NON-EUCLIDEAN SPACE

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Introduction

The reflections in this paper are drawn from the writings of Husserl and from the commentaries and critiques of his work by Becker, Strocker, and Claesges. The intent of the phenomenological analysis is to delimit the essential differences between the Euclidean and non-Euclidean understanding of space. The issues which will appear during the analysis are the following: (i) to what extent is the Euclidean geometry present or accessible to sense intuition and to what extent categorial intuition plays a role in its experience? (ii) what kinds of modes of presentation are required in Euclidean geometry? (iii) the role, if any of sense-intuition in non-Euclidean geometry and the mode of presentation of such a geometry. Indeed, other, and somewhat more periphery questions will appear in the discussion, yet the main focus will be on the deciphering the essence of phenomenological experience of Euclidean and non-Euclidean geometries. This means that all other modalities of having space, such as "lived-space," "practical space" and even what some call "aesthetic" and "psychological spaces, will be bracketed. While they may have a relationship of being founded in, for example, the geometry of the Euclidean space, such a founding will not be taken into consideration. Such a relationship of founding would take the discussion too far afield from its main task.

The discussion will avoid details and will adhere to the exposition of the essential aspects of the problematic of geometrical understanding of space. At the same time it will avoid technical-mathematical constructions, since such constructions constitute a domain of investigations of their own, with the added problematic of relating mathematical ideality to the reality of space.

Preliminary Considerations

Sense intuition is depth-intuition, and possible only for an intentionally directed corporeal entity. It is also a finite intuition, limited by a horizonal domain of space. What is known as the normal structural properties of space, linearity and plane are accessible from here. It is to be noted that linearity, or line, does not transcend but subtends all spatial measures. This means that it is not a metric but a topological concept. Analogously, the same can be said of the plane. Just like the line, it also is a topological concept; it is not derived from addition of parts.

The space of intuition is structured in terms of depth and width extension. The depth of a corporeally centered space is formally distinguishable from the breadth expansion insofar as it possesses a univocity of orientation. While in the extensional mode two points can be related in various ways, in the extension of depth this is possible only in one precise way: to be behind another. This being behind one another acquires its order directly from the intuiting corporeity. Simultaneously, corporeity has a univocal criterion for the fulfillment of this order to what phenomenally constitutes the "overlay" of two points laying one behind the other. Although this overlay may seem to be purely visual, the intuitive grasp is needed for the apperception of being one behind the other despite the visual coincidence of the two points. This is the originary constitution of straight linearity. The intuitive view would encounter any series of phenomenal points in the just described position if it could penetrate the ones in front.

Besides the depth intuition, as the originary apperception of straight linearity, the intuited space is structured extensionally comprising a manifold of pure relationships of things. Together with depth, it constitutes the structure of space in which the intuiting corporeal subject exists in a twofold manner: on the one hand, as being in and among, and on the other as being against the states of affairs.

This space is homogenous and open ended; it is conceivable in terms of the univocal continuation of thing relationships predominating in the finite domain. The characteristic relationships predominating among things contains the experience of one behind the other in spatial depth. In accordance with this experience, it can be continued indefinitely and hence such spatiality can be understood as an open-ended progression. Homogenization of space the center, as a corporeal point of departure, can be reduced to any phenomenal point of space—no longer having any special meaning. Point among points, it can be not only arbitrarily changed, but it must also forfeit any special character as a point of departure of something, if the homogeneity of space is to be maintained. What remains is purely formal meaning of straight linearity.
It no longer contains the postulation of a corporeal function and, while being without any point of inception, it is conceived as open ended in two directions, i,e., a straight line. In its free movement, it is generative of the all-sided and open ended plane as a structural element of the homogenous space of objects. That it is open-endless means nothing other than that it is a space structured in lines and planes.

This space of course corresponds topologically with the space of the natural consciousness of objects. This correspondence is seen in terms of the direct relationship between the constitutive structure of stretch and the intuitive form-characteristics of the spatial world of things. Stretch is the measurable moment of spatiality and indeed of the relationships among things. In this sense, the measurability of a stretch establishes for Euclidean space a direct founding in the natural space of objects, both metrically and topologically. This founding is what lends to all geometries their primordial meaning.

If we raise the question of intuitability in Euclidean geometry, the answer will have to be that intuitability here means of an open plane. This is of course distinct from a mere sense-intuition. Yet this intuitability is subtended by the present possibility to interpret the geometric states of affairs as morphological forms of the sensibly intuited world of things. The morphological forms are such experientially present phenomena as edges, corners, surfaces, curves, etc. In brief, the intuitability of plane of Euclidean geometry is directly tied to the topology of the natural space. This furthermore means that the Euclidean intuited geometry can correspond to and be symbolized pictorially by the use of morphological forms of the intuited world of things. The Euclidean space can be presented pictorially, even if in an inadequate way. Everything that can be depicted in pictorial symbolisms must be bound to two or three dimensions, if the change of position is to be accessible to pictorial presentations, using morphological forms. Any space, with higher than three dimensionality, cannot be presented in pictorial symbolisms using morphological forms derived from the natural space of things.

It is assumed that the normal space of objects is the three-dimensional Euclidean space. This normalcy can be extended toward R_n. It too can be of the topological type of open plane, and its metrics can be determined by the Euclidean invariance of a stretch. But what would stretch and movement mean in more than three dimensions? If for R_3 the meanings of stretch and movement were the results of ideational acquisition freeing from the sense intuited things and through the transposition of the meaning of their relationships into ideal relationships, then their conceptual apprehension would require still new moments of apprehension, transcending the meaning of R_3. The movement to more than three-dimensional geometry is constituted by significations entering universally into geometry through the analytic-algebraic method. This no longer requires pictorial symbolizations.

Yet insofar as it is not merely a vectorial algebra, but vectorially and algebraically pursued analytic geometry, the efforts of mathematization contain a capacity to have a spatially intuitive interpretation of the vectorial equations. Thus even higher-dimensioned geometries speak of "straight, plane distance and movement." They speak in concepts belonging to R_3 whose meaning may be the same, although in the latter case it may possess a symbolic intuitive fulfillment. The specific unintuitability of the n-dimensional geometry is inherent in the fact that it is a formal extension of the geometry of R_3. The concept of extension is here additive. What this means is that what is unintuitable here is not a complexity of properties characteristic of the basic concepts of geometry, which resist visualization, but rather the supersession of the number of dimensions.

It is to be noted that while the n-dimensioned Euclidean geometry may preclude intuitability, yet it is directly translatable to Σ_3; the non-Euclidean geometry, which is founded on the geometry of curved surfaces, cannot be intuited even within the R_3 domain. Although at the first sign it may seem that the curved surface is accessible to intuition, yet its accessibility is founded on the basis of a space which is structured as plane. Here the curved space is seen as "in" the plane space, i.e. imbedded in it as a curved surface. That it is presented as being "in" a plane space is not contested. What is being contested is that the depiction of such a curved space algorithmically occurs in space. For the algorithmic conception of the curved space, the notions of being in a surrounding space are metaphors attached to the conception of space as a container.

Why this is the case can be seen from the consideration of the mathematical space. While the coordinate system is freely choosable, i.e., while the geometric structures can move freely in contrast to the system, the movement here has nothing in common with the movements of a corporeal being. In a space without corporeity, the motive for movement is purely rational. The simplicity of a structure, as an aim of mathematical investigations in the choice of a coordinate system, corresponds to a specific principle of economy which is purely quantitative and in no wise geometric. It is determined by the rules of mathematical relationships. The simplicity does not relate to the geometric structure as such, established in its geometric form and its metric characteristics, but rather relates to its analytic form. By introducing number as something completely distinct from the spatially structured domain of quantum, geometry has accepted modes of understanding completely foreign to the intuited or pictorial symbolized space.

What is interesting is that the analytically constructed algebraic geometry enables a dimensional extension of the mathematical space. In this extension, the three-dimensionality is not accorded any privileged status in the framework of free geometric sciences. As a consequence of analytic geometry, i.e. geometry which no longer has its forms in spatial imagery but only in the algebraic coordination of signs, it is investigated in terms of pure algebraic coordination of signs which ·are viewed in terms of pure algebraic attributes (degrees, equations, number of parts, etc.). Here the question of dimensions also becomes meaningless. Once a coordinate system is established, with the possibility of symbolizing each point, whose meaning in elementary geometry is a stretch, into quadratic equations so that only the degree of this equation has a geometric relevance, then there is nothing that can hinder to interpret multi-membered quadratic equations geometrically as stretches.

In the sign-symbolism of vectorial calculus, the meaninglessness of the question of dimensions for geometry is most striking. It functions without any fixed number of vectorial components, and its symbolism is so construed, that these components do not even appear in calculus. Only specific limitations may freely interpret the meant and completed calculus for R_3, R_4, … R_n. For a geometry, constructed vectorio-analytically, there is always the free possibility to grasp it as a geometry of space, possessing an arbitrary, finite number of dimensions. This has to do with the possibilities of extension which may be misunderstood as generalized universal.

Yet the geometry of n-dimensional space is no more general than the three dimensional space. Insofar as n stands for any natural number, it does not subsume under itself as a genus subsumes a species, but only as a singular example of itself. Just as in the series of natural numbers, a greater number is no more general than a smaller. What legitimates the notion of abstraction in this domain is the commonly assumed understanding that the three-dimensional space is intuitable and pictorially symbolizable. In any case, the transition from three-dimensional geometry to more-dimensioned geometries, is not a process of generalization. A careful observation of what is contained in the genuine mathematical spaces and the mathematical intentions constituting them reveals that no abstraction takes place. To speak of mathematical spaces is to simply accept analytic geometry as such.

Strict and Vague Characterizations

Geometrically speaking, objects can be described morphologically by terms such as "cornered , notched, jagged, egg-shaped, flattened, etc." Such characterizations are vague and admit of various adjustments in their application to spatial objects. They suggest certain parameters, yet they are also open for a multitude of vague objects falling under them. Moreover, these terms are also vague with respect to the other features of objects which fall under them. Although the term "egg-shaped" suggests homogeneity, there is no strict delimitation of the parameters when something ceases to be egg-shaped. These terms are derivable "abstractively" in the sense of leaving aside a multitude of other features of the object. This is not to say that such terms do not possess a degree of universality: they can subsume object of various genera. As Stroker points out, these concepts or terms can be visualized pyramidically: the greater the extent of the term, the lesser is its content; it has its specifications "under" itself, but not within itself.

Another group of terms, distinct from the first, include "square , tri-angular, spherical, cubical, etc." which offer strict delimitations. Basically they are mathematical terms. This does not contradict their morphological employment in the intuited space. Indeed, in common usage they seem to function as morphological terms, yet upon reflection they reveal characteristics which are beyond those of morphological characteristics. They are mathematically universal terms which have necessary conceptual interconnections which are not apprehensible through factual enumeration of cases but through logical structurations. What this means is that an absence of its one factor. disrupts the total interconnection. In this sense they contain added determinations not only under themselves, but also within themselves. This means that it is possible to derive the entirety of their structures from each of its members.

The validity of these terms is not enhanced or diminished by addition or lack of empirical discoveries. The square does not become more universal by the discovery of more square objects. It either has the properties of orthogeneity and the equality of the length of bisectors and diagonals or it does not. These terms, thus do not correspond to the universality of genera. This implies that such terms are not derived by generic construction, where the more specific term contains the more universal, attainable by exclusion of the limitations of the specific term . The more universal thus has less content, the more specified contains richer content. What constitutes a "special case" in geometry is not an addition of new determinations to the universal, but rather a specification of the universal through the variable characteristics. Hence, the more universal concept here is a richer one. And the specification of the universal is a logical and not empirical process. Implicitly this means that the sphere of objects in this domain are not accessible in the same way as the objects in the morphological domain; they cannot be obtained abstractively.

Before proceeding to the question of sense intuition of geometric space, it is essential to add a number of distinctions within the domain of the geometric objectivity itself. At first sight it might appear that the domain of objectivity of the geometric terms lacks any corporeal-sense involvement. Yet to bridge the gap between the sensibly intuited and the geometric, Husserl has coined the concept of ideation. While confronted with a singular sense object, ideation brings forth something common to a multitude of given objects. Thus universal essentialities are obtained "on" an individual sense object. The sense object becomes an example of the essence. Yet the direct sense intuition presupposes the presence of the essence. Ideation, in this sense offers a new type of objectivity. This objectivity is apparent in categorial intuition, containing two layered intentionality: the sensible and the intelligible, the universal and the exemplifying. Hence the categorical intuition accounts for the claim that sense intuition is the source of geometry. At the same time it is to be pointed out that ideational intuition of geometrical essences is not identical with mathematical ideation. Mathematical ideation adds an idealization: what is grasped ideationally is conceived not only for itself, but also "exactly." Mathematical ideality is what adds this exactness. Moreover, mathematical ideation transgresses the boundaries of other forms of ideation, insofar as it can deal with a multitude of categorically intuited types. Keeping these distinctions in mind, it is possible to turn to the modes of sense-giveness of geometric objectivities.

Pictorial and Sign Representations

While the geometric categorical intuitions are given "on" a sense intuited object, the work with such categorical structures takes place in another sense-intuited medium: a picture in which the categorical structures appear in sense intuition. This visibility in another is not something external, but is constitutively essential for the geometric and mathematical domains. As will be seen later, all mathematical idealization is present only and insofar as it is accomplished in another, in a medium. Concerning the geometric structures, they are no longer apprehended "on" a sensibly intuited object, yet they are given in sensibility without being in it. The medium was once called "representation." Yet this term is highly misleading, since it assumes a correspondence between the categorical structure and the morphological figure given in sense experience. The geometrician does not work or remain with the morphological sketch but rather by means of it is concerned with the categorical structure. He does not "engage" in the picture, but in the form. Hence the morphological figure is pictorial not in a sense of representation but in the sense of medium through which something else appears.

The geometric object is not given in sense intuition, yet its fulfilment, its confirmation and validation, requires the complementary activity of sense intuition functioning at the same time. The pictorial is here the “rough," the morphological aspect, but it is not the object of geometry. The morphological aspect is here variable without the required variability of the geometrical object. To apprehend a geometric structure , means to orient oneself in a pure consciousness of an essence, and to be able to correlate various sensibly intuited pictures to it. This means, furthermore, that the correlation between the essential and the pictorial is reversible; the sensible can be constantly apprehended as a "picture of" the categorical or the essential. The picture, in turn, can be apprehended as a possible morphological form of sensibly intuited object.

The older geometry remained essentially at the level of categorical intuition and pictorial medium, with the possibility of the pictorial aspect to be morphological, i.e. to be derived from the presence of specific sense objects. The pictorially intuited figure presents the geometric structure itself, although in a limiting sense of mediation through the pictorial figure. This figure does not resist, so to speak, the pure geometric form. Yet with an emergence and introduction of analytic geometry, leading to non-Euclidean geometries, abolishes this direct relationship and the required pictorial sense intuition. The pictorial intuition is abolished in favor of a sign-symbolism. Moreover, it introduces the notion of constructability and thus modifies essentially the domain of what constitutes the geometric. The noetic acts must be different: for example, the discussion of a circle, where it is meant in a pictorial presentation or intuition, is different in its meaning from a circle when it is spoken of in terms of a “circle $x^2 + y^2 = a^{2u}$.” Of course, in the latter case the specific equation admits coordination to the circle as also meant with the pictorial intuition, yet it is no longer "pictured." What is here strictly manifest in the sign symbolism is not a circle but a functional equation of a circle. The circle is intended only mediately. In contrast to the pictorial intuition, the mathematical equation requires a specific intention to posit it as meaning "the circle." What this means is that the relationship between the sign and the signified, in contrast to that between picture and its object, is no longer direct. Moreover, this also implies that the sign-symbolism is not prescribed by the meant object. The pictorial intuition remains bound to the eidetically intuited structure, thus retaining the "picture" character, allowing morphological inexactness within the permissible limits, the sign-symbolism contains in principle a free choice. It allows that the sign-symbolism can be reached by agreement, as long as univocity of meaning is maintained.

What is clearly noticeable in this turn to sign symbolism is the functioning of intentionality. An entire series of intentionalities are required to determine the variously leveled mediation of the meant object. The latter is encountered through various and phenomenologically heterogenous levels of meaning, before it appears "in" the sign as the precisely meant object. After all, nothing hinders the exclusion of the specific geometric meaning of the sign symbols. The intentions could remain with the equations and functions as such, leading no further than an analysis of algebraic equations. The possibility of such an exclusion also dissolves a necessary relationship between the sense intuition of anything spatial and the pictorial sense intuition of geometry as a direct science of intuitable spatial shapes. The establishment of a sign symbolism becomes a matter of choice. Of course the mathematician can speak of a "geometrical index" of space, but only by a specific intentionality. In its own right, the sign symbolism does not contain any geometric motivation.

Sign Systems and Geometry

What has been said need not be taken as an abolition of geometry of the spatial. Rather, there appears a new sense of what constitutes the domain of the geometric. In a way the algebraic considerations constitute an extension of geometry, an extension impossible on the pictorially intuitive grounds. Moreover, what is implicit in the new sense of the geometric is a geometrically formalized theory. Yet the geometrical motivation is different. It is no longer based on the question how an objectivity, conceived purely spatially, can be mastered mathematically, but is dictated from a question as to what can a system of signs, constructed purely operationally, can offer if they are intended geometrically? This question completely reverses the pictorial sense intuition. The originary activity of mathematical symbolization, creating the signs from spatial objectivity, leading to pictorial and intuitive presentations, were meaningful and were akin to the original pictorial intuitions as presentations of an objective sense. But in calculus, the sign has become removed from the signified to such an extent, that the intention toward the meant objectivity is deemed irrelevant. If the pictorially intuited symbol stressed the "representative" function, making it stand for something, in the sign symbolism the sign is taken for itself. Indeed, the sign symbolism can subsume the geometric at will, yet it is meant as in itself and only as a sign without signifying anything. It exhausts itself in being purely operational.

A moment of sense intuitability, nonetheless appears in the sign system. This is precisely so if the sign is taken purely in itself and becomes an object of mathematical consideration. To the extent that it no longer has any objective direction, but is posited in its own meaning as purely operational, the sense intuition of such signs assumes an important significance. To say the least, it plays a leading role in the process of proof.

An operation with signs is not a process of thought which becomes subsequently spatialized; the operations are with and through the signs, and do not occur without them. One could say that they are constitutive of what occurs in and through them. Due to the fact that signs are spatially intuited structures, i.e., sensible morphological figures, each logical contradiction in calculation is apprehended in direct morphological intuition, and this means that the direct perception of signs becomes a necessary act of mathematical operations themselves. After all, although arbitrarily selected with regard to their morphological characteristics, they follow precise spatial series and rules of operation. The spatial alignment also means a temporal succession. Indeed, in geometric domain the pictorial sense intuition also constitutes temporal seriality; the seriality nonetheless is inessential to the order of geometrical depictions. In sign symbolism, in contrast, the succession of signs is irreversible. Here the logical succession is apprehended directly as a spatio-temporal succession. The logical succession escapes pictorial representation. The inferential process, involving the "then," escapes pictorial intuition.

In this domain something is only insofar as it is demonstratable. At this level, i.e., mediate level of geometric demonstrations, it cannot be assumed that the geometric states of affairs are and then they are demonstrated. They have their being in being demonstrated. If mathematically proven geometric states of affairs is invalid. then it is nothing. This suggests that the mathematized geometry proceeds constructively. There is no imitation of something pre-given; rather the geometric structures are generated through construction. This implies that if the "existence" of these structures rests on constructive generation, then the ultimate explication of them comes from the constitutive activity of the subject.

While earlier geometries used pictorial constructions as sensible symbols of geometric structures, contemporary geometry uses signs. The use of signs implies correlatively the requirement that the constructive steps be finite. Finitude here depends on the requirements of sense intuition. While pictorial sense intuition functioned in direct geometrical work to "picture" the geometric structures intuited ideationally, the sense intuition functions even at the level of mathematized geometries which, although not requiring pictorial representations, are very much involved in sense intuition for the construction of finite series of signs. What this suggests is that the mathematized or mediate geometrical understanding does not occur in pure consciousness, but involves intuitive sensibility, not merely as an auxiliary process, but as a necessary means of construction. Both, pure consciousness and sense intuition function together.

<u>Non-Euclidean Spaces</u>

The intuitability of Euclidean space depends on its three-dimensionality, consisting of planes and depth. What leads to the unintuitability in the non-Euclidean geometry is not a complexity of properties which resist perception, but rather the supersession of dimensions which precludes even pictorial representations. This is not to say that the supersession does not include some of the components achieved intuitively. Not only experienced stretches and angles, but also straight lines and planes have their meaning in a pre-mathematical treatment in the space of intuition. The mathematical sense is understandable only in terms of such a meaning. These topological structures, as scientific conceptions are founded in the achievements of a corporeal being with senses. Specifically, the oriented function of vision aids in the understanding of the topological structure of the space of intuition in terms of straight lines and planes. It could be maintained that geometry could not have proceeded in any other way, except with plane geometry, as long as it understood itself in terms of constructive generation through instruments and instrumental images, inclusive of pictorial representations.

The movement toward non-Euclidean geometry transforms methodically the basic conceptions of geometry. For example, the spherical surface becomes a source of demonstrations for non-Euclidean events. This leads to the notions such as a sphere with an imaginary radius which are not representable pictorially. All pictorial representations are here merely heuristic. The leading role is assumed solely by an algebraic symbol, the pictureless sign for spatiality which cannot be meaningfully discussed in pictorial terms. This leads to positive results: the possibility to signify exhaustively the spatial only in sign symbolisms, abolishes the opposition between "real-spatial" and pictorially intuitable meaning of spatiality; the opposition between plane and non-plane surfaces, between open endless and closed space. If pictorial intuition functions as a heuristic value and not as a fundamental requirement to "imagine" the spatial structures, then the distinctions between intuitable and non-intuitable vanish at the level of sign symbolism. Sign symbolism discards any necessary ties to the plane medium of presentation. As Stroker points out, for a geometry which is purely signitive and constructive in the modern sense, it is in principle irrelevant whether one operates in a medium of open endless extension or with a form closed upon itself. The plane of signs, in its constitution as plane, is as irrelevant for the characteristics and meaning of sign, as the color of a chalk is for the process of proof. Although this may seem "shallow," it nonetheless shows an important differentiation between pictorial and sign symbolisms in geometry. The first demands that the plane of the picture should at least minimally correspond to the meant structure. After all, the Euclidean elementary geometry is not constructible synthetically on a spherical surface. In contrast the sign symbolism allows the plane of signs, on which something is merely signified, to deviate topologically from the normal plane. What this means, furthermore, is that with the sign construction of geometrical manifolds plane and curved surface do not become non-differentiated; rather they assume equal structural value. To consider geometrically a curved surface as a plane, means to modify its conceptual sense. And this leads to the possibility of constructing new types of geometric spaces. Just as the plane plays a fundamental role in the intuition of Euclidean space, the curved surfaces play a role in the constitution of non-Euclidean spaces.

Take for example the concept of stretch. At the level of sign symbolism, the stretch may be seen operationally. This means to conceive of the stretch in terms of linear operations such as a+a+a+a+ ... is only one possibility of its application among others. In principle, the operations can be variable. The operations do not define a stretch; rather they assume it. Rather they define and prescribe a particular manner of proceeding with stretches. In this sense operational concepts do not define an objective concept, but a procedure which delimits what must be done with Euclidean stretches in non-Euclidean geometry. The new concept of stretch is not a concept of an object, given either pictorially or intuitively (even if these function mediately), but an operational concept. Although the Euclidean intuitive pictoriality remains as a representation of the ideationally constituted stretch, at least in the background, what appears here to be primary is the "meaning giving" activity as operational within which the sense of the signs becomes comprehensible. That is to say, the modified sense of the stretch is not apriori evident or derivable from that of the ideationally constituted meaning in the Euclidean geometry. It rather emerges with the constitutive operations and only in them. It seems that we are .touching upon genetic phenomenology. Important, and interesting as this may be, we shall not consider it in this discussion.

The operationally achieved constitution of the new sense of stretch is independent from and yet related to the intuitively present notion of Euclidean stretch. Yet what the operationally constituted stretch allows is the extension of the meaning of stretch to subsume not only the interval in the hyperbolic (properly non-Euclidean) geometry, but also is valid of the Euclidean interval, i.e. parabolic geometry, to the extent that the latter can be constructed in terms of sign symbolisms logarithmically. This operationally signitive mode of procedure constitutes for the new metric conception a characteristic mode of generalization which is mathematical. This "generalization" cannot be conceived in terms of a movement from particular to the universal. Such a generalization plays a role only at the morphological level; as was seen above, it does not even play any role in the strict geometrical concepts such as a square or a circle which can be determined mathematically. The form of generalization here is a transformation of operations which subsume under themselves other modes of operations,. and this in accordance with logarithmic functions. This can be simply seen that the non-Euclidean geometry is Euclidean in its smallest parts.

Euclidean and non-Euclidean Subject

The corporeal, intuiting subject, is Euclidean. It opens space and masters it mathematically on the basis of sensory intuition and pictorial symbolization; at the same time the subject may use mathematically ideated structures which lead away from pictorial symbolism to sign symbolism. Thus to this subject experiences the non-Euclidean geometric structures, such as those of conic section chord etc . as structures of a Euclidean plane. Yet through new means of sign symbolism, the meanings of the pictorially presented Euclidean structures are changed. Yet the change of meaning is coordinated to the original Euclidean meaning by a kind of mediation through analogy, i.e., by the new regulations of the measure of a stretch. This is calculated when the chord is taken as a straight line, conic section as a totality of infinitely distant points, the inside as a new total plane; the accomplished change of the original Euclidean concepts thus appears meaningful. Although it is possible to construct the subject and its functions in a non-Euclidean space, such a construction becomes redundant. The subject of the Euclidean space has a sign system which transforms the pictorial-Euclidean plane of intuition through pure algebraic analysis. The non-Euclidean relationships are understood by a Euclidean being not though an immersion into the non-Euclidean space in order to gain intuitive-sense understanding, but through two different lendings of meaning to the pictorially intuited structures in one and the same consciousness before and after specific operational requirements. The subject rules, so to speak, the translation of meanings without allowing the transition to have corresponding pictorial symbols. The geometric figures, such as the plane, line, etc., remain Euclidean, i.e., they retain the Euclidean meaning. They persevere mediately when consciousness is engaged in re-signification of these structures through a new sign system which lends such structures a different meaning through different operations.

What is pictured and what is meant assume a unique relationship in non-Euclidean geometry. The signified here is not the original geometric meaning of the pictorial symbol; rather it is constructed through the operation with signs; yet the pictorially symbolized geometric structures remain in the background of non-Euclidean signs; these are not pictorial but purely signitive. This means that the pictorial symbolism, understandable through intuition of Euclidean being, is here inappropriate and indeed inadequate for what is being signified. The plane is an intuitive surrogate for what is genuinely structured in the sign system. In place of the pictorially symbolic intuition, there is a "model representation." The model must be constituted independently and ahead of time before it can be presented in some sensible medium. Such a medium is usually seen as algorithm. Yet there are analogies to Euclidean space intuition. For example, a being in hyperbolic space, represented by a conic section, moves slower toward the edge without ever reaching it. The same experience is offered in Euclidean space in direct intuition. Someone moving away from the point of observation seems to move slower when he recedes toward the horizon. In a specific domain of nearness, Euclidean relationships seem to rule, while in analogy to hyperbolic geometry, the deviations are greater toward the edge.

Yet this is only an analogy, based on our picturing of a being in non-Euclidean space. Yet such picturing is precluded by the mathematical mode of construction of the hyperbolic space. The difficulty in picturing the hyperbolic geometry is seen in the requirements of a calculus as a sign system required to mediate the pictorially intuited geometric structures and the meaning giving which meant such structures in a novel way. Although the pictorially intuiting consciousness is founding for the transformation of geometric Euclidean relationships to the hyperbolic relationships, the latter are not dependent on intuitability in a pictorial sense. This means that the hyperbolic geometry is not a mathematical generalization and indeed idealization of the intuitable morphological world of things in the perspectival space of intuition. Even the intuited pure geometric structures, such as square or circle, presented in pictorial awareness, are transformed into a different meaning.

That the hyperbolic geometry can be applied to the intuited space does not mean that the intuited space is the sole foundation of the hyperbolic space. The founded and the founding have a necessary intentional interconnection. The first builds itself in acts "over" the latter, and requires the latter in order to be. An application has an entirely different structure. Something to be applied is entirely free in contrast to that on which it is being applied. In fact this freedom is required in order that application may be discovered; yet the finding does not touch the domain to which application is made. A mathematical objectivity or a meaning giving to an ordered sign system , preserve their purity even if no application is found. The hyperbolic system can be understood even if no corporeal application can be found for it. Its validity depends entirely to the mathematical meaning giving activity, and only from there and its modes of operation that it can be discussed as to its applicability.

Retrospect

While we have not dealt with the background distinctions between the visual and the intuited spaces, demonstrating the inherence of visual space in the intuited space , we took the intuited space as a point of departure to show that its pictorial presentation and intuitive ideality, although basic for the hyperbolic geometry, is nonetheless inadequate to yield such a geometry. The latter requires an entirely distinct modes of meaning giving. The hyperbolic geometry emerges correlatively with the establishment of a sign symbolism whose only intuitable requirement is the proper arrangement of freely chosen signs. At this level one works purely signitively and, so to speak, formally, using the means of Euclidean intuited space as auxiliary but not as representative factors. But these factors do not point to or signify anything similar to them. They reveal a meaning which is dependent on a meaning giving consciousness which operates free from the immersion in the intuited space of corporeity. The corporeity here appears only as a means of sign constitution, but not as a means of a constitution of the hyperbolic geometry.

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