of length* $r_k + x$, *and for all* i, $i \neq k$, $1 \leq i \leq H$, *an* i*th special rim hook of length* r_i. *Moreover,* $\text{sgn}(T_+) = \text{sgn}(T)\text{sgn}(\mu - \lambda)$.

Let $\nu \succ (n-y)$. *Then* $\lambda - \nu$ *is a rim hook of length* y *if and only if there is a unique* k, $1 \leq k \leq H$, *for which there exists a special rim hook tabloid* T_- *of shape* μ *having a* k*th special rim hook of length* $r_k - y$, *and for all* i, $i \neq k$, $1 \leq i \leq H$, *an* i*th special rim hook of length* r_i. *Moreover,* $\text{sgn}(T) = \text{sgn}(T_-)\text{sgn}(\lambda - \nu)$.

The proof of this lemma turns on the fact that a rim hook $\mu - \lambda$ that starts at a northeasterly line $(r,\cdot)_{E\times NE}$ and ends at a northeasterly line $(s,\cdot)_{E\times NE}$ can be added to a shape λ if and only if the northeasterly line $(r-1,\cdot)_{E\times NE}$ contains the rightmost square of some row of λ, and the northeasterly line $(s+1,\cdot)_{E\times NE}$ contains the uppermost square of some column of λ. This fact is illustrated by the example below.



This result holds even when the relevant row or column of λ has length zero.

The special rim hooks of T end on precisely those northeasterly lines which pass through the rightmost square of some row of λ. Upon adding a rim hook $\mu - \lambda$, the northeasterly line $(r-1,\bullet)_{E\times NE}$, which passed through the rightmost square of a row of μ, does not pass through the rightmost square of a row of λ. Also, the northeasterly line $(s,\bullet)_{E\times NE}$, which did not pass through the rightmost square of a row of λ, does pass through the rightmost square of a row of μ. The other northeasterly lines are unaffected in this regard. Therefore any special rim hook tabloid of shape μ may be thought of as being gotten from a special rim hook tabloid of shape λ by lengthening the special rim hook that ends at $(r-1,\bullet)_{E\times NE}$ by $x = s - r$ squares. Given T and $\mu - \lambda$, then appealing to $\psi_{\delta^{II}}(T)$ shows that there exists a unique augmented special rim hook tabloid U of shape $\delta^{II} \cup \lambda$ such that for all i, $i \neq k$, $1 \leq i \leq \ell(\lambda)$, the length of the ith rim hook of U equals the length of the ith rim hook of $\psi_{\delta^{II}}(T)$, and the length of the kth rim hook of U is x greater than the length of the kth rim hook of $\psi_{\delta^{II}}(T)$. The special rim hook tabloid $\psi_{\delta^{II}}^{-1}(U)$ exists, and in fact, $T_+ = \psi_{\delta^{II}}^{-1}(U)$ is unique.

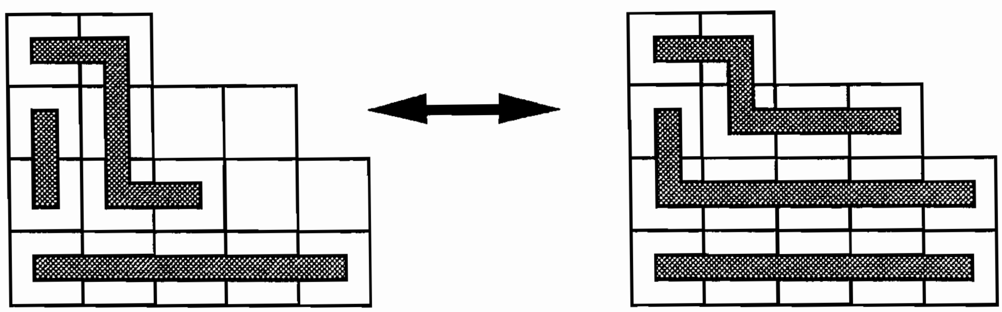
Conversely, suppose that there exists special rim hook tabloids T of shape λ and T_+ of shape μ, and that for all i, $i \neq k$, $1 \leq i \leq \ell(\lambda)$, the length of the ith special rim hook of T_+ equals the length of the ith special rim hook of T, and the length of the kth special rim hook of T is greater than the length of the kth special rim hook of T_+. Then $\mu - \lambda$ is a connected set that consists of one square per northeasterly line, and hence $\mu - \lambda$ is a rim hook.

In order to show that $\operatorname{sgn}(T_+) = \operatorname{sgn}(T)\operatorname{sgn}(\mu - \lambda)$, we first suppose that $k = \ell(\lambda)$. Then the end of the kth special rim hook of T occurs at the square to the immediate west of the start of the rim hook $\mu - \lambda$. Therefore the product of their signs equals the sign of the kth special rim hook of T_+, and $\operatorname{sgn}(T_+) = \operatorname{sgn}(T)\operatorname{sgn}(\mu - \lambda)$. Next, suppose that $k \neq \ell(\lambda)$. Then use Lemma 4.2.3 to switch the northeasterly lines at which the kth and $\ell(\lambda)$th special rim hooks of T end. This yields a special rim hook tabloid T_1 such that

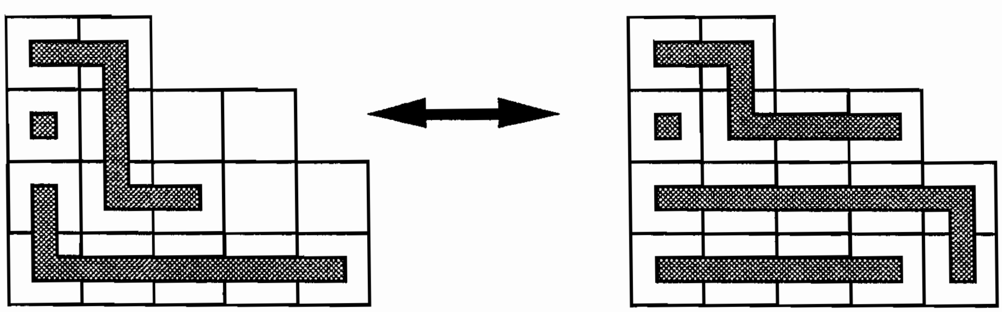
$\mathrm{sgn}(T_1) = -\mathrm{sgn}(T)$. Incorporate the rim hook $\mu - \lambda$ into the $\ell(\lambda)$th special rim hook of T_1. This yields a special rim hook tabloid T_2 such that $\mathrm{sgn}(T_2) = -\mathrm{sgn}(T)\mathrm{sgn}(\mu - \lambda)$. Finally, use Lemma 4.2.3 to switch the northeasterly lines at which the kth and $\ell(\lambda)$th special rim hooks of T_2 end. This yields the special rim hook tabloid T_+ and shows that $\mathrm{sgn}(T_+) = \mathrm{sgn}(T)\mathrm{sgn}(\mu - \lambda)$.

The proof of the first half of the lemma is complete. The second half is proved in essentially the same way. QED.

We illustrate Lemma 4.2.4 with two examples, one in which a special rim hook of nonzero length is lengthened,



and another in which a special rim hook of length zero is lengthened.



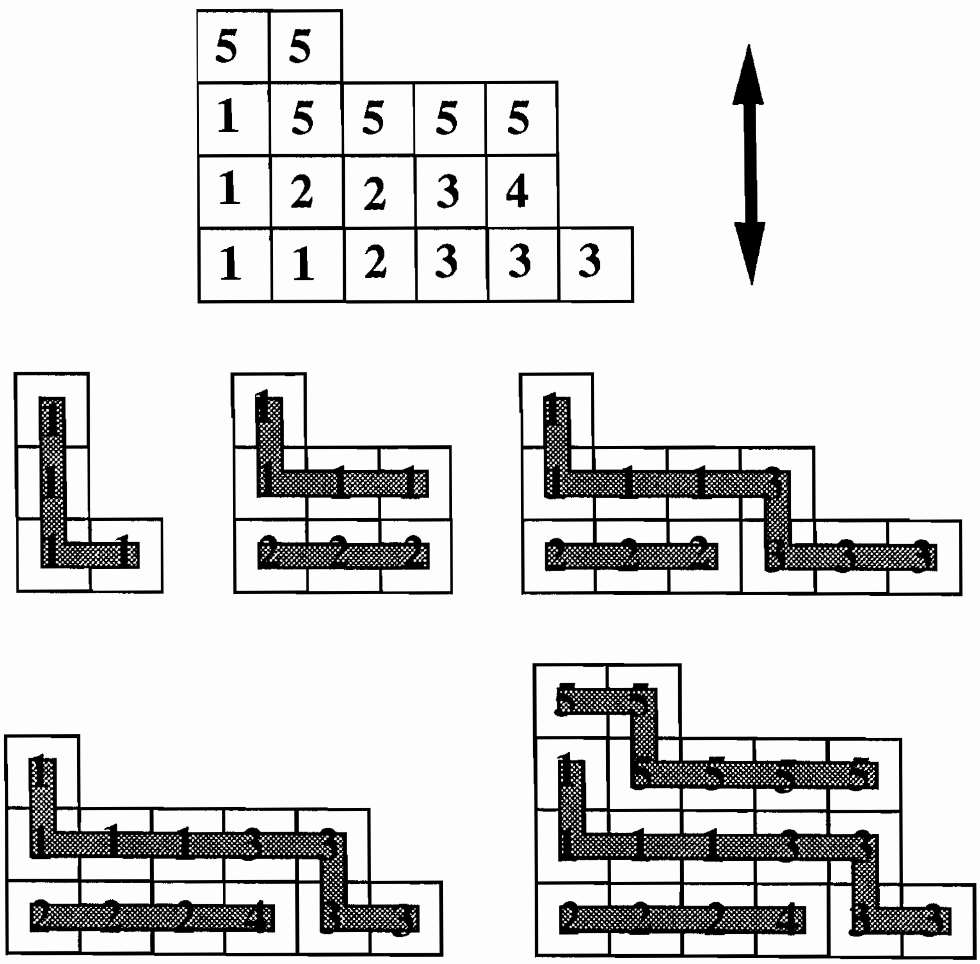
In the first example $k = 3$, and in the second $k = 1$. In either case we may visualize the start of the rim hook as sliding down a northeasterly line until it can be connected with the end of a special rim hook. Note that these same examples can be used to illustrate the shortening of a special rim hook. Observe also that the equation $\text{sgn}(T_+) = \text{sgn}(T)\text{sgn}(\mu - \lambda)$ holds.

Lemma 4.2.4 compares the incorporation of a rim hook with the lengthening of a special rim hook. Applying this lemma inductively allows us to compare the construction of a rim hook tableaux with that of a special rim hook tabloid. We record this relationship with the following lemma which in the next section will supply us with an important bijection in our proof of Theorem 4.3.1.

LEMMA 4.2.5 *Fix $\lambda \succ n$ and $x_1,\dots,x_r$. Let C be the set of sequences $T_0,T_1,\dots,T_r$ such that T_0 is the special rim hook tabloid of shape $\varnothing$, T_r is a special rim hook tabloid of shape λ, and there exists a sequence $i_1,\dots,i_r$ such that for all b, $1 \le b \le r$, lengthening the i_bth special rim hook of T_{b-1} by x_b squares gives rise to T_b. Let the sign of the sequence $T_0,T_1,\dots,T_r$ be the sign of the special rim hook tabloid T_r. Let D be the set of rim hook tableaux X such that X is given by a sequence of shapes $\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(r)} = \lambda$ such that for all b, $1 \le b \le r$, the length of the rim hook $\lambda^{(b)} - \lambda^{(b-1)}$ is x_b. Then there exists a sign preserving correspondence between the elements of C and D.*

This result is a consequence of Lemma 4.2.4. It is true when $r = 1$, and the result follows readily by induction on r. Suppose that the result is true for $r-1$. Given a rim hook tableau X from D, removing the rim hook $\lambda^{(r)} - \lambda^{(r-1)}$ leaves a rim hook tableaux X' with rim hooks of length $x_1,\dots,x_{r-1}$. Assuming the result is true for $r-1$, there exists a sequence of special rim hook tabloids $T_1',\dots,T_{r-1}'$ which corresponds to X' and such that T_{r-1}' has the same sign and shape as X'. By Lemma 4.2.4, the special rim hook tabloid T_{r-1}' has a unique special rim hook such that lengthening it by x_r gives a special rim hook tabloid T_r of shape λ. Lemma 4.2.4 also tells us that X and T_r have the same sign and shape. This establishes a sign preserving map $X \to T'_1,\ldots,T'_{r-1},T_r$ from D to C. This map is one-to-one and onto and is the desired sign preserving correspondence. QED.

We illustrate the correspondence of Lemma 4.2.5 with an example. The bth lengthening of a special rim hook is depicted by assigning the letter b to x_b of the squares of the i_bth special rim hook. The shapes of the special rim hook tabloids are the terms of the sequence of shapes $\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$ which defines the rim hook tableau.



In the next section this relation between special rim hook tabloids and rim hook tableaux will be used in the second half of our proof of Theorem 4.3.1 to express $s_\lambda(\xi_1,\ldots,\xi_N)$ in terms of rim hook tableaux.

SECTION 4.3 RIM HOOKS AND LYNDON WORDS

A central fact in the combinatorics of symmetric functions is that the Schur function s_λ generates the column strict tableaux of shape λ. The purpose of this section is to present an interpretation of $s_\lambda(\xi_1,\dots,\xi_N)$ which generalizes this fact. This interpretation is one of the two main results of this thesis. It follows from the equation $s_\lambda = \sum_{\nu\succ n} K^{-1}_{\nu,\lambda} h_\nu$. The functions $\{h_\nu\}_{\nu\succ n}$ are evaluated at the eigenvalues $\xi_1,\dots,\xi_N$ of an arbitrary $N\times N$ matrix $\mathbf{A}$ and interpreted as generating functions of multisets of Lyndon words. The integers $K^{-1}_{\nu,\lambda}$ are given a combinatorial interpretation due to Eğecioğlu and Remmel in terms of special rim hook tabloids. A sign reversing involution is defined on the objects generated by $\sum_{\nu\succ n} K^{-1}_{\nu,\lambda} h_\nu(\xi_1,\dots,\xi_N)$, and those that survive are expressed as rim hook tableaux. The rim hooks of the tableaux are associated with Lyndon words. Special conditions on the rim hooks ensure that if $a_{ij}=0$ for $i\neq j$, then the rim hook tableaux become column strict tableaux.

THEOREM 4.3.1 *Let M be the set of multisets of Lyndon words on $1<\cdots<N$ with a total of n places. Suppose that $m\in M$ consists of Lyndon words $\ell_1\le\ell_2\le\cdots\le\ell_s$ of length $r_1,\dots,r_s$, respectively. Let T_m be the set of rim hook tableaux of shape λ with rim hooks $R_1,R_2,\dots,R_s$ of length $r_1,\dots,r_s$, respectively, such that if $\ell_j=\ell_i$, $j>i$, then the start of R_j is in a column strictly to the right of the start of R_i. Then*

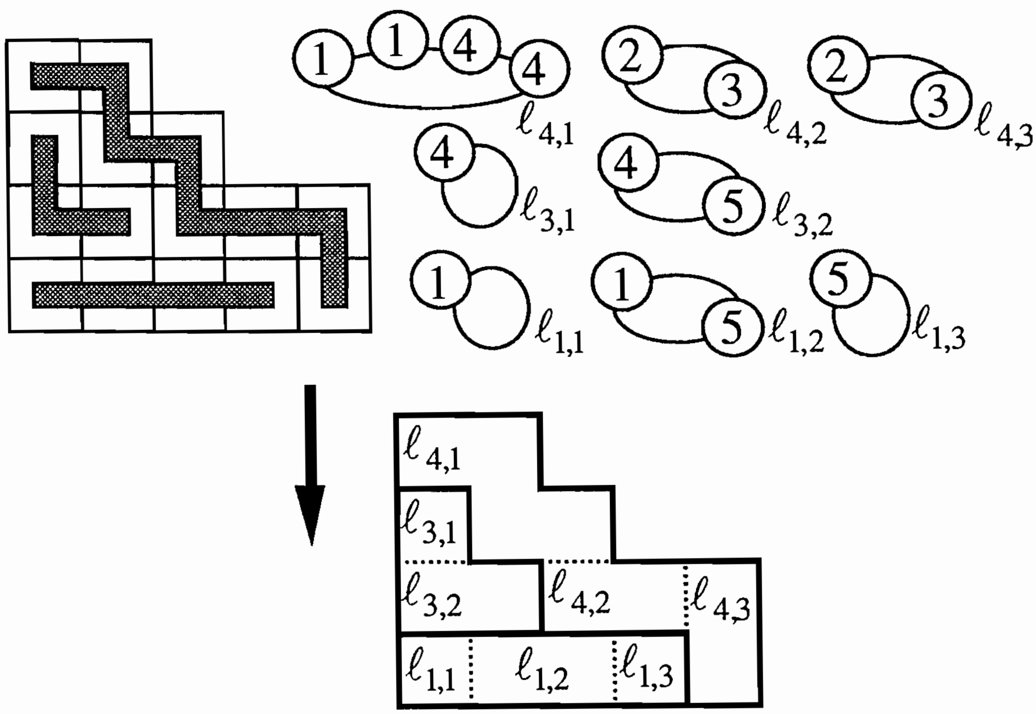
$$s_\lambda(\xi_1,\dots,\xi_N)=\sum_{\substack{m\in M\\ t\in T_m}} \operatorname{sgn}(t)\,\mathrm{W}_{\mathbf{A}}(m).$$

In this proof we rely heavily on the machinery that we developed in the previous section. Recall that the east by northeast coordinate system which we defined depended on an integer H. Fix $H \geq \ell(\lambda)$ and recall that the northeasterly line $\left(i+\delta_j^H,\bullet\right)_{E\times NE}$ passes through the square $(i,j)_{E\times N}$.

We stated in the previous section that $s_\lambda(\xi_1,\ldots,\xi_N) = \sum_{\nu\succ n} K_{\nu,\lambda}^{-1} h_\nu(\xi_1,\ldots,\xi_N)$ and that $K_{\nu,\lambda}^{-1} = \sum_{T\in U_{\nu,\lambda}} \operatorname{sgn}(T)$ where $U_{\nu,\lambda}$ is the set of special rim hook tabloids of shape λ and type ν. By Theorem 2.1.4 we know that $h_{\nu_i}(\xi_1,\ldots,\xi_N)$ is a generating function of multisets of Lyndon words on the letters $1,\ldots,N$ that use a total of ν_i places. We think of $s_\lambda(\xi_1,\ldots,\xi_N)$ as a generating function of pairs (T,M) that consist of a special rim hook tabloid T of shape λ and a sequence $M=(m_1,\ldots,m_n)$ of multisets of Lyndon words on the letters $1,\ldots,N$ for which for all i, $1\leq i\leq n$, the length of the ith special rim hook equals the number of places used by the multiset m_i.

Recall the lexicographic order on Lyndon words that we discussed in Section 1.3. Given M, for any i, $1\leq i\leq n$, consider the multiset m_i. Let $\ell_{i,1}\leq\ell_{i,2}\leq\ldots\leq\ell_{i,d(i)}$ denote the Lyndon words that comprise m_i, let $l_{i,1}\leq l_{i,2}\leq\ldots\leq l_{i,d(i)}$ denote their respective lengths, and define $x_{i,1}=l_{i,1}$ and $x_{i,j}=x_{i,j-1}+l_{i,j}$ for all j, $2\leq j\leq d(i)$. Furthermore, we require the convention that there exist Lyndon words $\varnothing$ and Z with letters unspecified for which $\varnothing<\ell<Z$ for all Lyndon words ℓ. Set $\ell_{i,0}=\varnothing$, $x_{i,0}=0$, $\ell_{i,d(i)+1}=Z$, and $x_{i,d(i)+1}=\infty$ for all i, $1\leq i\leq n$.

In the proof that we embark upon, we do not rely on pictures, but they will be helpful for illustrating the ideas involved. With this in mind, we associate each Lyndon word $\ell_{i,j}$ with the $x_{i,j-1}+1$th through $x_{i,j}$th squares of the ith special rim hook of T. For example, suppose that (T,M) is such that T is the special rim hook shown below on the left, and M is the sequence of multisets shown below on the right.

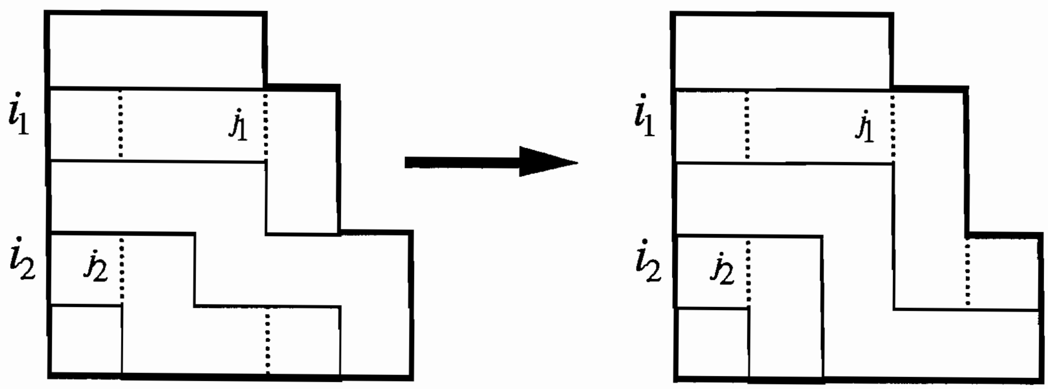


Then we can associate the Lyndon words with pieces of the special rim hooks, as illustrated above. The task at hand is to define a sign reversing weight preserving involution $(T,M)\to(\overline{T},\overline{M})$. Note that the sign of (T,M) is given by the sign of T. The idea behind the involution is to find two special rim hooks that pass through the same northeasterly line, to break off the continuations that extend beyond this line, and then to exchange these continuations, along with their respective pieces. Then T and $\overline{T}$ will have opposite sign, but M and $\overline{M}$ will have the same weight. In order for this to work we must describe a procedure for finding Lyndon words ℓ_{i_1,j_1} and ℓ_{i_2,j_2} for which $\ell_{i_1,j_1},\ell_{i_2,j_2}\le\ell_{i_1,j_1+1},\ell_{i_2,j_2+1}$ and $\left(x_{i_1j_1}+\delta^H_{i_1},\cdot\right)_{E\times NE}=\left(x_{i_2j_2}+\delta^H_{i_2},\cdot\right)_{E\times NE}$. In the example above, the only pairs of Lyndon words that meet all of these requirements are $\ell_{3,1}$, $\ell_{2,0}$ and $\ell_{4,1}$, $\ell_{1,1}$ and $\ell_{4,2}$, $\ell_{1,2}$. We now describe a procedure that selects a pair ℓ_{i_1,j_1}, ℓ_{i_2,j_2} of highest priority which remains such even after the exchange. This pair will be $\ell_{3,1}$, $\ell_{2,0}$ in the example above because the pieces of rim hooks that these Lyndon words correspond to are those which end on the northeasterly line that is furthest to the west.

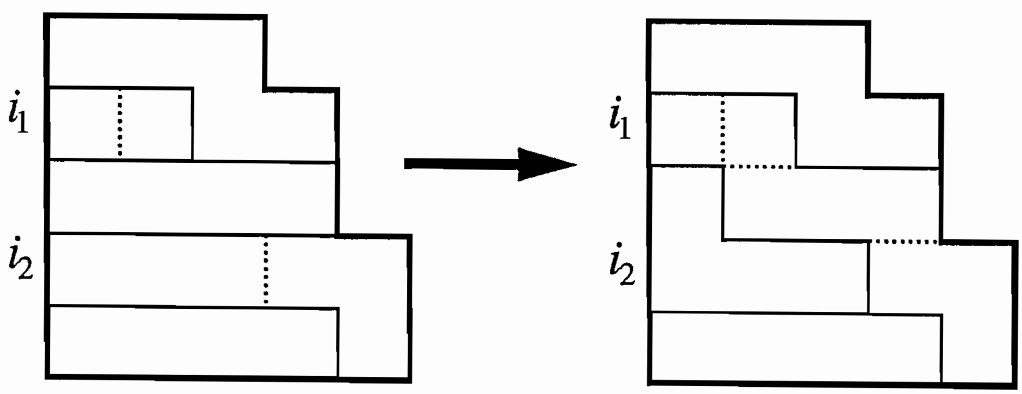
Given (T,M), let P be the set of all tuplets (a_1,a_2,b_1,b_2) for which $a_1 \neq a_2$, $1 \leq a_1,a_2 \leq n$, $0 \leq b_1 \leq d(a_1)+1$, $0 \leq b_2 \leq d(a_2)+1$, and such that $(x_{a_1b_1} + \delta^H_{a_1},\bullet)_{E\times NE} = (x_{a_2b_2} + \delta^H_{a_2},\bullet)_{E\times NE}$ and $\ell_{a_1,b_1},\ell_{a_2,b_2} \leq \ell_{a_1,b_1+1},\ell_{a_2,b_2+1}$. Then let q denote the smallest value of $x_{a_1b_1} + \delta^H_{a_1}$ attained by any $(a_1,a_2,b_1,b_2) \in P$. Let Q be the set of all pairs (a_1,a_2) for which there exist b_1 and b_2 such that $q = x_{a_1b_1} + \delta^H_{a_1}$ and $(a_1,a_2,b_1,b_2) \in P$. Note that the fact that $(a_1,a_2) \in Q$ determines b_1 and b_2. Among the pairs $(a_1,a_2) \in Q$ there is a smallest possible value for the Lyndon word ℓ_{a_1,b_1}, so let i_1 be the largest value of a_1 for which ℓ_{a_1,b_1} equals this Lyndon word. Likewise, among the pairs $(i_1,a_2) \in Q$ there is a smallest possible value for the Lyndon word ℓ_{a_2,b_2}, so let i_2 be the largest value of a_2 for which ℓ_{a_2,b_2} equals this Lyndon word. Define j_1 and j_2 so that $(q,\bullet)_{E\times NE} = (x_{i_1j_1} + \delta^H_{i_1},\bullet)_{E\times NE} = (x_{i_2j_2} + \delta^H_{i_2},\bullet)_{E\times NE}$. We have that $\ell_{i_1,j_1},\ell_{i_2,j_2} \leq \ell_{i_1,j_1+1},\ell_{i_2,j_2+1}$. Furthermore, note that $\ell_{i_1,j_1} \leq \ell_{i_2,j_2}$.

If P is empty, then set $\overline{T} = T$ and $\overline{M} = M$. Otherwise i_1, i_2 and j_1, j_2 exist. Exchange the northeasterly lines at which the i_1th and i_2th special rim hooks of T end, as in Lemma 4.2.3. Let $\overline{T}$ be the resulting special rim hook tabloid, and note that $\text{sgn}(\overline{T}) = -\text{sgn}(T)$. Define $\overline{M} = (\overline{m}_1,\ldots,\overline{m}_n)$ as the sequence of multisets of Lyndon words such that $\overline{m}_u = m_u$ for all u, $u \neq i_1$, $u \neq i_2$, $1 \leq u \leq n$, and $\overline{m}_{i_1} = \ell_{i_1,1} \leq \ell_{i_1,2} \leq \ldots \leq \ell_{i_1,j_1} \leq \ell_{i_2,j_2+1} \leq \ell_{i_2,j_2+2} \leq \ldots \leq \ell_{i_2,d(i_2)}$, and $\overline{m}_{i_2} = \ell_{i_2,1} \leq \ell_{i_2,2} \leq \ldots \leq \ell_{i_2,j_2} \leq \ell_{i_1,j_1+1} \leq \ell_{i_1,j_1+2} \leq \ldots \leq \ell_{i_1,d(i_1)}$. Then $\overline{M}$ is well defined because $\ell_{i_1,j_1},\ell_{i_2,j_2} \leq \ell_{i_1,j_1+1},\ell_{i_2,j_2+1}$. It has the same total weight as M because the Lyndon words are the same, although distributed among different multisets. In summary, $(\overline{T},\overline{M})$ is well defined and has the same weight as (T,M) but has the opposite sign.

We illustrate the construction of $(\overline{T},\overline{M})$ with a picture. The Lyndon word ℓ_{i_1,j_1} is associated with the rim hook that consists of the $x_{i_1,j_1-1}+1$th through x_{i_1,j_1}th squares of the i_1th special rim hook, and likewise the Lyndon word ℓ_{i_2,j_2} is associated with the rim hook that consists of the $x_{i_2,j_2-1}+1$th through x_{i_2,j_2}th squares of the i_2th special rim hook. This pair of words is chosen so that the northeasterly line at which the associated rim hooks end is as far to the west as possible. After this line is found, the pair (i_1, i_2) is chosen so that ℓ_{i_1,j_1} is as small as possible with respect to the lexicographical order, and then i_1 is as large as possible, and then ℓ_{i_2,j_2} is as small as possible, and then i_2 is as large as possible. Exchanging the the northeasterly lines at which the i_1th and i_2th special rim hooks end yields the special rim hook tabloid $\overline{T}$, as is shown below.



Note that one but not both of ℓ_{i_1,j_1} and ℓ_{i_2,j_2} can equal $\varnothing$, because otherwise there would be two special rim hooks of T ending on the same northeasterly line. For this same reason one but not both of ℓ_{i_1,j_1+1} and ℓ_{i_2,j_2+1} can equal Z. Below we present an example for which $\ell_{i_2,j_2} = \varnothing$ and $\ell_{i_1,j_1+1} = Z$.



From these examples it is apparent that the exchange does not affect anything at or above the line $\left(x_{i_1j_1}+\delta^H_{i_1},\cdot\right)_{E\times NE}$, and therefore we expect $\left(\overline{\overline{T}},\overline{\overline{M}}\right)=(T,M)$. After we prove this, we will move on to consider the fixed points of the involution.

Given the pair $\left(\overline{T},\overline{M}\right)$ with $\overline{M}=\left(\overline{m}_1,\ldots,\overline{m}_n\right)$, let $\overline{m}_i$ consist of the Lyndon words $\overline{\ell}_{i,1}\leq\ldots\leq\overline{\ell}_{i,\overline{d}(i)}$, and define $\overline{x}_{i,j}$ just as we did $x_{i,j}$. Observe that $\overline{T}=T$ if and only if $(T,M)=\left(\overline{T},\overline{M}\right)$ is a fixed point. Suppose that $\overline{T}\neq T$. We show that $\overline{\overline{T}}=T$ by demonstrating that the same northeasterly line $\left(x_{i_1j_1}+\delta^H_{i_1},\cdot\right)_{E\times NE}$ and the same pair $\left(i_1,i_2\right)$ that were chosen in the construction of $\overline{T}$ from T are also chosen in the construction of $\overline{\overline{T}}$ from $\overline{T}$. In order to do this we make use of the fact that T and $\overline{T}$ are identical with respect to all northeasterly lines west of $\left(x_{i_1j_1}+1+\delta^H_{i_1},\cdot\right)_{E\times NE}$. We start by observing that $\overline{\ell}_{i_1,j_1},\overline{\ell}_{i_2,j_2}\leq\overline{\ell}_{i_1,j_1+1},\overline{\ell}_{i_2,j_2+1}$ because $\ell_{i_1,j_1},\ell_{i_2,j_2}\leq\ell_{i_1,j_1+1},\ell_{i_2,j_2+1}$ and $\left(\overline{x}_{i_1j_1}+\delta^H_{i_1},\cdot\right)_{E\times NE}=\left(\overline{x}_{i_2j_2}+\delta^H_{i_2},\cdot\right)_{E\times NE}$ because $\overline{x}_{i_1,j_1}=x_{i_1,j_1}$ and $\overline{x}_{i_2,j_2}=x_{i_2,j_2}$. Therefore $\overline{\overline{T}}\neq\overline{T}$ and there exists a pair $\left(a_1,a_2\right)$ such that $\left(\overline{\overline{T}},\overline{\overline{M}}\right)$ is gotten by exchanging continuations of the a_1th and a_2th special rim hooks of $\overline{T}$. It must be that the line $\left(\overline{x}_{a_1b_1}+1+\delta^H_{a_1},\cdot\right)_{E\times NE}$ at which the exchange is executed is the same as $\left(\overline{x}_{i_1j_1}+1+\delta^H_{i_1},\cdot\right)_{E\times NE}=\left(x_{i_1j_1}+1+\delta^H_{i_1},\cdot\right)_{E\times NE}$, because if it were any further to the west, than $\left(a_1,a_2\right)$ or some other pair would have been chosen instead of $\left(i_1,i_2\right)$ in constructing $\left(\overline{T},\overline{M}\right)$. For the same reason, if $\overline{\ell}_{a_1,b_1}<\overline{\ell}_{i_1,j_1}$, then $\ell_{a_1,b_1}=\overline{\ell}_{a_1,b_1}$ and $\ell_{a_1,b_1}<\ell_{i_1,j_1}$, ℓ_{i_2,j_2}, ℓ_{i_1,j_1+1}, ℓ_{i_2,j_2+1}, so that a_1 would have been chosen instead of i_1 or i_2 in constructing $\left(\overline{T},\overline{M}\right)$. However, if $\overline{\ell}_{i_1,j_1}<\overline{\ell}_{a_1,b_1}$, then $\left(i_1,i_2\right)$or some other pair would have been chosen instead of $\left(a_1,a_2\right)$ in constructing $\left(\overline{\overline{T}},\overline{\overline{M}}\right)$. Therefore $\overline{\ell}_{a_1,b_1}=\overline{\ell}_{i_1,j_1}$. This same method of reasoning shows that $a_1=i_1$, $\overline{\ell}_{a_2,b_2}=\overline{\ell}_{i_2,j_2}$, and $a_2=i_2$. Therefore $\left(\overline{\overline{T}},\overline{\overline{M}}\right)$ is gotten by exchanging the northeasterly lines at which the i_1th and i_2th special rim hooks of $\overline{T}$ end. This reverses the effects of the construction of $\left(\overline{T},\overline{M}\right)$ and shows that $\left(\overline{\overline{T}},\overline{\overline{M}}\right)=(T,M)$. Therefore $(T,M)\to\left(\overline{T},\overline{M}\right)$ is a weight preserving sign reversing involution.

The fixed points of this involution are those (T,M) for which there are no tuplets (a_1,a_2,b_1,b_2) for which $a_1 \neq a_2$, $1 \le a_1,a_2 \le n$, $0 \le b_1 \le d(a_1)+1$, $0 \le b_2 \le d(a_2)+1$, and such that $\left(x_{a_1b_1} + \delta^H_{a_1},\bullet\right)_{E\times NE} = \left(x_{a_2b_2} + \delta^H_{a_2},\bullet\right)_{E\times NE}$ and $\ell_{a_1,b_1},\ell_{a_2,b_2} \le \ell_{a_1,b_1+1},\ell_{a_2,b_2+1}$. It is always true that $\ell_{a_1,b_1} \le \ell_{a_1,b_1+1}$ and $\ell_{a_2,b_2} \le \ell_{a_2,b_2+1}$. Therefore for a fixed point (T,M) we conclude that $\left(x_{a_1b_1} + \delta^H_{a_1},\bullet\right)_{E\times NE} = \left(x_{a_2b_2} + \delta^H_{a_2},\bullet\right)_{E\times NE}$ and $\ell_{a_1,b_1} \le \ell_{a_2,b_2}$ imply $\ell_{a_1,b_1} \le \ell_{a_1,b_1+1} < \ell_{a_2,b_2} \le \ell_{a_2,b_2+1}$. In particular, this shows that there is no line $\left(x_{a_1b_1} + \delta^H_{a_1},\bullet\right)_{E\times NE} = \left(x_{a_2b_2} + \delta^H_{a_2},\bullet\right)_{E\times NE}$ such that $\ell_{a_1,b_1} = \ell_{a_2,b_2}$. This allows us to establish a total order on all of the pairs (a,b), $1 \le a \le n$, $0 \le b \le d(a)+1$. We declare that $(a_1,b_1) < (a_2,b_2)$ if $\ell_{a_1,b_1} < \ell_{a_2,b_2}$ or if $\ell_{a_1,b_1} = \ell_{a_2,b_2}$ and $x_{a_1b_1} + \delta^H_{a_1} < x_{a_2b_2} + \delta^H_{a_2}$.

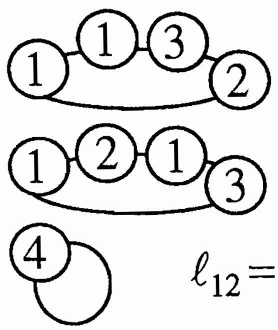
Fix M. Given M, list out the pairs (a,b), $1 \le a \le n$, $0 \le b \le d(a)+1$, in increasing order, so that we have $(c_1,d_1) < (c_2,d_2) < \ldots < (c_y,d_y)$. We note that a correspondence can be made between tabloids T for which (T,M) is a fixed point and sequences of special rim hooks that are very similar to those considered in Lemma 4.2.5. Define T_0 as the special rim hook tabloid of shape $\varnothing$, and by induction define T_i as the special rim hook tabloid that is gotten from by lengthening the c_ith special rim hook of T_{i-1} by l_{c_i,d_i} squares, where l_{c_i,d_i} denotes the length of the Lyndon word ℓ_{c_i,d_i}, as we stated in the beginning of the proof. We claim that the special rim hook tabloid T_i always exists, for otherwise, suppose not. Then there exists a smallest i such that T_i does not exist. It must be that lengthening the c_ith special rim hook of T_i by l_{c_i,d_i} has it end on the same northeasterly line as some other special rim hook. Suppose that this other special rim hook was extended to this line at a stage $j < i$, so that it is the c_jth special rim hook, and was lengthened by l_{c_j,d_j} squares. Then $(c_j,d_j) < (c_i,d_i)$, and the fact that (T,M) is a fixed point implies that there exists a (c_k,d_k) such that $c_k = c_j$, $d_k = d_{j+1}$, and $(c_j,d_j) < (c_k,d_k) < (c_i,d_i)$, contradicting our assumption about the c_jth special rim hook. Therefore the induction on i proceeds undisturbed and in fact $T_y = T$, which is true because for all s, $1 \le s \le n$, the sth special rim hooks of T and T_y have the same length.

This shows that T gives rise to a sequence $T_0, T_1, \ldots, T_y$ of special rim hook tabloids such that for all i, $1 \le i \le y$, lengthening the c_ith special rim hook of T_{i-1} by l_{c_i,d_i} squares gives T_i. These tabloids have the additional property that if $\ell_{c_i,d_i} = \ell_{c_j,d_j}$, then $x_{c_j d_j} + \delta^H_{c_j} <$ $x_{c_i d_i} + \delta^H_{c_i}$, because (T,M) is a fixed point. But given any such sequence $T_0, T_1, \ldots, T_y$ with this additional property, and by the fact that M is fixed, we see that it is the unique sequence which ends in $T_y = T$. This is true because it is completely dictated by the order $(c_1,d_1) < (c_2,d_2) < \ldots < (c_y,d_y)$, which is determined by M and is independent of T.

We have demonstrated that $s_\lambda(\xi_1,\ldots,\xi_N)$ generates all pairs (L,X) of the following kind. L is a multiset of Lyndon words $\ell_1 \le \ell_2 \le \cdots \le \ell_s$ on the letters $1,\ldots,N$ of length $r_1,\ldots,r_s$, respectively, with a total of n places. X is a sequence $T_0, T_1, \ldots, T_s$ of special rim hook tabloids such that T_0 has shape $\varnothing$, T_s has shape λ, and for which there exists a sequence $i_1,\ldots,i_s$ such that for all t, $1 \le t \le s$, lengthening the i_tth special rim hook of T_{t-1} by r_t gives T_t. Furthermore, $\ell_{t_1} = \ell_{t_2}$, $t_1 < t_2$, implies that the t_1th special rim hook of T_{t_1-1} ends on a northeasterly line to the west of that at which the t_2th special rim hook of T_{t_2-1} ends. Finally, each pair is generated with sign, and the sign of X is the sign of T_s.

Fix L. Consider all sequences X such that (L,X) is generated by $s_\lambda(\xi_1,\ldots,\xi_N)$. Lemma 4.2.5 tells us how to associate to each such sequence X a rim hook tableau T of shape λ such that for all t, $1 \le t \le s$, the tth rim hook has length r_t. In each case the rim hook tableau T is such that if $\ell_i = \ell_j$, $i < j$, then the northeasterly line at which the ith rim hook of T starts is strictly above the northeasterly line at which jth rim hook of T starts. However, as there are no further restrictions on X, there are no further restrictions on T. Finally, by Lemma 4.2.5 we know that $\mathrm{sgn}(X) = \mathrm{sgn}(T)$. Therefore $s_\lambda(\xi_1,\ldots,\xi_N)$ generates all pairs (L,X), with sign, where X is a rim hook tableau of shape λ with rim hooks $R_1, R_2, \ldots, R_s$ such that for all t, $1 \le t \le s$, R_t has length r_t, and such that if $\ell_i = \ell_j$, $i < j$, then the northeasterly line at which R_i starts is strictly above the northeasterly line at which R_j starts. Equivalently, the start of R_j is in a column to the right of the start of R_i. This follows from the fact that the end of R_i is at the end of a row of the rim hook tableau consisting of the rim hooks $R_1,\ldots,R_i$. This final point brings us to the statement of the theorem. QED.

We illustrate a typical term generated by $s_\lambda(\xi_1,\ldots,\xi_N)$ when $\lambda = 16,13,10,7,5$. Let the multiset m consist of the Lyndon words below.



$\ell_1=\ell_2$

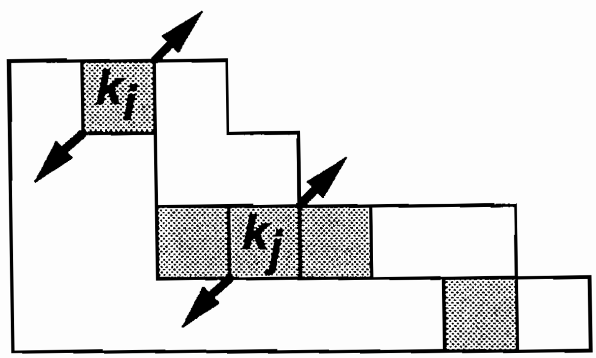
$\ell_3=\ell_4=\ell_5=\ell_6=\ell_7=\ell_8=\ell_9=\ell_{10}=\ell_{11}$

$\ell_{12}=\ell_{13}=\ell_{14}=\ell_{15}=\ell_{16}=\ell_{17}=\ell_{18}$

3	4	4	5	12											
3	3	4	5	6	7	13									
1	3	4	5	6	7	8	14	15	16						
1	2	2	5	6	7	8	9	9	10	10	17	18			
1	1	2	2	6	7	8	8	9	9	10	10	11	11	11	11

It can be paired with the rim hook tableau above. The weight of the term is given by the weight of the multiset, which is $(a_{11}a_{13}a_{32}a_{21})^2(a_{12}a_{21}a_{13}a_{31})^9(a_{44})^7$. The sign of the term is given by the sign of this tableau, which is $(-1)^{20}=+1$. Note that rim hooks which correspond to the same Lyndon word are layed down so that if $\ell_j=\ell_i$, $j>i$, then the start of the jth rim hook is in a column to the right of the start of the ith rim hook.

Having proven Theorem 4.3.1, we examine its various consequences. The most satisfying consequence is arrived at upon setting $a_{ij} = 0$ for all $i \neq j$, $1 \leq i, j \leq N$. Then the only Lyndon words possible are $a_{11}, \ldots, a_{NN}$, and therefore any rim hook tableau generated by $s_\lambda(\xi_1, \ldots, \xi_N)$ consists of rim hooks of length one, which is to say it is a standard tableau, and has positive sign. For all i, $1 \leq i \leq n$, the ith square of this standard tableau is associated with a Lyndon word $a_{k_i k_i}$, and $i < j$ implies that $k_i \leq k_j$. Furthermore, if $i < j$ and $k_i = k_j$, then the jth square is located to the right of the ith square, as shown below.



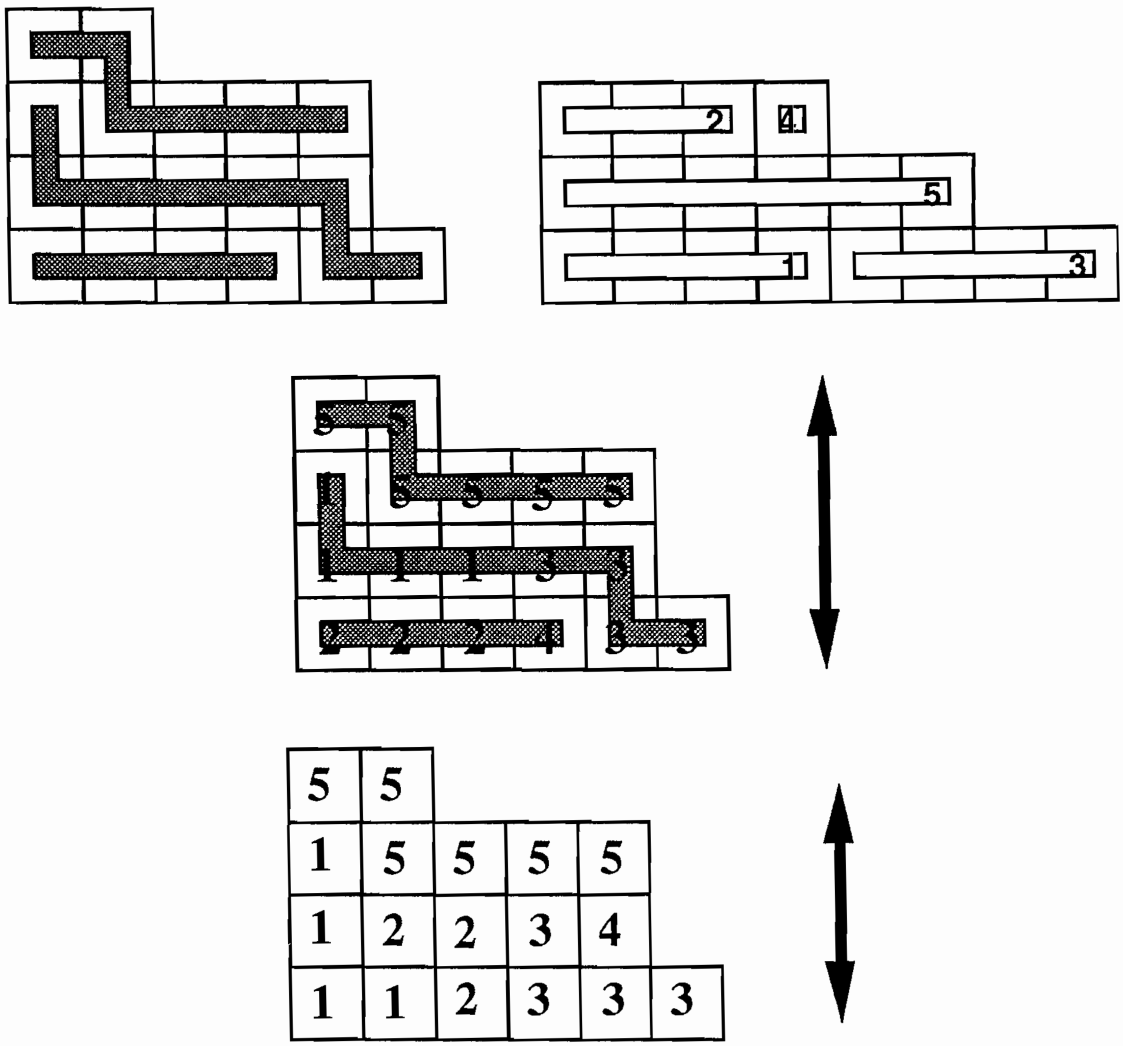
Associating the letter k_i to the ith square for all i, $1 \leq i \leq n$, and setting $x_{k_i} = a_{k_i k_i}$ for all i, $1 \leq i \leq N$, recovers the fact that s_λ generates the column strict tableaux of shape λ.

As usual, we may consider what Theorem 4.3.1 says about the special cases $\lambda = n$ and $\lambda = 1^n$. If $\lambda = n$, then given a multiset m of Lyndon words $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_s$ there is exactly one rim hook tableau of shape n with rim hooks of length $r_1, \ldots, r_s$, and this tableau has positive sign. Therefore $s_n(\xi_1, \ldots, \xi_N)$ generates all multisets of Lyndon words, and we have $h_n(\xi_1, \ldots, \xi_N)$. If $\lambda = 1^n$, then given a multiset m of Lyndon words $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_s$ there is exactly one rim hook tableau of shape 1^n with rim hooks of length $r_1, \ldots, r_s$, and this tableau has sign $\operatorname{sgn}(\nu)$, where ν is the partition that gives the lengths of the Lyndon words. If the Lyndon words of each multiset are listed in nonincreasing order, then it is possible to show that this function equals $e_n(\xi_1,\ldots,\xi_N)$ by performing an involution very similar to the second involution that we employed in our proof of $e_n(\xi_1,\ldots,\xi_N)=\sum_{\mu\succ n} f_\mu(\xi_1,\ldots,\xi_N)$ at the end of Section 3.3.

The method by which we proved Theorem 4.3.1 can be used to give combinatorial proofs of several other results. One of these is a theorem which Goulden and Jackson [GJ] recently reexamined and which we as Theorem 3.1.5. If we let $\xi_1,\ldots,\xi_N$ be the eigenvalues of $\mathbf{AZ}=(a_{ij}z_j)_{1\le i,j\le N}$, then this theorem states that the coefficient of $z_1z_2\cdots z_N$ in $\sum_{\nu\succ n} K_{\nu,\lambda}^{-1} h_\nu(\xi_1,\ldots,\xi_N)$ equals $\mathrm{Imm}_{\chi^\lambda}\mathbf{A}$. Our proof of Theorem 4.3.1 gives a combinatorial means of demonstrating that the coefficient of $z_1z_2\cdots z_N$ in $\sum_{\nu\succ n} K_{\nu,\lambda}^{-1} h_\nu(\xi_1,\ldots,\xi_N)$ equals the coefficient of $z_1z_2\cdots z_N$ in $s_\lambda(\xi_1,\ldots,\xi_N)$. But the statement of Theorem 4.3.1 shows that the coefficient of $z_1z_2\cdots z_N$ in $s_\lambda(\xi_1,\ldots,\xi_N)$ can be gotten by considering those multisets of Lyndon words which consist of disjoint cycles. These cycles are all distinct, and therefore there are no restrictions on the rim hook tableaux, except that the lengths of the cycles match the lengths of the rim hooks. But these are precisely the objects generated by $\mathrm{Imm}_{\chi^\lambda}\mathbf{A}$ when $\chi^\lambda(\sigma)$ is interpreted in terms of rim hook tableaux.

Similarly, the method by which we proved Theorem 4.3.1 can also be used to give a combinatorial proof of the equation $\chi^\beta(\alpha)=\sum_{\lambda\succ n}\eta^\lambda(\alpha)K_{\lambda,\beta}^{-1}$. This equation relates the characters $\chi^\beta=\mathrm{ch}^{-1}(s_\beta)$ and $\eta^\lambda=\mathrm{ch}^{-1}(h_\lambda)$ and is the image of the Jacobi-Trudi identity $s^\beta=\sum_{\lambda\succ n} h_\lambda K_{\lambda,\beta}^{-1}$ under the map ch^{-1} defined in Section 1.1. The character table $\eta^\lambda(\alpha)$ has an interpretation due to Eğecioğlu and Remmel [ER2] as the number of ordered brick tabloids of shape λ and type α. The product $\eta^\lambda(\alpha)K_{\lambda,\beta}^{-1}$ generates all pairs consisting of an ordered brick tabloid of shape β and type λ, and a special rim hook tabloid of shape λ and type α. The sign of such a pair is defined to be the sign of the special rim hook tabloid. As in the proof of Theorem 4.3.1, an involution based on Lemma 4.2.3 cancels away many of these pairs, and a bijection based on Lemma 4.2.5

associates the pairs that survive with rim hook tableaux. The latter bijection is illustrated below.



In a combinatorial proof of $\chi^{\beta}(\alpha)=\sum_{\lambda\vdash n}\eta^{\lambda}(\alpha)K^{-1}_{\lambda,\beta}$, the bricks of the ordered brick tabloids play the role that cycles do in our combinatorial proof of Goulden and Jackson's theorem.

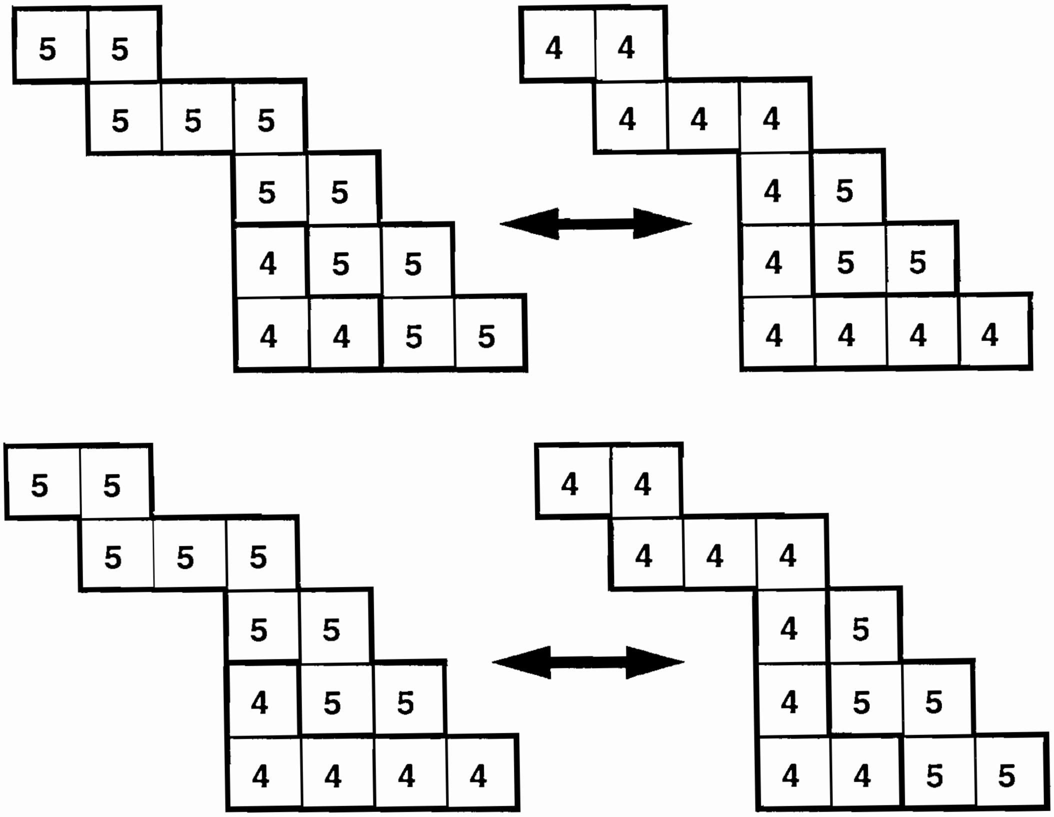
This last proof, using ordered brick tableaux, can be used to give a combinatorial proof of the fact that if T_{id} is the set of rim hook tableaux of shape β and with rim hooks of length $r_1,\ldots,r_{\ell(\alpha)}$, if T_{σ} is the set of rim hook tableaux of shape β and with rim hooks of length $r_{\sigma(1)},\ldots,r_{\sigma(\ell(\alpha))}$, and if $\sigma \in S_{\ell(\alpha)}$, then $\sum_{t\in T_{\rm id}} \operatorname{sgn}(t) = \sum_{t\in T_\sigma} \operatorname{sgn}(t)$. Specifically, given our proof of $\chi^\beta(\alpha) = \sum_{\lambda \vdash n} \eta^\lambda(\alpha) K^{-1}_{\lambda,\beta}$, it is enough to observe that if $\sigma \in S_{\ell(\alpha)}$, then the number of ordered brick tabloids of shape λ and with bricks of length $r_1,\ldots,r_{\ell(\alpha)}$ equals the number of ordered brick tabloids of shape λ and with bricks of length $r_{\sigma(1)},\ldots,r_{\sigma(\ell(\alpha))}$.

The disadvantage of the argument above is that it does not apply to the rim hook tableaux directly, but to a more numerous set of objects, namely, pairs consisting of an ordered brick tabloid and a special rim hook tabloid. However, if we analyze the special case when $\sigma \in S_{\ell(\alpha)}$ is a transposition $\sigma = (i\ i+1)$, then we are led to an argument due to Stanton and White which applies directly to the rim hook tableaux [SW][W]. Remark 4.3.2 provides a sketch of their argument. We include it for the sake of completeness. It is applicable to the rim hook tableaux generated by $s_\lambda(\xi_1,\ldots,\xi_N)$ in Theorem 4.3.1 and may be used to generalize the Knuth-Bender correspondence for column strict tableaux.

REMARK 4.3.2 Given $\sigma = (i\ i+1)$, we demonstrate that $\sum_{t\in T_{\rm id}} \operatorname{sgn}(t) = \sum_{t\in T_\sigma} \operatorname{sgn}(t)$. To do this we define subsets $U_{\rm id} \subset T_{\rm id}$ and $U_\sigma \subset T_\sigma$ and a sign reversing involution from $T_{\rm id}$ onto $T_{\rm id}$ which demonstrates that $\sum_{t\in T_{\rm id}} \operatorname{sgn}(t) = \sum_{t\in U_{\rm id}} \operatorname{sgn}(t)$. Likewise, our construction allows us to argue that $\sum_{t\in U_\sigma} \operatorname{sgn}(t) = \sum_{t\in T_\sigma} \operatorname{sgn}(t)$. We also define a sign preserving involution from $U_{\rm id}$ onto U_σ. It then follows that $\sum_{t\in T_{\rm id}} \operatorname{sgn}(t) = \sum_{t\in U_{\rm id}} \operatorname{sgn}(t) = \sum_{t\in U_\sigma} \operatorname{sgn}(t) = \sum_{t\in T_\sigma} \operatorname{sgn}(t)$.

Suppose that $t \in T_{\rm id}$ is a rim hook tableau with rim hooks $R_1,\ldots,R_{\ell(\alpha)}$ that start at $(s_1,\ast)_{E\times NE},\ldots,(s_{\ell(\alpha)},\ast)_{E\times NE}$ and end at $(s_1+r_1-1,\ast)_{E\times NE},\ldots,(s_{\ell(\alpha)}+r_{\ell(\alpha)}-1,\ast)_{E\times NE}$. We construct a rim hook tableau $\bar{t}$ with rim hooks $\overline{R}_1,\ldots,\overline{R}_{\ell(\alpha)}$ such that $\overline{R}_j = R_j$ for all $j \neq i, i+1$ and such that either $\bar{t} \in T_{\rm id}$ and $\operatorname{sgn}(\bar{t}) = -\operatorname{sgn}(t)$, or $\bar{t} \in T_\sigma$ and $\operatorname{sgn}(\bar{t}) = \operatorname{sgn}(t)$. In the first case it is understood that $t, \bar{t} \in T_{\rm id} - U_{\rm id}$ and in the second case it is understood that $t \in U_{\rm id}$, $\bar{t} \in U_\sigma$.

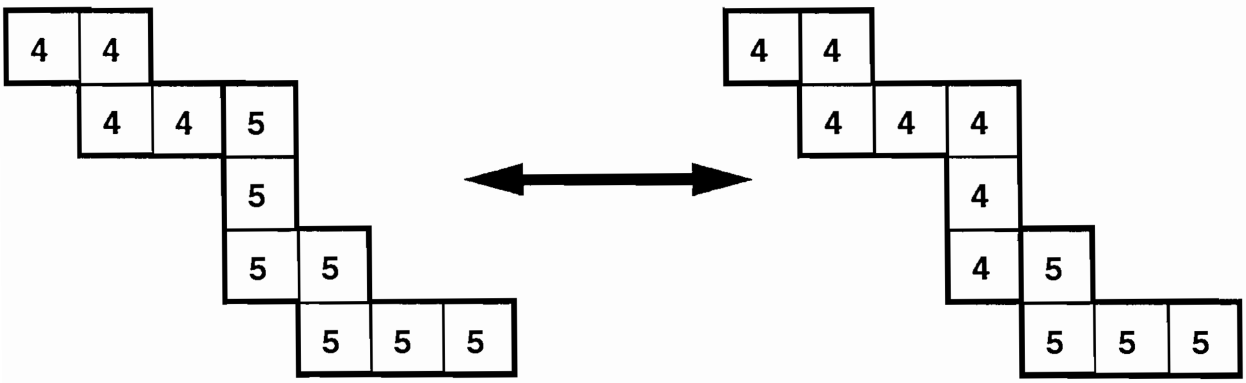
Given $t \in T_{\text{id}}$, if $s_i + r_i + 1 \neq s_{i+1}$ and $s_{i+1} + r_{i+1} + 1 \neq s_i$, then let $\bar{t} \in T_\sigma$ be the rim hook tableau for which the squares of R_i and $\overline{R}_{i+1}$ are on the same northeasterly lines, and likewise, the squares of R_{i+1} and $\overline{R}_i$ are on the same northeasterly lines, as illustrated below, where $i = 4$.



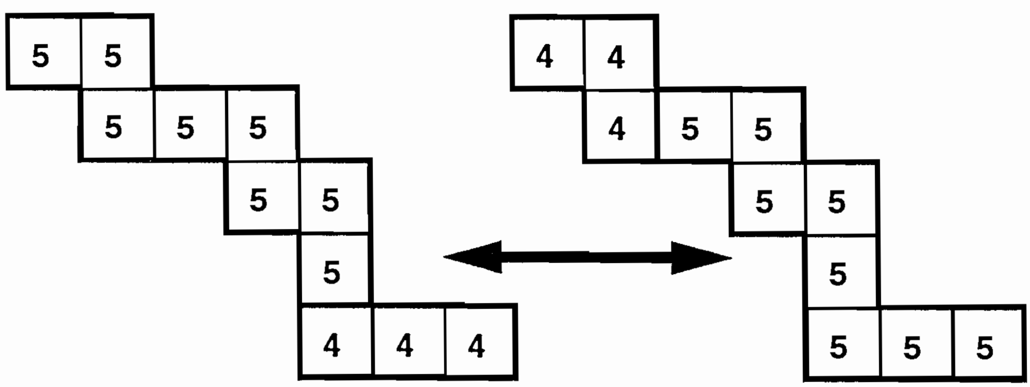
It can be shown that in this case $\text{sgn}(\bar{t}) = \text{sgn}(t)$ and we say that $t \in U_{\text{id}}$, $\bar{t} \in U_\sigma$. We think of these instructions as changing the order of the ith and $i+1$st bricks in an ordered brick tabloid, as discussed above. They deal with the case when the ith and $i+1$st bricks belong to different rows of the ordered brick tabloid.

If $s_i + r_i + 1 = s_{i+1}$, then let $\bar{t}$ be the rim hook tableau for which $\overline{R}_i$ and $\overline{R}_{i+1}$ are defined so that one starts at $(s_i, \cdot)_{E \times NE}$ and ends at $(s_i + r_{i+1} - 1, \cdot)_{E \times NE}$, and the other starts at $(s_i + r_{i+1}, \cdot)_{E \times NE}$ and ends at $(s_{i+1} + r_i - 1, \cdot)_{E \times NE}$. Likewise, if $s_{i+1} + r_{i+1} + 1 = s_i$, then let

$\bar{t}$ be the rim hook tableau for which $\overline{R}_i$ and $\overline{R}_{i+1}$ are defined so that one starts at $(s_{i+1},\bullet)_{E\times NE}$ and ends at $(s_{i+1}+r_i-1,\bullet)_{E\times NE}$, and the other starts at $(s_{i+1}+r_i,\bullet)_{E\times NE}$ and ends at $(s_{i+1}+r_i-1,\bullet)_{E\times NE}$. We may understand these instructions as changing the order of the ith and $i+1$st bricks in the case when they belong to the same row of an ordered brick tabloid. It can then be shown that either $\bar{t}\in T_\sigma$ and $\operatorname{sgn}(\bar{t})=\operatorname{sgn}(t)$, or $\bar{t}\in T_{\text{id}}$ and $\operatorname{sgn}(\bar{t})=-\operatorname{sgn}(t)$. In the first case the number of vertical crossings does not change, as illustrated below, and we say that $t\in U_{\text{id}}$, $\bar{t}\in U_\sigma$.



In the second case it changes by one, as illustrated below, and we say that $t,\bar{t}\in T_{\text{id}}-U_{\text{id}}$.



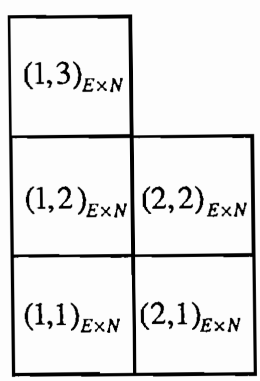
It is this last case which may be said to define $T_{\text{id}}-U_{\text{id}}$ and U_{id} and to show that $\sum_{t\in T_{\text{id}}}\operatorname{sgn}(t)=\sum_{t\in U_{\text{id}}}\operatorname{sgn}(t)$. An analogous chain of thought defines $T_\sigma-U_\sigma$ and U_σ and makes it possible to claim that $\sum_{t\in U_\sigma}\text{sgn}(t)=\sum_{t\in T_\sigma}\text{sgn}(t)$. Moreover, it can be shown that $\bar{\bar{t}}=t$ in each of the cases that we have considered. It then follows that there is a a sign preserving involution by which $\sum_{t\in U_{\text{id}}}\text{sgn}(t)=\sum_{t\in U_\sigma}\text{sgn}(t)$ and therefore $\sum_{t\in T_{\text{id}}}\text{sgn}(t)=\sum_{t\in T_\sigma}\text{sgn}(t)$. QED.

The conspicuous role of the Lyndon words in Theorem 4.3.1 and the fact that $\chi^\lambda(\sigma)$ has a combinatorial interpretation in terms of rim hook tableaux together suggest that we could have used the equation $s_\lambda(\xi_1,\ldots,\xi_N)=\frac{1}{n!}\sum_{\sigma\in S_n}\chi^\lambda(\sigma)p_\sigma(\xi_1,\ldots,\xi_N)$ to prove Theorem 4.3.1. This is the approach that worked in Chapter 3, and indeed, this is how we first conjectured our result. Although it seemed likely that this approach would succeed, we never determined whether in fact it does succeed. Instead we ended up taking a course that appears to require less effort. To anyone interested in starting from the equation $s_\lambda(\xi_1,\ldots,\xi_N)=\frac{1}{n!}\sum_{\sigma\in S_n}\chi^\lambda(\sigma)p_\sigma(\xi_1,\ldots,\xi_N)$, we provide three recommendations. First, in any sequence of circular walks, with labels, order the walks by the Lyndon word of which they are products, as we did in Theorem 3.3.1, and construct the rim hook tableaux so that the ith hook has the length of the ith walk. Second, study the equation $K_{\lambda,\beta}=\frac{1}{n!}\sum_{\sigma\in S_n}\chi^\lambda(\sigma)\eta^\beta(\sigma)$, which can be given a com-bi-natorial interpretation. Apply the involution used in this interpretation to those rim hooks which correspond to circular walks that are products of the same Lyndon word and start on the same northeasterly lines $\bmod r$, where the Lyndon word has length r. Third, remove the labels from the surviving objects with the help of the following lemma: "Let T be a rim hook tableau of shape $\lambda\succ n$ with rim hooks $R_1,\ldots,R_m$ such that for all i, $1\le i\le m$, R_i starts at $(r_i,\bullet)_{E\times NE}$ and ends at $(s_i,\bullet)_{E\times NE}$. Let $\tau\in S_m$ be a permutation for which $r_i=r_j$ or $s_i=s_j$ or $r_i=s_j+1$ implies that $i<j$ if and only if $\tau(i)<\tau(j)$. Then there is a unique rim hook tableau T' of shape λ with rim hooks $Q_1,\ldots,Q_m$ such that for all i, $1\le i\le m$, $Q_{\tau(i)}$ starts at $(r_i,\bullet)_{E\times NE}$ and ends at $(s_i,\bullet)_{E\times NE}$. Moreover, $\text{sgn}(T)=\text{sgn}(T')$."

SECTION 4.4 IRREDUCIBLE REPRESENTATIONS OF $GL_N(\mathbb{C})$

The most important fact about the Schur function $s_\lambda(\xi_1,\ldots,\xi_N)$, evaluated at the eigenvalues $\xi_1,\ldots,\xi_N$ of an $N \times N$ matrix $A \in GL_N(\mathbb{C})$, is that it is an irreducible character of the general linear group $GL_N(\mathbb{C})$, that is, it is the trace of an irreducible representation of $GL_N(\mathbb{C})$. It is actually possible to describe the entries of these representations, and hence, of their traces. The reader interested in the symmetric functions of eigenvalues will surely be curious to see an explicit description of these representations. However, they are not to be found in many modern textbooks. In order to satisfy this curiosity, we provide a description taken from Littlewood's book The Theory of Group Characters [L]. We refer the reader to this book, and also to a paper by Garsia and McLarnan which has proven helpful [GM]. When λ is a hook shape, then this leads to a description of $s_\lambda(\xi_1,\ldots,\xi_N)$ that is quite straightforward, but otherwise this description becomes rather complicated. Our goal in this section is not to prove any new result, but simply to include one last description of the Schur function $s_\lambda(\xi_1,\ldots,\xi_N)$ and thereby complete the combinatorial picture presented in this thesis.

The combinatorial objects that we will work with typically have a shape $\lambda \succ n$. We use an east by north coordinate system to refer to the squares of λ with, as shown below.



A λ-tableau W is an assignment of the letters $1^{m_1}2^{m_2}\cdots N^{m_N}$, $m_1+m_2+\ldots+m_N=n$, to the n squares in the shape λ. A column strict tableau T of shape λ is a λ-tableau T such that if $i<i'$ and $j<j'$, then the letter which T assigns to $(i,j)_{E\times N}$ is less than or equal to the letter which it assigns to $(i',j)_{E\times N}$, and the letter which it assigns to $(i,j)_{E\times N}$ is strictly less than the letter which it assigns to $(i,j')_{E\times N}$. That is, the letters of T increase strictly as we go up a column, but increase weakly as we go across a row from left to right. An injective tableau U of shape λ is a λ-tableau U for which $n=N$ and $m_1=m_2=\ldots=m_N=1$. A standard tableau Q of shape λ is a column strict tableau Q of shape λ that is an injective tableau. Consequently, the letters of Q increase strictly as we go up a column, and also increase strictly as we go across a row from left to right.

Given two λ-tableau T and U of the same shape λ, let $t(i,j)$ be the letter that T assigns to each square $(i,j)_{E\times N}\in\lambda$, and let $u(i,j)$ be the letter that U assigns to each square of $(i,j)_{E\times N}\in\lambda$. Then we define $G_A[T,U]=\prod_{(i,j)_{E\times N}\in\lambda} a_{t(i,j),u(i,j)}$. We may think of $G_A[T,U]$ as a means of comparing two λ-tableau T and U square by square. In the example below,

T

1	
3	1
1	2

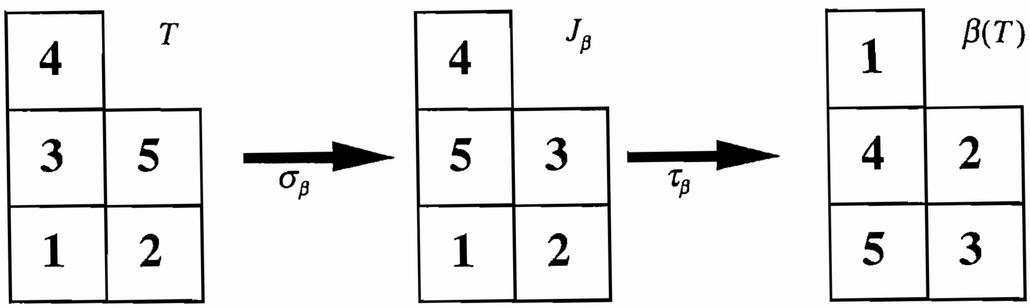
U

1	
5	5
4	2

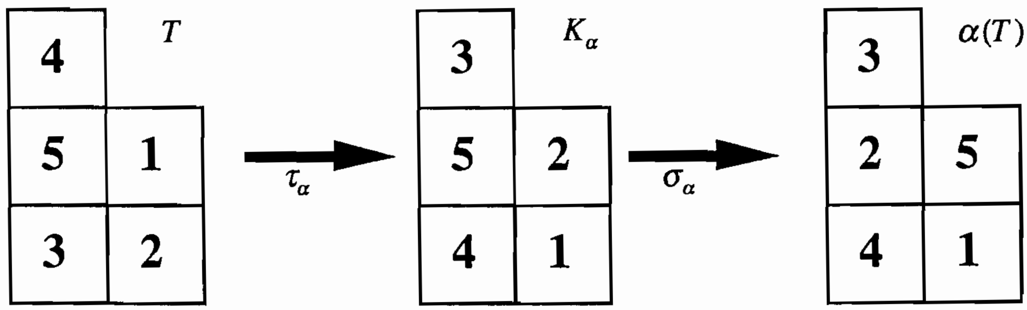
$G_A[T,U]=a_{11}a_{14}a_{15}a_{22}a_{35}$.

Z_λ is the set of bijections $\beta:(i,j)_{E\times N}\to(i_\beta,j_\beta)_{E\times N}$ on the squares of the shape λ such that if $(i,j)_{E\times N}\neq(k,l)_{E\times N}$, then $(i_\beta,j)_{E\times N}\neq(k_\beta,l)_{E\times N}$. We let $\beta\in Z_\lambda$ act on a

λ−tableau T by having $\beta(T)$ denote the unique λ−tableau with the property that if a letter appears at $(i,j)_{E\times N}$ in T, then it also appears at $\left(i_\beta, j_\beta\right)_{E\times N}$ in $\beta(T)$. Indeed, if we fix an injective tableau T, then the elements of Z_λ correspond to the permutations $\beta = \tau_\beta \sigma_\beta$ such that σ_β permutes the letters in the rows of T, but keeps the columns fixed, creating an injective tableau J_β, and then τ_β permutes the letters in the columns of J_β, but keeps the rows fixed, giving rise to $\beta(T)$, as illustrated below, where $\sigma_\beta = (35)$ and $\tau_\beta = (154)(23)$:



We define the sign of $\beta \in Z_\lambda$ to be $\mathrm{sgn}(\beta) = \mathrm{sgn}\left(\tau_\beta\right)$. In the example above, $\mathrm{sgn}(\beta) = -1$. We also define Z_λ^{-1} as the set of bijections $\alpha\!:\!(i,j)_{E\times N} \to \left(i_\alpha, j_\alpha\right)_{E\times N}$ on the squares of the shape λ such that if $(i,j)_{E\times N} \neq (k,l)_{E\times N}$, then $\left(i, j_\alpha\right)_{E\times N} \neq \left(k, l_\alpha\right)_{E\times N}$. If $\alpha \in Z_\lambda^{-1}$, then let $\alpha(T)$ denote the unique λ-tableau with the property that if a letter appears at $(i,j)_{E\times N}$ in T, then it also appears at $\left(i_\alpha, j_\alpha\right)_{E\times N}$ in $\alpha(T)$. Indeed, if we fix an injective tableau T, then the elements of Z_λ^{-1} correspond to the permutations $\alpha = \sigma_\alpha \tau_\alpha$ such that τ_α permutes the letters in the columns of T, creating an injective tableau K_α, and then σ_α permutes the letters in the rows of K_α, giving rise to $\alpha(T)$, as shown below, where $\tau_\alpha = (12)(34)$ and $\sigma_\alpha = (25)$:



The sign of $\alpha \in Z_\lambda^{-1}$ is defined to be $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\tau_\alpha)$.

In comparison with Littlewood [L], we point out that given $\beta \in Z_\lambda$, $\beta = \tau_\beta \sigma_\beta$, he does not consider β as a single bijection, acting on any λ–tableau, but rather as a product of two permutations τ and σ, acting on a fixed standard tableau T. Given T, if $\tau \in S_n$ is a permutation that permutes the letters in the columns of T, but fixes the rows, and if $\sigma \in S_n$ is a permutation that permutes the letters in the rows of T, but fixes the columns, then it can be shown that there exists a unique $\beta \in Z_\lambda$ such that $\beta = \tau_\beta \sigma_\beta$ and $\beta = \tau\sigma$. Moreover, $\operatorname{sgn}(\tau) = \operatorname{sgn}(\tau_\beta)$. Likewise, given $\beta \in Z_\lambda$, it can be shown that for any standard tableau T there is a unique permutation $\tau \in S_n$ that permutes the letters in the columns of T, but fixes the rows of T, and a unique permutation $\sigma \in S_n$ that permutes the letters in the rows of T, but fixes the columns of T, such that $\beta = \tau\sigma$. It is important to point out, however, that both τ and σ are defined with respect to T, and therefore it is difficult to think of $\tau\sigma$ as a single action, whereas τ_β is defined with respect to $\sigma_\beta(T)$, as we mentioned above. It is for this reason that we prefer to work with elements of Z_λ.

We are now in a position to describe the irreducible representations of $\mathrm{GL}_N(\mathbb{C})$ which Littlewood [L] constructs in his book. There is one irreducible representation for each partition $\lambda \succ n$, where $n \geq 1$ and $\ell(\lambda) \leq N$. The degree of the λth irreducible representation is given by the number of column strict tableaux of shape λ with letters from $1,\ldots,N$.

We start by drawing special attention to the cases when λ is a hook shape or $\lambda = 22$, as the description of the irreducible representations is then especially straightforward [L].

THEOREM 4.4.1 *Fix $\lambda \succ n$, where $\lambda = a1^{n-a}$, $1 \le a \le n$, or $\lambda = 22$. Fix $N \ge \ell(\lambda)$. Let $t_1, t_2, \ldots, t_s$ be the column strict tableaux of shape λ with letters from $1, \ldots, N$, and for all i, $1 \le i \le s$, let $1^{i_1} 2^{i_2} \cdots N^{i_N}$ denote the type of t_i. Let $\mathbf{A} = (a_{ij})_{1 \le i,j \le N} \in \mathrm{GL}_N(\mathbb{C})$, and let $\mathrm{M}_\lambda(\mathbf{A})$ be the $s \times s$ matrix with entries*

$$\mathrm{M}_\lambda(\mathbf{A})_{i,j} = \sum_{\beta \in Z_\lambda} \frac{1}{i_1! i_2! \cdots i_N!} \mathrm{G}_{\mathbf{A}}\left[\beta(t_i), t_j\right] \cdot \mathrm{sgn}(\beta)$$

where $1 \le i, j \le s$. Then $\mathrm{M}_\lambda(\mathbf{A})$ is the irreducible representation of $\mathrm{GL}_N(\mathbb{C})$ associated with the partition λ.

We may use the above theorem to consider some examples of representations of the general linear group $\mathrm{GL}_N(\mathbb{C})$. The simplest of these occurs when $n = 1$ and $\lambda = 1$. Order the column strict tableaux of shape 1 so that t_i has the letter i, and note that $i_1! i_2! \cdots i_N! = 1$. Then Z_1 has a single element id and therefore $\mathrm{M}_1(\mathbf{A})_{i,j} = \mathrm{G}_{\mathbf{A}}\left[t_i, t_j\right] = a_{ij}$. This means that $\mathrm{M}_1(\mathbf{A}) = \mathbf{A}$, and in particular, for all $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_N(\mathbb{C})$, it is true that $(\mathbf{A})(\mathbf{B}) = (\mathbf{AB})$.

If $\lambda = 1^N$, then there is a single column strict tableau t_1 of shape 1^N, and $i_1! i_2! \cdots i_N! = 1$ for all i, $1 \le i \le s$. The fact that t_1 consists of one column of length N means that each element $\beta \in Z_{1^N}$ corresponds to a permutation $\tau_\beta \in S_N$ such that $\mathrm{sgn}(\beta) = \mathrm{sgn}(\tau_\beta)$. Therefore $\mathrm{M}_{1^N}(\mathbf{A})_{1,1} = \sum_{\beta \in Z_{1^N}} \mathrm{G}_{\mathbf{A}}\left[\beta(t_1), t_1\right] \mathrm{sgn}(\beta) =$

$\sum_{\tau\in S_N} a_{\tau(1),1}a_{\tau(2),2}\cdots a_{\tau(N),N}\,\mathrm{sgn}(\tau)=\det\mathbf{A}$. We have that $\mathrm{M}_{1^N}(\mathbf{A})=(\det\mathbf{A})$, and in particular, for determinants it is indeed true that $(\det\mathbf{A})(\det\mathbf{B})=(\det\mathbf{AB})$.

If $\lambda=1^n$, then there are $\binom{N}{n}$ column strict tableaux of shape 1^n, and $i_1!i_2!\cdots i_N!=1$ for all i, $1\le i\le s$. If t_i consists of the letters $x_1(i)<x_2(i)<\ldots<x_n(i)$, and t_j consists of the letters $x_1(j)<x_2(j)<\ldots<x_n(j)$, then

$$\mathrm{M}_{1^n}(\mathbf{A})_{i,j}=\sum_{\tau\in S_n} a_{x_{\tau(1)}(i),x_1(j)}a_{x_{\tau(2)}(i),x_2(j)}\cdots a_{x_{\tau(n)}(i),x_n(j)}\,\mathrm{sgn}(\tau)\;=\det\left(a_{x_u(i),x_v(j)}\right)_{1\le u,v\le n}.$$

The entries of $\mathrm{M}_{1^n}(\mathbf{A})$ are precisely the determinants of $n\times n$ minors of $\mathbf{A}$.

If $\lambda=n$, then there are $\binom{N+n-1}{n}$ column strict tableaux of shape n. The fact that t_i consists of one row of length N means that each element $\beta\in Z_{1^N}$ corresponds to a permutation $\tau_\beta\in S_N$, and $\mathrm{sgn}(\beta)=+1$. If t_i consists of the letters $x_1(i)\le x_2(i)\le \ldots\le x_n(i)$, and t_j consists of the letters $x_1(j)\le x_2(j)\le\ldots\le x_n(j)$, then

$$\mathrm{M}_n(\mathbf{A})_{i,j}=\sum_{\tau\in S_n}\frac{1}{i_1!i_2!\cdots i_n!}a_{x_{\tau(1)}(i),x_1(j)}a_{x_{\tau(2)}(i),x_2(j)}\cdots a_{x_{\tau(n)}(i),x_n(j)}\;=\frac{1}{i_1!i_2!\cdots i_n!}\mathrm{per}\left(a_{x_u(i),x_v(j)}\right)_{1\le u,v\le n}$$

The i,jth entry of $\mathrm{M}_n(\mathrm{A})$ may be thought of as comparing permutations of the sequence $x_1(i)\le x_2(i)\le\ldots\le x_n(i)$ with the sequence $x_1(j)\le x_2(j)\le\ldots\le x_n(j)$. In particular, when $i=j$ this brings to mind the circuits by which we interpret $h_n(\xi_1,\ldots,\xi_n)$ in Theorem 2.2.2.

Of interest to us, with regard to this thesis, are the traces of the irreducible representations, because the trace of the irreducible representation $\mathrm{M}_\lambda(\mathbf{A})$ equals $s_\lambda(\xi_1,\ldots,\xi_N)$. For the cases that we have been considering, when λ is a hook shape or $\lambda=22$, we can record the following theorem [L].

THEOREM 4.4.2 *Fix $\lambda \succ n$, where $\lambda = a1^{n-a}$, $1 \le a \le n$, or $\lambda = 22$. Fix $N \ge \ell(\lambda)$. Let $t_1, t_2, \ldots, t_s$ be the column strict tableaux of shape λ with letters from $1, \ldots, N$, and for all i, $1 \le i \le s$, let $1^{i_1} 2^{i_2} \cdots N^{i_N}$ denote the type of t_i. Then*

$$s_\lambda(\xi_1, \ldots, \xi_N) = \sum_{\substack{\beta \in Z_\lambda \\ 1 \le i \le s}} \frac{1}{i_1! i_2! \cdots i_N!} \mathrm{G}_{\mathrm{A}}[\beta(t_i), t_i] \cdot \mathrm{sgn}(\beta)$$

The above expression allows us to think of the Schur function $s_\lambda(\xi_1, \ldots, \xi_N)$ as comparing, for all column strict tableaux $t_1, t_2, \ldots, t_s$ of shape λ, the tableau t_i with all of the tableaux $\beta(t_i)$ that are gotten from it by actions $\beta \in Z_\lambda$. For example, if $\lambda = n$, then this gives us the Vere-Jones identity that we mentioned in our discussion of Theorem 3.1.1.

In general, however, the expression that Littlewood gives for $\mathrm{M}_\lambda(\mathrm{A})$ is a bit more complicated, as is the trace $\mathrm{tr}(\mathrm{M}_\lambda(\mathrm{A})) = s_\lambda(\xi_1, \ldots, \xi_N)$. Given a column strict tableau t_i of type $1^{i_1} 2^{i_2} \cdots N^{i_N}$, let $\hat{t}_i$ denote the standard tableau gotten from t_i by replacing the occurrences of q with the letters $i_1 + i_2 + \ldots + i_{q-1} + 1, \ldots, i_1 + i_2 + \ldots + i_{q-1} + i_q$, from left to right, for all q, $1 \le q \le N$. The following results are taken from Littlewood's [L] book, albeit presented in a notation different from his.

THEOREM 4.4.3 *Fix $\lambda \succ n$, $N \ge \ell(\lambda)$. Let $t_1, t_2, \ldots, t_s$ be the column strict tableaux of shape λ with letters from $1, \ldots, N$, and for all i, $1 \le i \le s$, let $1^{i_1} 2^{i_2} \cdots N^{i_N}$ denote the type of t_i. For all j, $1 \le j \le s$, let B_j denote the set of finite sequences $(\beta_1, \beta_2, \ldots, \beta_r, \beta_{r+1})$, $r \ge 0$, such that $\beta_{r+1} = id \in Z_\lambda^{-1}$ and if $1 \le h \le r$, then $\beta_h \in Z_\lambda^{-1}$, $\beta_h \ne id$, and $\beta_h \beta_{h+1} \cdots \beta_r(\hat{t}_j)$ is a standard tableau. Let $\mathrm{A} = (a_{ij})_{1 \le i, j \le N} \in \mathrm{GL}_N(C)$, and let $\mathrm{M}_\lambda(\mathrm{A})$ be the $s \times s$ matrix with entries*

$$\mathrm{M}_\lambda(\mathrm{A})_{i,j} = \sum_{\substack{\beta_0 \in Z_\lambda \\ (\beta_1,\beta_2,\ldots,\beta_r,id)\in B_i}} \left(\frac{1}{i_1!i_2!\cdots i_N!} \mathrm{G}_\mathrm{A}\left[\beta_0(t_i),\beta_1\beta_2\cdots\beta_r\cdot id(t_j)\right]\cdot(-1)^r \prod_{k=0}^{r} \mathrm{sgn}(\beta_k) \right)$$

where $1 \le i,j \le s$. *Then* $\mathrm{M}_\lambda(\mathrm{A})$ *is the irreducible representation of* $\mathrm{GL}_N(\mathbb{C})$ *associated with the partition* λ. *Furthermore,*

$$s_\lambda(\xi_1,\ldots,\xi_N) = \sum_{\substack{\beta_0 \in Z_\lambda \\ (\beta_1,\beta_2,\ldots,\beta_r,id)\in B_i \\ 1\le i \le s}} \left(\frac{1}{i_1!i_2!\cdots i_N!} \mathrm{G}_\mathrm{A}\left[\beta_0(t_i),\beta_1\beta_2\cdots\beta_r\cdot id(t_i)\right]\cdot(-1)^r \prod_{k=0}^{r} \mathrm{sgn}(\beta_k) \right)$$

If λ is a hook shape or $\lambda = 22$, then for all j, $1 \le j \le s$, it is not too hard to show that the set B_j consists of the sole element (id), and the above theorem reduces to Theorems 4.4.1 and 4.4.2. But it is true even in the general case that for many j the only element in B_j is (id), in which case the entries $\mathrm{M}_\lambda(\mathrm{A})_{i,j}$ can be calculated as in Theorem 4.4.1. A case where B_j is not trivial is when t_j is the column strict tableau

5	
3	**4**
1	**2**

Then B_j contains not only (id), but also the element (β_1, id) where $\beta_1(t_j)$ is the column strict tableau shown below

3	
2	5
1	4

With regard to this thesis, it is not clear to us whether the formula for $s_\lambda(\xi_1,\ldots,\xi_N)$ in Theorem 4.4.3 tells us anything more than if we use an appropriate interpretation of $\chi^\lambda(\sigma)$ with the formula

$$s_\lambda(\xi_1,\ldots,\xi_N)=\sum_{\substack{w\in\mathbf{N}^n\\ w=1^{m_1}2^{m_2}\cdots N^{m_N}}}\frac{1}{m_1!m_2!\cdots m_N!}\cdot \mathrm{Imm}_{\chi^\lambda}\mathbf{A}_w,$$

from Section 3.1, due to Littlewood [L]. However, on behalf of Theorem 4.4.3 we may point out that the structure of the set B_j is not as daunting as it may at first seem. The reason is that given standard tableau T and U, if there exists a sequence $\beta_1,\beta_2,\ldots,\beta_r$ of elements of Z_λ^{-1} such that $U=\beta_1\beta_2\cdots\beta_r(T)$, then there does not exist any sequence $\theta_1,\theta_2,\ldots,\theta_w$ of elements of Z_λ^{-1} such that $T=\theta_1\theta_2\cdots\theta_w(U)$.

We may also remark that irreducible representations of S_N may be gotten from entries of $\mathrm{M}_\lambda(\mathbf{A})$ in the following way. Let $\lambda\succ n=N$. If $t_1,t_2,\ldots,t_s$ are the column strict tableaux of shape λ, then let $t_1,t_2,\ldots,t_d$ be the standard tableaux of shape λ. For all $\sigma\in S_N$, define $\mathbf{M}(\sigma)_{i,j}=1$ if $j=\sigma(i)$ and $\mathbf{M}(\sigma)_{i,j}=0$ if $j\neq\sigma(i)$. If we set $\mathbf{C}_\lambda(\sigma)_{i,j}=\mathrm{M}_\lambda(\mathbf{M}(\sigma))_{i,j}$ for all i, j, $1\le i,j\le d$, then $\mathbf{C}_\lambda(\sigma)$ is an irreducible representation of S_N. In fact, $\{\mathbf{C}_\lambda(\sigma)\}_{\lambda\succ N}$ are the irreducible representations of S_N known as Young's natural representations of S_N.

It is evident that the irreducible representations $\{\mathrm{M}_\lambda(\mathrm{A})\}_{\substack{\lambda \vdash n \\ 1\le n\le N}}$ of the general linear group $\mathrm{GL}_N(\mathbb{C})$ capture a greater amount of combinatorial information than do the irreducible representations $\{\mathrm{C}_\lambda(\sigma)\}_{\lambda\vdash n}$ of the symmetric groups S_n, $1\le n\le N$. Analogously, the symmetric functions $s_\lambda(\xi_1,\ldots,\xi_N)=\mathrm{tr}(\mathrm{M}_\lambda(\mathrm{A}))$ capture a greater amount of combinatorial information than do the irreducible characters $\chi^\lambda(\sigma)=\mathrm{tr}(\mathrm{C}_\lambda(\sigma))$.

The representations of $\mathrm{GL}_N(\mathbb{C})$ that we have described in this section appear rather straightforward when λ is a hook shape, but in general seem overly complicated. Historically, the representations $\{\mathrm{M}_\lambda(\mathrm{A})\}_{\substack{\lambda \vdash n \\ 1\le n\le N}}$ of $\mathrm{GL}_N(\mathbb{C})$ came later than the representations $\{\mathrm{C}_\lambda(\sigma)\}_{\lambda\vdash n}$ of S_n. It is the combinatorics of the symmetric functions which ultimately helped make possible the representation theory of S_n. But evaluating these same functions at the eigenvalues $\xi_1,\ldots,\xi_n$ of an $N\times N$ matrix $\mathbf{A}$, as we have done in this thesis, may present clues as to how more straightforward representations of $\mathrm{GL}_N(\mathbb{C})$ might be discovered. Such representations would in turn simplify the representation theory of S_n, and provide us with deeper intuition about the symmetric group and the symmetric functions as well.

CHAPTER 5

CONCLUSION

Our goal in this thesis has been to evaluate six bases of the symmetric functions at the eigenvalues $\xi_1,\dots,\xi_N$ of an $N \times N$ matrix $\mathbf{A}$. We have presented four combinatorial interpretations for the homogeneous symmetric functions $h_n(\xi_1,\dots,\xi_N)$, two for the forgotten symmetric functions $f_\lambda(\xi_1,\dots,\xi_N)$, two for the Schur functions $s_\lambda(\xi_1,\dots,\xi_N)$, and one each for the elementary $e_n(\xi_1,\dots,\xi_N)$, power $p_n(\xi_1,\dots,\xi_N)$, and monomial $m_\lambda(\xi_1,\dots,\xi_N)$ symmetric functions. In each case the specialization $a_{ij}=0$, $i \neq j$, recovers the usual definitions of the six bases.

Surprisingly, of all the six bases it is the basis of the forgotten symmetric functions which has arguably the most elegant interpretation upon evaluation at the eigenvalues $\xi_1,\dots,\xi_N$ of $\mathbf{A}$. Theorem 3.3.1 describes the forgotten symmetric functions $f_\lambda(\xi_1,\dots,\xi_N)$ in terms of Lyndon words on an alphabet of upper case and lower case letters. It is satisfying to observe how these functions interpolate between the closed walks generated by $p_n(\xi_1,\dots,\xi_N)$ and the multisets of Lyndon words generated by $h_n(\xi_1,\dots,\xi_N)$. It is true that the bases $p_n(\xi_1,\dots,\xi_N)$, $h_n(\xi_1,\dots,\xi_N)$, and $e_n(\xi_1,\dots,\xi_N)$ also have elegant combinatorial descriptions, but these bases are all multiplicative and therefore no such interpolation takes place. Furthermore, the terms of $f_\lambda(\xi_1,\dots,\xi_N)$ all have the same sign $\operatorname{sgn}(\lambda)$, and this is not true of either $s_\lambda(\xi_1,\dots,\xi_N)$ or $m_\lambda(\xi_1,\dots,\xi_N)$. The prominent role that Lyndon words play in this thesis draws further attention to the forgotten symmetric functions. It seems worthwhile to examine the relations between the homogeneous symmetric functions and the forgotten symmetric functions with the free Lie Algebra, the free monoid, and the Poincaré-Birkhoff-Witt theorem [G]. Indeed, Lyndon words on an alphabet of upper case and lower case letters arise in the study of the hyperoctahedral group B_n [B]. In summary, we suggest that the forgotten symmetric functions no longer suffer the neglect that they have in the past.

Another surprise of this thesis is the grace with which Theorem 4.1.1 and Theorem 4.3.1 generalize the usual definitions of the Schur functions as a quotient of alternants and as a generating function for column strict tableaux, respectively. The fact that Schur functions $s_\lambda(\xi_1,\ldots,\xi_N)$ are characters of the general linear group $GL_n(\mathbb{C})$ suggests that these results may be helpful in the search for more satisfactory representations of the general linear group and also the symmetric group.

Although the results of this thesis hold for arbitrary $\mathbf{A} \in GL_N(\mathbb{C})$, the matrix $\mathbf{A}$ may be profitably specialized. For example, suppose that $\mathbf{A}$ is the matrix below, in rational canonical form.

$$\mathbf{A} = \begin{pmatrix} b_1 & 1 & & 0 & 0 \\ b_2 & 0 & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & 1 & 0 \\ b_{N-1} & 0 & & 0 & 1 \\ b_N & 0 & \cdots & 0 & 0 \end{pmatrix}$$

If the eigenvalues of $\mathbf{A}$ are $\xi_1,\ldots,\xi_N$, then $b_i = (-1)^{i-1} e_i(\xi_1,\ldots,\xi_N)$ for all i, $1 \le i \le N$. The only cycles with nonzero weight are $a_{12}a_{23}\cdots a_{i1} = (-1)^{i-1} e_i(\xi_1,\ldots,\xi_N)$ where $1 \le i \le N$. The fact that this is true for all values of $\xi_1,\ldots,\xi_N$ means that by specializing in this way we are expressing any symmetric function $F(\xi_1,\ldots,\xi_N)$ in terms of the elementary symmetric functions $\{e_\mu(\xi_1,\ldots,\xi_N)\}_{\mu \vdash n, n \ge 0}$. For example, if F is f_λ, then from Theorem 3.3.1 we find a combinatorial interpretation for the transition matrix in the equation $f_\lambda = \sum_{\mu \vdash n} \mathbf{M}(f,e)_{\lambda,\mu} e_\mu$. Applying the involution ω then gives us a long sought interpretation for the entries of the transition matrix in the equation $m_\lambda = \sum_{\mu \vdash n} \mathbf{M}(m,h)_{\lambda,\mu} h_\mu$. Likewise, by Theorem 3.4.1 we find a combinatorial interpretation for the transition matrices in the equations $m_\lambda = \sum_{\mu \succ n} \mathbf{M}(m,e)_{\lambda,\mu} e_\mu$ and $f_\lambda = \sum_{\mu \succ n} \mathbf{M}(f,h)_{\lambda,\mu} h_\mu$. In a similar spirit, this specialization transforms the quotient formula of Theorem 4.1.1 into a determinantal identity in terms of Schur functions of hook shape. If $\mathbf{A}$ is defined as above, then $\left(\mathbf{A}^k\right)_{jj} = s_{(k-(j-1)),1^{(j-1)}}$ whenever $1 \le j \le k$, and $\left(\mathbf{A}^k\right)_{jj} = 0$ if $j > k$. Therefore

$$s_\lambda = \det\left(s_{(\lambda_i+\delta_i-(j-1)),1^{(j-1)}}\right)_{1\le i,j\le N} \Big/ \left(e_0 e_1 \cdots e_{N-1}\right)$$

with the understanding that if $\lambda_i + \delta_i = 0$, then $s_{(\lambda_i+\delta_i-(j-1)),1^{(j-1)}} = (-1)^{j-1}$, and otherwise, if $j > \lambda_i + \delta_i$, then $s_{(\lambda_i+\delta_i-(j-1)),1^{(j-1)}} = 0$. We emphasize that these results hold independently of $\mathbf{A}$ because they are true for all values of $\xi_1,\ldots,\xi_N$. This makes all the more remarkable the power of specialization. It is our hope that specializing the matrix $\mathbf{A}$ in this and other ways will add a new tool to Algebra's tool chest.

Finally, this thesis suggests that evaluating symmetric functions at the eigenvalues $\xi_1,\ldots,\xi_N$ of a matrix $\mathbf{A}$ offers a unifying framework for the discussion of many of the results of matrix algebra. If nothing else, it provides a way of teaching these results that makes them tangible, but also makes evident their place with regard to deeper algebraic issues, such as the theory of representations. The research problems suggested by this approach are concrete, reasonable, of beauty, and of consequence.

APPENDIX: SCHUR FUNCTIONS GENERATE COLUMN STRICT TABLEAUX

The purpose of this appendix is to document a new proof of the old fact that the Schur function $s_\mu(x_1,\ldots,x_N)$ is a generating function for the column strict tableaux of shape μ. We take the definition of the Schur function to be the quotient of alternants, and then expand it as an infinite series. All but finitely many of these terms cancel away, and the surviving terms correspond to column strict tableau.

We define the Schur function to be the quotient of alternants $s_\mu(x_1,\ldots,x_N) = \det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \Big/ \det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$, as in Section 1.1. Here $\delta_j = N-j$. Our aim is to give a combinatorial proof of the following result.

THEOREM A.1 *Let* U *be the set of sequences* $\nu=(\nu_1,\nu_2,\ldots,\nu_N)$ *for which* $\nu_i \ge 0$, $1\le i\le N$, *and* $\nu_1+\nu_2+\ldots+\nu_N = n$. *Let* $K_{\mu,\nu}$ *be the number of column strict tableaux of shape* $\mu \succ n$ *and type* $1^{\nu_1}2^{\nu_2}\cdots N^{\nu_N}$. *Then*

$$s_\mu(x_1,\ldots,x_N) = \det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \Big/ \det\left(x_i^{\delta_j}\right)_{1\le i,j\le N} = \sum_{\nu\in U} K_{\mu,\nu} x_1^{\nu_1}x_2^{\nu_2}\cdots x_N^{\nu_N}.$$

The proof that we present is new in that it does not require that we first cross multiply by $\det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$. We start by recalling from Section 1.1 the well known identity $\prod_{1\le i<j\le N}(x_i-x_j) = \det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$. Our idea is to expand the denominator $\prod_{1\le i<j\le N} 1/(x_i-x_j)$ as an infinite series. We then divide top and bottom by $x_1x_2\cdots x_N$, which serves a technical purpose. This gives

$$\begin{aligned} s_\mu &= \det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \prod_{1\le i<j\le N} \frac{1}{(x_i-x_j)} \\ &= \det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \frac{1}{\left(x_1^{\delta_1}x_2^{\delta_2}\cdots x_N^{\delta_N}\right)} \prod_{1\le i<j\le N} \frac{1}{\left(1-x_j/x_i\right)} \end{aligned}$$

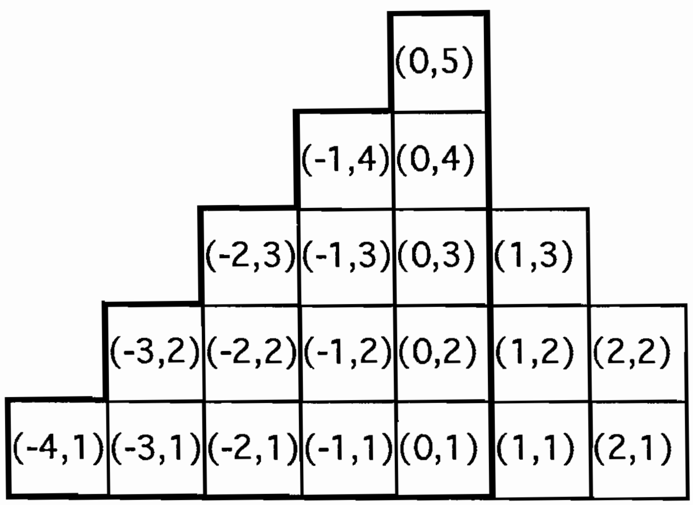
$$= \det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \frac{1}{\left(x_1^{\delta_1}x_2^{\delta_2}\cdots x_N^{\delta_N}\right)} \prod_{1\le i<j\le N}\left(1+\frac{x_j}{x_i}+\frac{x_j^2}{x_i^2}+\ldots\right)$$

$$= \left[\det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N} \prod_{1\le i<j\le N}\left(1+\frac{x_j}{x_i}+\frac{x_j^2}{x_i^2}+\ldots\right)\right]\cdot\left[\frac{x_1x_2\cdots x_N}{\left(x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}\right)}\right]$$

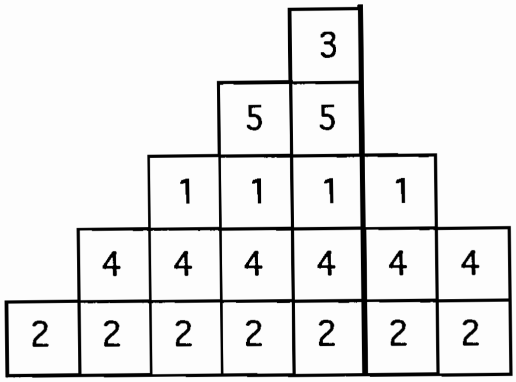
$$= \left[\det\left(x_i^{\mu_j+\delta_j+1}\right)_{1\le i,j\le N} \prod_{1\le i<j\le N}\left(1+\frac{x_j}{x_i}+\frac{x_j^2}{x_i^2}+\ldots\right)\right]\cdot\left[\frac{1}{\left(x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}\right)}\right]$$

Consider the left factor of the last product. We give a combinatorial interpretation for the infinitely many elements in this factor, and then give a bijection to show how they cancel away, leaving only finitely many. Those that are left are all divisible by $x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}$, which we divide away, and the terms that we are left with are interpreted as column strict tableaux.

We use an $E\times N$ coordinate system to locate the squares of an augmented shape $\Delta_N\cup\mu$. For all j, $1\le j\le N$, the jth row of $\Delta_N\cup\mu$ extends the jth row of μ by $N+1-j$ squares, as shown below.

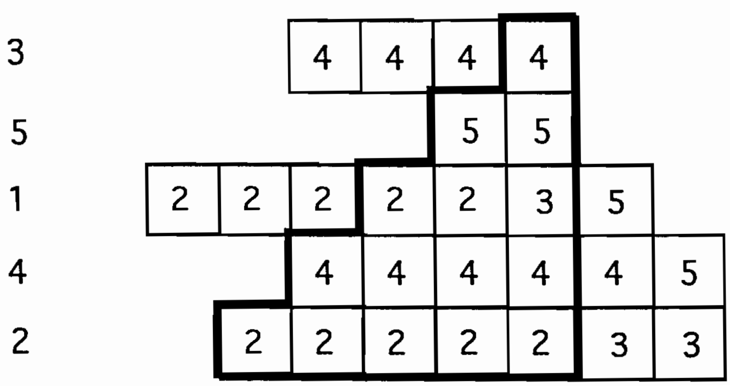


We interpret the term $\operatorname{sgn}(\sigma)x_{\sigma^{-1}(1)}^{\delta_1+1+\mu_1}x_{\sigma^{-1}(2)}^{\delta_2+1+\mu_2}\cdots x_{\sigma^{-1}(N)}^{\delta_N+1+\mu_N}$ from $\det\left(x_i^{\mu_j+\delta_j+1}\right)_{1\le i,j\le N}$ as filling the $\sigma(i)$th row of $\Delta_N\cup\mu$ with i's, as shown below.



We keep in mind the labels i associated with the $\sigma(i)$th row because they determine the sign of the monomial and because we are about to change the contents of the rows. This is because the quotients x_j/x_i work as raising operators and $(x_j/x_i)^k$ will be understood as replacing k of the elements of the row of i's with j's.

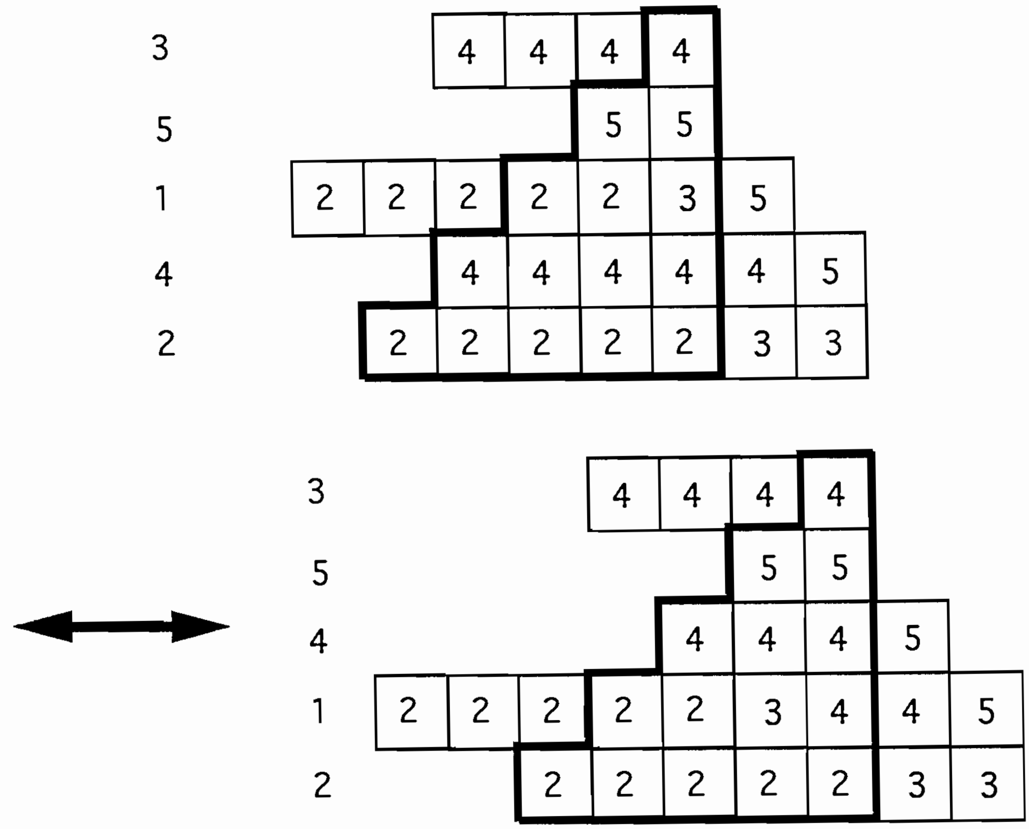
To make sense of this last statement, consider a term in our expression for s_μ and the quotients x_j/x_i that contribute to it, where i is fixed and $i < j$. The product of these quotients may be written $\prod_j (x_j/x_i)^{k_j}$. In the row with the i's remove the rightmost $\sum_{1\le j\le m} k_j$ i's, or as many as there are if there are fewer. Write in their place the letters $i+1$ through N, with k_j of the j's, starting with the N's at the rightmost square of the row, and weakly decreasing going from right to left, possibly extending beyond the leftmost square of $\Delta_N \cup \mu$. Do likewise for all of the rows, as illustrated below.



This yields a description of s_μ as a sum of weights on an infinite set $\{X\}_{\mu,N}$ of combinatorial objects. The sign of each object is given by $\mathrm{sgn}(\sigma)$.

Note that the squares outside of the augmented shape $\Delta_N \cup \mu$ are weighted differently than those within it. If a letter j is in $\Delta_N \cup \mu$, then its weight is x_j. If a letter j is to the left of $\Delta_N \cup \mu$, then its weight x_j/x_i depends on the label i that σ has assigned to its row. The most crucial point in the argument to follow is that if the row with label j contains j, then that row does not extend to the west of the augmented shape.

We now describe an involution $X \to \overline{X}$ by which the objects of $\{X\}_{\mu,N}$ cancel away so that only finitely many are left, all with positive sign. Given X, for each square $(e,n)_{E\times N}$ let $X(e,n)_{E\times N}$ be the letter which appears at that square. Look for pairs of squares $(e,n)_{E\times N}$, $(e,n+1)_{E\times N}$ such that $X(e,n)_{E\times N} > X(e,n+1)_{E\times N}$. If no such pair exists, then set $\overline{X} = X$. Otherwise, find the pairs for which e is largest, and among these the pair such that n is smallest. Derive $\overline{X}$ from X by letting $\overline{X}(c-1,n)_{E\times N} = X(c,n+1)_{E\times N}$ for all $c \le e$, $\overline{X}(c,n+1)_{E\times N} = X(c-1,n)_{E\times N}$ for all $c \le e$, and $\overline{X}(c,d)_{E\times N} = X(c,d)_{E\times N}$ for all other squares $(c,d)_{E\times N}$. This has the effect of switching along northeasterly lines the letters in the nth and $n+1$th rows that are to the left of $(e,n)_{E\times N}$. In addition, switch the labels on the nth and $n+1$th rows, setting $\overline{\sigma}^{-1}(n) = \sigma^{-1}(n+1)$, $\overline{\sigma}^{-1}(n+1) = \sigma^{-1}(n)$, and $\overline{\sigma}^{-1}(d) = \sigma^{-1}(d)$ for all other d, $1 \le d \le N$, as illustrated below, where $e = 0$ and $n = 2$.

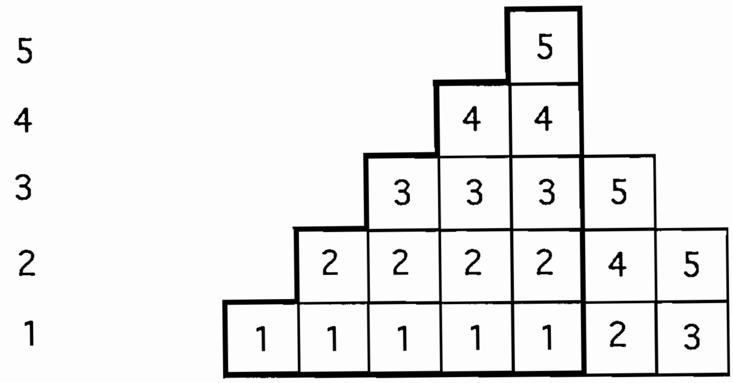


Assuming that the pair $(e,n)_{E\times N}$, $(e,n+1)_{E\times N}$ exists, we show that $\overline{X} \in \{X\}_{\mu,N}$. Are the numbers in the rows of $\overline{X}$ weakly increasing from left to right ? It is enough to show that $\overline{X}(e-1,n)_{E\times N} \leq \overline{X}(e,n)_{E\times N}$ and $\overline{X}(e,n+1)_{E\times N} \leq \overline{X}(e+1,n+1)_{E\times N}$. The first equation follows from the fact that $\overline{X}(e-1,n)_{E\times N} = X(e,n+1)_{E\times N} \leq X(e,n)_{E\times N} = \overline{X}(e,n)_{E\times N}$. Assuming that $(e+1,n+1)_{E\times N}$ exists, we prove the second equation by noting that $X(e+1,n)_{E\times N} < X(e+1,n+1)_{E\times N}$, because otherwise the pair $(e+1,n)_{E\times N}$, $(e+1,n+1)_{E\times N}$ would have been chosen instead of $(e,n)_{E\times N}$, $(e,n+1)_{E\times N}$. This observation and the facts $\overline{X}(e,n+1)_{E\times N} = X(e-1,n)_{E\times N}$ and $X(e+1,n+1)_{E\times N} = \overline{X}(e+1,n+1)_{E\times N}$ together imply $\overline{X}(e,n+1)_{E\times N} = X(e-1,n)_{E\times N} \leq X(e+1,n)_{E\times N} < \overline{X}(e+1,n+1)_{E\times N}$. Note also that the labels n and $n+1$ are switched accordingly, and therefore it is still true that for all j, $1 \leq j \leq N$, the letters in the row with label j are all greater than or equal to j.

Observe that X and $\overline{X}$ have the same weight. This is because the letters involved are not changed, but simply moved along northeasterly lines, which means that if they were outside of $\Delta_N \cup \mu$, they stay outside, and if they were inside, they stay inside. Furthermore, if j is the label for a row, then the number of squares outside of $\Delta_N \cup \mu$ that are in the row associated with label j remains the same for all j, $1 \leq j \leq N$. Observe also that X and $\overline{X}$ have opposite sign because σ and $\overline{\sigma}$ do.

This construction of $\overline{X}$ from X defines an involution because $\overline{X}(e,n+1)_{E\times N} = X(e+1,n)_{E\times N} \leq X(e,n)_{E\times N} = \overline{X}(e,n)_{E\times N}$ and because all of the squares disturbed are strictly to the left of e, except for $(e,n+1)_{E\times N}$, which is in the same column as $(e,n)_{E\times N}$, but above $(e,n)_{E\times N}$. Therefore $(e,n)_{E\times N}$, $(e,n+1)_{E\times N}$ remains the pair of highest priority and a second application of the construction returns us from $\overline{X}$ to X. In summary, $X \to \overline{X}$ is a sign reversing weight preserving involution.

What are the fixed points of the involution ? Only those objects survive for which $X(e,n)_{E\times N} < X(e,n+1)_{E\times N}$ for all $(e,n)_{E\times N}$, $(e,n+1)_{E\times N}$. This property is what is meant by the words "column strict": the letters increase strictly as we move north along a column. We now note that in X there is a column with N squares. It is the first column to the left of the shape μ, and the squares are $(0,1)_{E\times N},\ldots,(0,N)_{E\times N}$. The letters $X(0,1)_{E\times N},\ldots,X(0,N)_{E\times N}$ in this column must be all distinct, and they must be chosen from $1,\ldots,N$. Therefore $X(0,k)_{E\times N} = k$ for all k, $1 \leq k \leq N$. This is the crucial point, for it implies by induction on k that $\sigma^{-1}(k) = k$ and therefore σ is necessarily the identity permutation. But recall that if a row is associated with k and the letter k appears in that row, then the letters in that row do not extend outside of the augmented shape $\Delta_N \cup \mu$. Therefore, given a fixed point of the involution, no row extends outside of the augmented shape. Furthermore, the letters in the kth row to the left of $(0,k)_{E\times N}$ must all be equal to $X(0,k)_{E\times N} = k$, as shown below.



This defines a unique way of assigning letters to the squares that are to the left of the shape μ. The part of the weight of X that comes from these squares is $x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}$. Moreover, $\mathrm{sgn}(X)=+1$.

As for the squares in μ, if X is a fixed point, then the letters in the squares of μ must define a column strict tableau. Each tableau occurs at most once, because the only permutation that can be associated with X is the identity permutation. But each tableau occurs exactly once, because if the letters in the shape μ define a column strict tableau, and the letters in the rest of $\Delta_N \cup \mu$ are as described above, then X is fixed by the involution.

Given any fixed point X, we may remove the squares to the left of μ, which taken together have weight $x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}$, and cancel $x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}$ with $1/x_1^{\delta_1+1}x_2^{\delta_2+1}\cdots x_N^{\delta_N+1}$. The weight of what remains is given by the product of the weights of the letters in the shape μ. Therefore $\det\left(x_i^{\mu_j+\delta_j}\right)_{1\le i,j\le N}\Big/\det\left(x_i^{\delta_j}\right)_{1\le i,j\le N}$ $=\sum_{\nu\in U} K_{\mu,\nu}x_1^{\nu_1}x_2^{\nu_2}\cdots x_N^{\nu_N}$, as in the statement of the theorem. QED

The idea to divide by $x_1x_2\cdots x_N$ belongs to Carbonara. It spares the need for an additional involution.

REFERENCES

[B] N.Bergeron, (1990). On the hyperoctahedral group, Ph.D. thesis, University of California, San Diego.

[BR] R. A. Brualdi and H. J. Ryser, (1991). Combinatorial Matrix Theory, *Encyclopedia of mathematics and its applications*, Vol.39, Cambridge University Press, Cambridge.

[CF] P.Cartier and D.Foata, (1969). Problèmes combinatoires de commutation et réarrangements, *Springer Lecture Notes* **85**, 1-88.

[C] W.C.Chu, (1983). Some algebraic identities concerning determinants and permanents, *Linear Alg. Applics.*, **116**, 35-40.

[ER1] O. Eğecioğlu and J. B. Remmel, (1990). A combinatorial interpretation of the inverse Kostka matrix, *Linear and Multilinear Algebra,* **26**, 59-84.

[ER2] O. Eğecioğlu and J. B. Remmel, (1990). The monomial symmetric functions and the Frobenius map, *Journal of Combinatorial Theory (Series A),* **54**, 272-295.

[G] A. M. Garsia, (1988). Combinatorics of the free Lie algebra and the symmetric group, to appear in a volume commemorating J.Moser's sixtieth birthday.

[GM] A. M. Garsia and T. J. McLarnan, (1987). Relations between Young's natural and the Kazhdan-Lusztig representations of S_n, *Advances in Mathematics,* preprint.

[Go] C. D. Godsil, (1993). *Algebraic combinatorics*, Chapman & Hall, New York.

[GJ] I. P. Goulden and D. M. Jackson, (1992). Immanants, Schur functions, and the MacMahon Master theorem, *Proceedings of the American Mathematical Society,* **115**, 605-612.

[H] I.N.Herstein, (1975). *Topics in Algebra*, 2nd edn., John Wiley&Sons, New York.

[L] D. E. Littlewood, (1950). *The theory of group characters*, 2nd edn., Oxford University Press.

[Lo] M. Lothaire, (1983). *Combinatorics on words,* Addison-Wesley, Reading, Mass..

[M] I.G.Macdonald, (1979). *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford.

[R] D.E.Rutherford, (1964). The Cayley-Hamilton theorem for semirings, *Proc. Roy. Soc. Edinburgh Sec.A,* **66**, 211-215.

[Se] J-P.Serre, (1977). Linear Representations of Finite Groups, Springer-Verlag, New York.

[S] R.P.Stanley, (1986). *Enumerative Combinatorics*, Vol.1, Wadsworth& Brooks/Cole, Monterrey.

[SW] D.W.Stanton and D.E.White, (1985). A Schensted algorithm for rim hook tableaux, *Journal of Combinatorial Theory (Series A)*, **40**, 211-247.

[St1] J.R.Stembridge, (1989). On the eigenvalues of representations of reflection groups and wreath products, *Pacific Journal of Mathematics*, **2**, 353-396.

[St2] J.R.Stembridge, (1990). Some conjectures for immanants, second preliminary draft.

[Str] H.Straubing, (1983). A combinatorial proof of the Cayley-Hamilton theorem, *Discrete Mathematics*, **43**, 273-279.

[VJ] D.Vere-Jones, (1984). An identity involving permanents, *Linear Alg.Applics.*, **63**, 267-270.

[W] D.E.White, (1983). A bijection proving orthogonality of the characters of S_n, *Advances in Mathematics*, **50**, 160-186.

[Z] D.Zeilberger, (1985). A combinatorial approach to matrix algebra, *Discrete Mathematics*, **56**, 61-72.