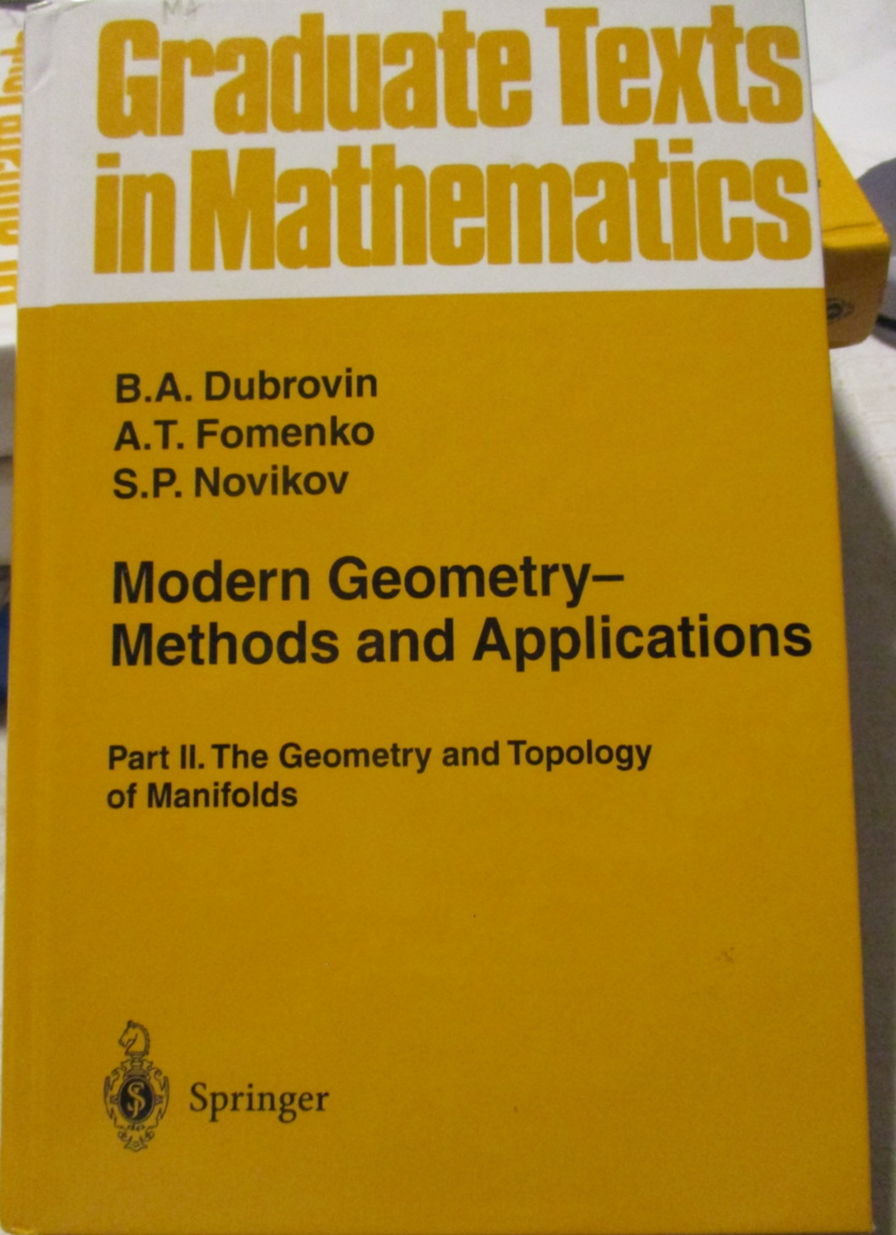

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B.A. Dubrovin
A.T. Fomenko
S.P. Novikov

Modern Geometry– Methods and Applications

Part II. The Geometry and Topology of Manifolds

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Part II. The Geometry and Topology of Manifolds

Translated by Robert G. Burns

With 126 Illustrations

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Preface

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Preface

Up until recently, Riemannian geometry and basic topology were not included, even by departments or faculties of mathematics, as compulsory subjects in a university-level mathematical education. The standard courses in the classical differential geometry of curves and surfaces which were given instead (and still are given in some places) have come gradually to be viewed as anachronisms. However, there has been hitherto no unanimous agreement as to exactly how such courses should be brought up to date, that is to say, which parts of modern geometry should be regarded as absolutely essential to a modern mathematical education, and what might be the appropriate level of abstractness of their exposition.

The task of designing a modernized course in geometry was begun in 1971 in the mechanics division of the Faculty of Mechanics and Mathematics of Moscow State University. The subject-matter and level of abstractness of its exposition were dictated by the view that, in addition to the geometry of curves and surfaces, the following topics are certainly useful in the various areas of application of mathematics (especially in elasticity and relativity, to name but two), and are therefore essential: the theory of tensors (including covariant differentiation of them); Riemannian curvature; geodesics and the calculus of variations (including the conservation laws and Hamiltonian formalism); the particular case of skew-symmetric tensors (i.e. "forms") together with the operations on them; and the various formulae akin to Stokes' (including the all-embracing and invariant "general Stokes formula" in n dimensions). Many leading theoretical physicists shared the mathematicians' view that it would also be useful to include some facts about manifolds, transformation groups, and Lie algebras, as well as the basic concepts of visual topology. It was also agreed that the course should be given in as simple and concrete a language as possible, and that wherever practicable the terminology should be that used by physicists. Thus it was along these lines that the archetypal course was taught. It was given more permanent form as duplicated lecture notes published under the auspices of Moscow State University as:

Differential Geometry, Parts I and II, by S. P. Novikov, Division of Mechanics, Moscow State University, 1972.

Subsequently various parts of the course were altered, and new topics added. This supplementary material was published (also in duplicated form) as:

Differential Geometry, Part III, by S. P. Novikov and A. T. Fomenko, Division of Mechanics, Moscow State University, 1974.

The present book is the outcome of a reworking, re-ordering, and extensive elaboration of the above-mentioned lecture notes. It is the authors' view that it will serve as a basic text from which the essentials for a course in modern geometry may be easily extracted.

To S. P. Novikov are due the original conception and the overall plan of the book. The work of organizing the material contained in the duplicated lecture notes in accordance with this plan was carried out by B. A. Dubrovin. This accounts for more than half of Part I; the remainder of the book is essentially new. The efforts of the editor, D. B. Fuks, in bringing the book to completion, were invaluable.

The content of this book significantly exceeds the material that might be considered as essential to the mathematical education of second- and third-year university students. This was intentional: it was part of our plan that even in Part I there should be included several sections serving to acquaint (through further independent study) both undergraduate and graduate students with the more complex but essentially geometric concepts and methods of the theory of transformation groups and their Lie algebras, field theory, and the calculus of variations, and with, in particular, the basic ingredients of the mathematical formalism of physics. At the same time we strove to minimize the degree of abstraction of the exposition and terminology, often sacrificing thereby some of the so-called "generality" of statements and proofs: frequently an important result may be obtained in the context of crucial examples containing the whole essence of the matter, using only elementary classical analysis and geometry and without invoking any modern "hyperinvariant" concepts and notations, while the result's most general formulation and especially the concomitant proof will necessitate a dramatic increase in the complexity and abstractness of the exposition. Thus in such cases we have first expounded the result in question in the setting of the relevant significant examples, in the simplest possible language appropriate, and have postponed the proof of the general form of the result, or omitted it altogether. For our treatment of those geometrical questions more closely bound up with modern physics, we analysed the physics literature:

books on quantum field theory (see e.g. [35], [37]) devote considerable portions of their beginning sections to describing, in physicists' terms, useful facts about the most important concepts associated with the higher-dimensional calculus of variations and the simplest representations of Lie groups; the books [41], [43] are devoted to field theory in its geometric aspects; thus, for instance, the book [41] contains an extensive treatment of Riemannian geometry from the physical point of view, including much useful concrete material. It is interesting to look at books on the mechanics of continuous media and the theory of rigid bodies ([42], [44], [45]) for further examples of applications of tensors, group theory, etc.

In writing this book it was not our aim to produce a "self-contained" text: in a standard mathematical education, geometry is just one component of the curriculum; the questions of concern in analysis, differential equations, algebra, elementary general topology and measure theory, are examined in other courses. We have refrained from detailed discussion of questions drawn from other disciplines, restricting ourselves to their formulation only, since they receive sufficient attention in the standard programme.

In the treatment of its subject-matter, namely the geometry and topology of manifolds, Part II goes much further beyond the material appropriate to the aforementioned basic geometry course, than does Part I. Many books have been written on the topology and geometry of manifolds: however, most of them are concerned with narrowly defined portions of that subject, are written in a language (as a rule very abstract) specially contrived for the particular circumscribed area of interest, and include all rigorous foundational detail often resulting only in unnecessary complexity. In Part II also we have been faithful, as far as possible, to our guiding principle of minimal abstractness of exposition, giving preference as before to the significant examples over the general theorems, and we have also kept the interdependence of the chapters to a minimum, so that they can each be read in isolation insofar as the nature of the subject-matter allows. One must however bear in mind the fact that although several topological concepts (for instance, knots and links, the fundamental group, homotopy groups, fibre spaces) can be defined easily enough, on the other hand any attempt to make nontrivial use of them in even the simplest examples inevitably requires the development of certain tools having no forbears in classical mathematics. Consequently the reader not hitherto acquainted with elementary topology will find (especially if he is past his first youth) that the level of difficulty of Part II is essentially higher than that of Part I; and for this there is no possible remedy. Starting in the 1950s, the development of this apparatus and its incorporation into various branches of mathematics has proceeded with great rapidity. In recent years there has appeared a rash, as it were, of nontrivial applications of topological methods (sometimes in combination with complex algebraic geometry) to various problems of modern theoretical physics: to the quantum theory of specific fields of a geometrical nature (for example, Yang–Mills and chiral fields), the theory of fluid crystals and superfluidity, the general theory of relativity, to certain physically important nonlinear wave equations (for instance, the Korteweg–de Vries and sine-Gordon equations); and there have been attempts to apply the theory of knots and links in the statistical mechanics of certain substances possessing "long molecules". Unfortunately we were unable to include these applications in the framework of the present book, since in each case an adequate treatment would have required a lengthy preliminary excursion into physics, and so would have taken us too far afield. However, in our choice of material we have taken into account which topological concepts and methods are exploited in these applications, being aware of the need for a topology text which might be read (given strong enough motivation) by a young theoretical physicist of the modern school, perhaps with a particular object in view.

The development of topological and geometric ideas over the last 20 years has brought in its train an essential increase in the complexity of the algebraic apparatus used in combination with higher-dimensional geometrical intuition, as also in the utilization, at a profound level, of functional analysis, the theory of partial differential equations, and complex analysis; not all of this has gone into the present book, which pretends to being elementary (and in fact most of it is not yet contained in any single textbook, and has therefore to be gleaned from monographs and the professional journals).

Three-dimensional geometry in the large, in particular the theory of convex figures and its applications, is an intuitive and generally useful branch of the classical geometry of surfaces in 3-space; much interest attaches in particular to the global problems of the theory of surfaces of negative curvature. Not being specialists in this field we were unable to extract its essence in sufficiently simple and illustrative form for inclusion in an elementary text. The reader may acquaint himself with this branch of geometry from the books [1], [4] and [16].

Of all the books on the topology and geometry of manifolds, the classical works *A Textbook of Topology* and *The Calculus of Variations in the Large*, of Siefert and Threlfall, and also the excellent more modern books [10], [11] and [12], turned out to be closest to our conception in approach and choice of topics. In the process of creating the present text we actively mulled over and exploited the material covered in these books, and their methodology. In fact our overall aim in writing Part II was to produce something like a modern analogue of Seifert and Threlfall's *Textbook of Topology*, which would however be much wider-ranging, remodelled as far as possible using modern techniques of the theory of smooth manifolds (though with simplicity of language preserved), and enriched with new material as dictated by the contemporary view of the significance of topological methods, and of the kind of reader who, encountering topology for the first time, desires to learn a reasonable amount in the shortest possible time. It seemed to us sensible to try to benefit (more particularly in Part I, and as far as this is possible in a book on mathematics) from the accumulated methodological experience of the physicists, that is, to strive to make pieces of nontrivial mathematics more

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comprehensible through the use of the most elementary and generally familiar means available for their exposition (preserving, however, the format characteristic of the mathematical literature, wherein the statements of the main conclusions are separated out from the body of the text by designating them "theorems", "lemmas", etc.). We hold the opinion that, in general, understanding should precede formalization and rigorization. There are many facts the details of whose proofs have (aside from their validity) absolutely no role to play in their utilization in applications. On occasion, where it seemed justified (more often in the more difficult sections of Part II) we have omitted the proofs of needed facts. In any case, once thoroughly familiar with their applications, the reader may (if he so wishes), with the help of other sources, easily sort out the proofs of such facts for himself. (For this purpose we recommend the book [21].) We have, moreover, attempted to break down many of these omitted proofs into soluble pieces which we have placed among the exercises at the end of the relevant sections.

In the final two chapters of Part II we have brought together several items from the recent literature on dynamical systems and foliations, the general theory of relativity, and the theory of Yang–Mills and chiral fields. The ideas expounded there are due to various contemporary researchers; however in a book of a purely textbook character it may be accounted permissible not to give a long list of references. The reader who graduates to a deeper study of these questions using the research journals will find the relevant references there.

Homology theory forms the central theme of Part III.

In conclusion we should like to express our deep gratitude to our colleagues in the Faculty of Mechanics and Mathematics of M.S.U., whose valuable support made possible the design and operation of the new geometry courses; among the leading mathematicians in the faculty this applies most of all to the creator of the Soviet school of topology, P. S. Aleksandrov, and to the eminent geometers P. K. Raševskiĭ and N. V. Efimov.

We thank the editor D. B. Fuks for his great efforts in giving the manuscript its final shape, and A. D. Aleksandrov, A. V. Pogorelov, Ju. F. Borisov, V. A. Toponogov and V. I. Kuz'minov, who in the course of reviewing the book contributed many useful comments. We also thank Ja. B. Zel'dovič for several observations leading to improvements in the exposition at several points, in connexion with the preparation of the English and French editions of this book.

We give our special thanks also to the scholars who facilitated the task of incorporating the less standard material into the book. For instance the proof of Liouville's theorem on conformal transformations, which is not to be found in the standard literature, was communicated to us by V. A. Zorič. The editor D. B. Fuks simplified the proofs of several theorems. We are grateful also to O. T. Bogojavlenskiĭ, M. I. Monastyrskiĭ, S. G. Gindikin, D. V. Alekseevskiĭ, I. V. Gribkov, P. G. Grinevič, and E. B. Vinberg.

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CHAPTER 1

Examples of Manifolds

§1. The Concept of a Manifold

1.1. Definition of a Manifold

The concept of a manifold is in essence a generalization of the idea, first formulated in mathematical terms by Gauss, underlying the usual procedure used in cartography (i.e. the drawing of maps of the earth's surface, or portions of it).

The reader is no doubt familiar with the normal cartographical process: The region of the earth's surface of interest is subdivided into (possibly overlapping) subregions, and the group of people whose task it is to draw the map of the region is subdivided into as many smaller groups in such a way that:

(i) each subgroup of cartographers has assigned to it a particular subregion (both labelled *i*, say); and
(ii) if the subregions assigned to two different groups (labelled *i* and *j* say) intersect, then these groups must indicate accurately on their maps the rule for translating from one map to the other in the common region (i.e. region of intersection). (In practice this is usually achieved by giving beforehand specific names to sufficiently many particular points (i.e. land-marks) of the original region, so that it is immediately clear which points on different maps represent the same point of the actual region.)

Each of these separate maps of subregions is of course drawn on a flat sheet of paper with some sort of co-ordinate system on it (e.g. on "squared" paper). The totality of these flat "maps" forms what is called an "atlas" of the region of the earth's surface in question. (It is usually further indicated on each map how to calculate the actual length of any path in the subregion represented by that map, i.e. the "scale" of the map is given. However the basic concept of a manifold does *not* include the idea of length; i.e. as it is usually defined, a manifold does not *ab initio* come endowed with a metric; we shall return to this question subsequently.)

The above-described cartographical procedure serves as motivation for the following (rather lengthy) general definition.

1.1.1. Definition. A *differentiable n-dimensional manifold* is an arbitrary set M (whose elements we call "points") together with the following structure on it. The set M is the union of a finite or countably infinite collection of subsets U_q with the following properties.

(i) Each subset U_q has defined on it co-ordinates x_q^α, $\alpha = 1, \dots, n$ (called *local co-ordinates*) by virtue of which U_q is identifiable with a region of Euclidean n-space with Euclidean co-ordinates x_q^α. (The U_q with their co-ordinate systems are called *charts* (rather than "maps") or *local co-ordinate neighbourhoods*.)

(ii) Each non-empty intersection $U_p \cap U_q$ of a pair of such subsets of M thus has defined on it (at least) two co-ordinate systems, namely the restrictions of (x_p^α) and (x_q^α); it is required that under each of these co-ordinatizations the intersection $U_p \cap U_q$ is identifiable with a region of Euclidean n-space, and further that each of these two co-ordinate systems be expressible in terms of the other in a one-to-one differentiable manner. (Thus if the *transition* or *translation functions* from the co-ordinates x_q^α to the co-ordinates x_p^α and back, are given by

$$\begin{aligned} x_p^\alpha &= x_p^\alpha(x_q^1, \dots, x_q^n), \qquad \alpha = 1, \dots, n; \\ x_q^\alpha &= x_q^\alpha(x_p^1, \dots, x_p^n), \qquad \alpha = 1, \dots, n, \end{aligned} \tag{1}$$

then in particular the Jacobian $\det(\partial x_p^\alpha / \partial x_q^\beta)$ is non-zero on the region of intersection.) The general smoothness class of the transition functions for all intersecting pairs U_p, U_q, is called the *smoothness class of the manifold M* (with its accompanying "atlas" of charts U_q).

Any Euclidean space or regions thereof provide the simplest examples of manifolds. A region of the complex space $\mathbb{C}^n$ can be regarded as a region of the Euclidean space of dimension $2n$, and from this point of view is therefore also a manifold.

Given two manifolds $M = \bigcup_q U_q$ and $N = \bigcup_p U_p$, we construct their *direct product* $M \times N$ as follows: The points of the manifold $M \times N$ are the ordered pairs (m, n), and the covering by local co-ordinate neighbourhoods is given by

$$M \times N = \bigcup_{p,q} U_q \times V_p,$$

where if x_q^α are the co-ordinates on the region U_q, and y_p^β the co-ordinates on V_p, then the co-ordinates on the region $U_q \times V_p$ are (x_q^α, y_p^β).

These are just a few (ways of obtaining) examples of manifolds; in the sequel we shall meet with many further examples.

It should be noted that the scope of the above general definition of a manifold is from a purely logical point of view unnecessarily wide; it needs to be restricted, and we shall indeed impose further conditions (see below). These conditions are most naturally couched in the language of general topology, with which we have not yet formally acquainted the reader. This could have been avoided by defining a manifold at the outset to be instead a smooth non-singular surface (of dimension n) situated in Euclidean space of some (perhaps large) dimension. However this approach reverses the logical order of things; it is better to begin with the abstract definition of manifold, and then show that (under certain conditions) every manifold can be realized as a surface in some Euclidean space.

We recall for the reader some of the basic concepts of general topology.

(1) A *topological space* is by definition a set X (of "points") of which certain subsets, called the *open sets* of the topological space, are distinguished; these open sets are required to satisfy the following three conditions: first, the intersection of any two (and hence of any finite collection) of them should again be an open set; second, the union of any collection of open sets must again be open; and thirdly, in particular the empty set and the whole set X must be open.

The complement of any open set is called a *closed set* of the topological space.

The reader doubtless knows from courses in mathematical analysis that, exceedingly general though it is, the concept of a topological space already suffices for continuous functions to be defined: A map $f: X \rightarrow Y$ of one topological space to another is *continuous* if the complete inverse image $f^{-1}(U)$ of every open set $U \subseteq Y$ is open in X. Two topological spaces are *topologically equivalent* or *homeomorphic* if there is a one-to-one and onto map between them such that both it and its inverse are continuous.

In Euclidean space $\mathbb{R}^n$, the "Euclidean topology" is the usual one, where the open sets are just the usual open regions (see Part I, §1.2). Given any subset $A \subset \mathbb{R}^n$, the *induced topology* on A is that with open sets the intersections $A \cap U$, where U ranges over all open sets of $\mathbb{R}^n$. (This definition extends quite generally to any subset of any topological space.)

1.1.2. Definition. The *topology* (or *Euclidean topology*) *on a manifold* M is given by the following specification of the open sets. In every local co-ordinate neighbourhood U_q, the open (Euclidean) regions (determined by the given identification of U_q with a region of a Euclidean space) are to be open in the topology on M; the totality of open sets of M is then obtained by admitting as open also arbitrary unions of countable collections of such regions, i.e. by closing under countable unions.

With this topology the continuous maps (in particular real-valued functions) of a manifold M turn out to be those which are continuous in the usual sense on each local co-ordinate neighbourhood U_q. Note also that any open subset V of a manifold M inherits, i.e. has induced on it, the structure of a manifold, namely $V=\bigcup_q V_q$, where the regions V_q are given by

$$V_q = V \cap U_q. \tag{2}$$

(2) "Metric spaces" form an important subclass of the class of all topological spaces. A *metric space* is a set which comes equipped with a "distance function", i.e. a real-valued function $\rho(x, y)$ defined on pairs x, y of its elements ("points"), and having the following properties:

(i) $\rho(x, y) = \rho(y, x)$;
(ii) $\rho(x, x) = 0$, $\rho(x, y) > 0$ if $x \neq y$;
(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (the "triangle inequality").

For example n-dimensional Euclidean space is a metric space under the usual Euclidean distance between two points $x=(x^1, \ldots, x^n)$, $y=(y^1, \ldots, y^n)$:

$$\rho(x, y) = \sqrt{\sum_{\alpha=1}^{n} (x^\alpha - y^\alpha)^2}.$$

A metric space is topologized by taking as its open sets the unions of arbitrary collections of "open balls", where by *open ball* with centre x_0 and radius ε we mean the set of all points x of the metric space satisfying $\rho(x_0, x) < \varepsilon$. (For n-dimensional Euclidean space this topology coincides with the above-defined Euclidean topology.)

An example important for us is that of a manifold endowed with a Riemannian metric. (For the definition of the distance between two points of a manifold with a Riemannian metric on it, see §1.2 below.)

(3) A topological space is called *Hausdorff* if any two of its points are contained in disjoint open sets.

In particular any metric space X is Hausdorff; for if x, y are any two distinct points of X then, in view of the triangle inequality, the open balls of radius $\frac{1}{2}\rho(x, y)$ with centres at x, y, do not intersect.

We shall henceforth assume implicitly that all topological spaces we consider are Hausdorff. Thus in particular we now supplement our definition of a manifold by the further requirement that it be a Hausdorff space.

(4) A topological space X is said to be *compact* if every countable collection of open sets covering X (i.e. whose union is X) contains a finite subcollection already covering X. If X is a metric space then compactness is equivalent to the condition that from every sequence of points of X a convergent subsequence can be selected.

(5) A topological space is (*path-*)*connected* if any two of its points can be joined by a continuous path (i.e. map from $[0, 1]$ to the space).

(6) A further kind of topological space important for us is the "space of

mappings" $M \to N$ from a given manifold M to a given manifold N. The topology in question will be defined later on.

The concept of a manifold might at first glance seem excessively abstract. In fact, however, even in Euclidean spaces, or regions thereof, we often find ourselves compelled to introduce a change of co-ordinates, and consequently to discover and apply the transformation rule for the numerical components of one entity or another. Moreover it is often convenient in solving a (single) problem to carry out the solution in different regions of a space using different co-ordinate systems, and then to see how the solutions match on the region of intersection, where there exist different co-ordinate systems. Yet another justification for the definition of a manifold is provided by the fact that not all surfaces can be co-ordinatized by a single system of co-ordinates without singular points (e.g. the sphere has no such co-ordinate system).

An important subclass of the class of manifolds is that of "orientable manifolds".

1.1.3. Definition. A manifold M is said to be *oriented* if for every pair U_p, U_q of intersecting local co-ordinate neighbourhoods, the Jacobian $J_{pq} = \det(\partial x_p^\alpha / \partial x_q^\beta)$ of the transition function is positive.

For example Euclidean n-space $\mathbb{R}^n$ with co-ordinates $x^1, \ldots, x^n$ is by this definition oriented (there being only one local co-ordinate neighbourhood). If we assign different co-ordinates $y^1, \ldots, y^n$ to the points of the same space $\mathbb{R}^n$, we obtain another manifold structure on the same underlying set. If the co-ordinate transformation $x^\alpha = x^\alpha(y^1, \ldots, y^n)$, $\alpha = 1, \ldots, n$, is smooth and non-singular, then its Jacobian $J = \det(\partial x^\alpha / \partial y^\beta)$, being never zero, will have fixed sign.

1.1.4. Definition. We say that the co-ordinate systems x and y define the *same orientation* of $\mathbb{R}^n$ if $J > 0$, and *opposite orientations* if $J < 0$.

Thus Euclidean n-space possesses two possible orientations. In the sequel we shall show that more generally any connected orientable manifold has exactly two orientations.

1.2. Mappings of Manifolds; Tensors on Manifolds

Let $M = \bigcup_p U_p$, with co-ordinates x_p^α, and $N = \bigcup_q V_q$, with co-ordinates y_q^β, be two manifolds of dimensions n and m respectively.

1.2.1. Definition. A mapping $f: M \to N$ is said to be *smooth of smoothness class k*, if for all p, q for which f determines functions $y_q^\beta(x_p^1, \ldots, x_p^n) = f(x_p^1, \ldots, x_p^n)_q^\beta$, these functions are, where defined, smooth of smoothness class k (i.e. all their partial derivatives up to those of kth order exist and are continuous). (It follows that the smoothness class of f cannot exceed the maximum class of the manifolds.)

Note that in particular we may have $N = \mathbb{R}$, the real line, whence $m = 1$, and f is a real-valued function of the points of M. The situation may arise where a smooth mapping (in particular a real-valued function) is not defined on the whole manifold M, but only on a portion of it. For instance each local co-ordinate x_p^α (for fixed α, p) is such a real-value function of the points of M, since it is defined only on the region U_p.

1.2.2. Definition. Two manifolds M and N are said to be *smoothly equivalent* or *diffeomorphic* if there is a one-to-one, onto map f such that both $f\colon M \to N$ and $f^{-1}\colon N \to M$, are smooth of some class $k \geq 1$. (It follows that the Jacobian $J_{pq} = \det(\partial y_q^\beta/\partial x_p^\alpha)$ is non-zero wherever it is defined, i.e. wherever the functions $y_q^\beta = f(x_p^1, \dots, x_p^n)_q^\beta$ are defined.)

We shall henceforth tacitly assume that the smoothness class of any manifolds, and mappings between them, which we happen to be considering, are sufficiently high for the particular aim we have in view. (The class will always be assumed at least 1; if second derivatives are needed, then assume class ≥ 2, etc.)

Suppose we are given a curve segment $x = x(\tau)$, $a \leq \tau \leq b$, on a manifold M, where x denotes a point of M (namely that point corresponding to the value τ of the parameter). That portion of the curve in a particular co-ordinate neighbourhood U_p with co-ordinates x_p^α is described by the parametric equations

$$x_p^\alpha = x_p^\alpha(\tau), \qquad \alpha = 1, \dots, n,$$

and in U_p its *velocity* (or *tangent*) *vector* is given by

$$\dot{x} = (\dot{x}_p^1, \dots, \dot{x}_p^n).$$

In regions $U_p \cap U_q$ where two co-ordinate systems apply we have the two representations $x_p^\alpha(\tau)$ and $x_q^\beta(\tau)$ of the curve, where of course

$$x_p^\alpha(x_q^1(\tau), \dots, x_q^n(\tau)) \equiv x_p^\alpha(\tau).$$

Hence the relationship between the components of the velocity vector in the two systems is expressed by

$$\dot{x}_p^\alpha = \sum_\beta \frac{\partial x_p^\alpha}{\partial x_q^\beta} \dot{x}_q^\beta. \tag{3}$$

As for Euclidean space, so also for general manifolds this formula provides the basis for the definition of "tangent vector".

1.2.3. Definition. A *tangent vector to an n-manifold M* at an arbitrary point x is represented in terms of local co-ordinates x_p^α by an n-tuple (ξ^α) of "components", which are linked to the components in terms of any other system x_q^β of local co-ordinates (on a region containing the point) by the formula

$$\xi_p^\alpha = \sum_{\beta=1}^{n} \left(\frac{\partial x_p^\alpha}{\partial x_q^\beta} \right)_x \xi_q^\beta \tag{4}$$

The set of all tangent vectors to an n-dimensional manifold M at a point x forms an n-dimensional linear space $T_x = T_x M$, the *tangent space* to M at the point x. We see from (3) that the velocity vector at x of any smooth curve on M through x is a tangent vector to M at x. From (4) it can be seen that for any choice of local co-ordinates x^α in a neighbourhood of x, the operators $\partial/\partial x^\alpha$ (operating on real-valued functions on M) may be thought of as forming a basis $e_\alpha = \partial/\partial x^\alpha$ for the tangent space T_x.

A smooth map f from a manifold M to a manifold N gives rise for each x, to an *induced linear map of tangent spaces*

$$f_*: T_x \to T_{f(x)},$$

defined as sending the velocity vector at x of any smooth curve $x = x(t)$ (through x) on M, to the velocity vector at $f(x)$ to the curve $f(x(t))$ on the manifold N. In terms of local co-ordinates x^α in a neighbourhood of $x \in M$, and local co-ordinates y^β in a neighbourhood of $f(x) \in N$, the map f may be written as

$$y^\beta = f^\beta(x^1, \ldots, x^n), \qquad \beta = 1, \ldots, m.$$

It then follows from the above definition of the induced linear map f_* that its matrix is the Jacobian matrix $(\partial y^\beta/\partial x^\alpha)_x$ evaluated at x, i.e. that it is given by

$$\xi^\alpha \to \eta^\beta = \frac{\partial f^\beta}{\partial x^\alpha} \xi^\alpha. \tag{5}$$

For a real-valued function $f: M \to \mathbb{R}$, the induced map f_* corresponding to each $x \in M$ is a real-valued linear function (i.e. linear functional) on the tangent space to M at x; from (5) (with $m = 1$) we see that it is represented by the gradient of f at x and is thus a covector. Interpreting the differential of a function at a point in the usual way as a linear map of the tangent space, we see that f_* at x is just df.

1.2.4. Definition. A *Riemannian metric on a manifold* M is a point-dependent, positive-definite quadratic form on the tangent vectors at each point, depending smoothly on the local co-ordinates of the points. Thus at each point $x = (x_p^1, \ldots, x_p^n)$ of each region U_p with local co-ordinates x_p^α, the metric is given by a symmetric matrix $(g_{\alpha\beta}^{(p)}(x_p^1, \ldots, x_p^n))$, and determines a (symmetric) scalar product of pairs of tangent vectors at the point x:

$$\langle \xi, \eta \rangle = g_{\alpha\beta}^{(p)} \xi_p^\alpha \eta_p^\beta = \langle \eta, \xi \rangle,$$

$$|\xi|^2 = \langle \xi, \xi \rangle,$$

where as usual summation is understood over indices recurring as superscript and subscript. Since this scalar product is to be co-ordinate-independent, i.e.

$$g^{(p)}_{\alpha\beta}\xi^{\alpha}_{p}\eta^{\beta}_{p}=g^{(q)}_{\gamma\delta}\xi^{\gamma}_{q}\eta^{\delta}_{q},$$

it follows from the transformation rule for vectors that the coefficients $g^{(p)}_{\alpha\beta}$ of the quadratic form transform (under a change to co-ordinates x^{γ}_{q}) according to the rule

$$g^{(q)}_{\gamma\delta}=\frac{\partial x^{\alpha}_{p}}{\partial x^{\gamma}_{q}}g^{(p)}_{\alpha\beta}\frac{\partial x^{\beta}_{p}}{\partial x^{\delta}_{q}}. \tag{6}$$

The definition of a *pseudo-Riemannian metric on a manifold M* is obtained from the above by replacing the condition that the quadratic form be at each point positive definite, by the weaker requirement that it be non-degenerate. (It then follows from the smoothness assumption that, provided M is connected, the index of inertia of the quadratic form is constant (cf. §3.2 of Part I).)

1.2.5. Definition. A *tensor of type* (k, l) *on a manifold* is given in each local co-ordinate system x^{α}_{p} by a family of functions

$$^{(p)}T^{i_1\ldots i_k}_{j_1\ldots j_l}(x)$$

of the points x. In other local co-ordinates x^{β}_{q} (embracing the point x) the components $^{(q)}T^{s_1\ldots s_k}_{t_1\ldots t_l}(x)$ of the (same) tensor are related to its components in the system x^{α}_{p} by the transformation rule

$$^{(q)}T^{s_1\ldots s_k}_{t_1\ldots t_l}=\frac{\partial x^{s_1}_{q}}{\partial x^{i_1}_{p}}\cdots\frac{\partial x^{s_k}_{q}}{\partial x^{i_k}_{p}}\frac{\partial x^{j_1}_{p}}{\partial x^{t_1}_{q}}\cdots\frac{\partial x^{j_l}_{p}}{\partial x^{t_l}_{q}}\,^{(p)}T^{i_1\ldots i_k}_{j_1\ldots j_l}. \tag{7}$$

All of the definitions and results of Chapter 3 of Part I pertaining to tensors defined on regions of Cartesian n-space, now apply without change to tensors on manifolds.

A metric $g_{\alpha\beta}$ on a manifold provides an example of a tensor of type (0, 2) (compare (6) and (7)). On an oriented manifold such a metric gives rise to a *volume element*

$$T_{\alpha_1\ldots\alpha_n}=\sqrt{|g|}\,\varepsilon_{\alpha_1\ldots\alpha_n},\qquad g=\det(g_{\alpha\beta}),$$

where $\varepsilon_{\alpha_1\ldots\alpha_n}$ is the skew-symmetric tensor of rank n such that $\varepsilon_{12\ldots n}=1$ (see §18.2 of Part I). It follows (as in §18.2 of Part I) that the volume element is a tensor with respect to co-ordinate changes with positive Jacobian, and so is indeed a tensor on our manifold-with-orientation. As in Part I, so also in the present context of general manifolds, it is convenient to write the volume element in the notation of differential forms (in arbitrary co-ordinates defining the same orientation):

$$\Omega=\sqrt{|g|}\,dx^1\wedge\cdots\wedge dx^n.$$

A Riemannian metric dl^2 on a (connected) manifold M gives rise to a metric space structure on M with distance function $\rho(P, Q)$ defined by

$$\rho(P, Q) = \min_{\gamma} \int_{\gamma} dl,$$

where the infimum is taken over all piecewise smooth arcs joining the points P and Q. We leave it to the reader to verify that the topology on M defined by this metric-space structure coincides with the Euclidean topology on M.

It follows from the results of §29.2 of Part I, that any two points of a manifold (with a Riemannian metric defined on it) sufficiently close to one another can be joined by a geodesic arc. For points far apart this may in general not be possible, though if the manifold is connected such points can be joined by a broken geodesic.

1.3. Embeddings and Immersions of Manifolds. Manifolds with Boundary

1.3.1. Definition. A manifold M of dimension m is said to be *immersed* in a manifold N of dimension $n \geq m$, if there is given a smooth map $f: M \to N$ such that the induced map f_* is at each point a one-to-one map of the tangent plane (or in other words if in terms of local co-ordinates the Jacobian matrix of the map f at each point has rank m). The map f is called an *immersion* of the manifold M into the manifold N. (In its image in N, self-intersections of M may occur.)

An immersion of M into N is called an *embedding* if it is one-to-one. Abusing language slightly, we shall then call M a *submanifold* of N.

We shall always assume that any submanifold M we consider is defined in each local co-ordinate neighbourhood U_p of the containing manifold N by a system of equations

$$\left.\begin{array}{r} f_p^1(x_p^1, \ldots, x_p^n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f_p^{n-m}(x_p^1, \ldots, x_p^n) = 0, \end{array}\right\} \quad \text{where} \quad \operatorname{rank}\left(\frac{\partial f_p^\alpha}{\partial x_p^\beta}\right) = n - m,$$

with the property that on each intersection $U_q \cap U_p$, the systems $(f_p^\alpha = 0)$ and $(f_q^\alpha = 0)$ have the same set of zeros. It follows that throughout each neighbourhood U_p of N we can introduce new local co-ordinates $y_p^1, \ldots, y_p^n$ satisfying

$$y_p^{m+1} = f_p^1(x_p^1, \ldots, x_p^n), \ldots, y_p^n = f_p^{n-m}(x_p^1, \ldots, x_p^n).$$

In terms of these co-ordinates the submanifold M is in each U_p given by the equations

$$y_p^{m+1}=0, \dots, y_p^n=0,$$

while $y_p^1, \dots, y_p^m$ will serve as local co-ordinates on the submanifold M.

1.3.2. Definition. A closed region A of a manifold M, defined by an inequality of the form $f(x) \leq 0$ (or $f(x) \geq 0$) where f is a smooth real-valued function on M, is called a *manifold-with-boundary*. (It is assumed here that the boundary ∂A, given by the equation $f(x)=0$, is a non-singular submanifold of M, i.e. that the gradient of the function f does not vanish on that boundary.)

Let A and B be manifolds with boundary, both given, as in the preceding definition, as closed regions of manifolds M and N respectively. A map $\varphi: A \to B$ is said to be a *smooth map of manifolds-with-boundary* if it is the restriction to A of a smooth map

$$\tilde{\varphi}: U \to N, \qquad \tilde{\varphi}|_A = \varphi,$$

of an open region U of M, containing A. (If A is defined in M by the inequality $f(x) \leq 0$, then U is usually taken to be $U_\varepsilon = \{x | f(x) < \varepsilon\}$ where $\varepsilon > 0$.

We conclude this section by mentioning yet another widely used term: a compact manifold without boundary is called *closed*.

§2. The Simplest Examples of Manifolds

2.1. Surfaces in Euclidean Space. Transformation Groups as Manifolds

A non-singular surface of dimension k in n-dimensional Euclidean space is given by a set of $n-k$ equations

$$f_i(x^1, \dots, x^n) = 0, \qquad i=1, \dots, n-k, \tag{1}$$

where for all x the matrix $(\partial f_i/\partial x^j)$ has rank $n-k$. If at a point $(x_0^1, \dots, x_0^n)$ on this surface the minor $J_{j_1 \dots j_{n-k}}$ made up of those columns of the matrix $(\partial f_i/\partial x^j)$ indexed by $j_1, \dots, j_{n-k}$, is non-zero, then as local co-ordinates on a neighbourhood of the surface about the point we make take

$$(y^1, \dots, y^k) = (x^1, \dots, \hat{x}^{j_1}, \dots, \hat{x}^{j_{n-k}}, \dots, x^n), \tag{2}$$

where the hatted symbols are to be omitted (see §7.1 of Part I). Since the surface is presupposed non-singular, it follows that it is covered by the regions of the form $U_{j_1 \dots j_{n-k}}$, where this symbol denotes the set of all points of the surface at which the minor $J_{j_1 \dots j_{n-k}}$ does not vanish.

2.1.1. Theorem. *The covering of the surface* (1) *by the regions*

$$U_{j_1 \dots j_{n-k}}, \qquad 1 \le j_1 < \dots < j_{n-k} \le n,$$

each furnished with local co-ordinates (2), *defines on the surface the structure of a smooth manifold.*

PROOF. Throughout the region $U_{j_1 \dots j_{n-k}}$ of the surface (1) equations of the following form hold:

$$x^{j_i} = \varphi^i(y^1, \dots, y^k), \qquad i = 1, \dots, n-k,$$

where the φ^i are (smooth) functions. Similarly, in the region $U_{s_1 \dots s_{n-k}}$ with coordinates

$$(z^1, \dots, z^k) = (x^1, \dots, \hat{x}^{s_1}, \dots, \hat{x}^{s_{n-k}}, \dots, x^n),$$

we have

$$x^{s_i} = \psi^i(z^1, \dots, z^k), \qquad i = 1, \dots, n-k,$$

where again the ψ^i are smooth functions. Throughout the region of intersection of $U_{j_1 \dots j_{n-k}}$ and $U_{s_1 \dots s_{n-k}}$, we have the following smooth transition functions $y \to z$ and $z \to y$ (where for ease of expression we are assuming $1 < j_1 < s_1 < j_2 < \cdots$; the general case is clear from this):

$$\begin{aligned}
y^1 &= z^1 && (= x^1), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y^{j_1 - 1} &= z^{j_1 - 1} && (= x^{j_1 - 1}), \\
\varphi^1(y^1, \dots, y^k) &= z^{j_1} && (= x^{j_1}), \\
y^{j_1} &= z^{j_1 + 1} && (= x^{j_1 + 1}), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y^{s_1 - 2} &= z^{s_1 - 1} && (= x^{s_1 - 1}), \\
y^{s_1 - 1} &= \psi^1(z^1, \dots, z^k) && (= x^{s_1}), \\
y^{s_1} &= z^{s_1} && (= x^{s_1 + 1}), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y^k &= z^k && (= x^n).
\end{aligned} \tag{3}$$

It is immediate that the two transition functions displayed here are mutual inverses, completing the proof of the theorem. □

Remark 1. It is not difficult to calculate the Jacobian of the transition function $y \to z$: it is given (up to sign) by

$$J_{(y) \to (z)} = \pm \frac{J_{s_1 \dots s_{n-k}}}{J_{j_1 \dots j_{n-k}}} \neq 0.$$

Remark 2. It is easy to see (much as in §7.2 of Part I) that the tangent space to the manifold (1) is identifiable with the linear subspace of $\mathbb{R}^n$ consisting of the solutions of the system of equations

$$\begin{array}{c}\dfrac{\partial f_1}{\partial x^\alpha}\xi^\alpha=0,\\ \cdots\cdots\cdots\cdots\\ \dfrac{\partial f_{n-k}}{\partial x^\alpha}\xi^\alpha=0.\end{array} \tag{4}$$

The (co)vectors grad $f_i=(\partial f_i/\partial x^\alpha)$, $i=1,\dots,n-k$, are orthogonal (in the sense of the standard Euclidean metric on $\mathbb{R}^n$) to the surface at each point.

Our next goal will be that of showing that a non-singular surface in Euclidean space can be oriented. For this purpose we need to introduce an alternative definition of an orientation of a manifold.

To begin with, consider at any point x of an n-manifold M the various frames (i.e. ordered bases) τ for the tangent space to M at x each consisting, of course, of n independent tangent vectors in some order. Any two such frames τ_1, τ_2 are linked to one another via a non-singular linear transformation A which sends the vectors in τ_2 to those in τ_1 in order. We shall say that the ordered bases τ_1, τ_2 *lie in the same orientation class* if $\det A>0$, and *lie in opposite orientation classes* if $\det A<0$. (Thus at each point x of the manifold M, there are exactly two orientation classes of ordered bases of the tangent space at x.) Since a frame τ for the tangent space at x can be moved continuously from x to take up the positions of frames for the tangent spaces at nearby points, it makes sense to speak of an orientation class as depending continuously on the points of the manifold. We are now ready for our alternative definition of orientation.

2.1.2. Definition. A manifold is said to be *orientable* if it is possible to choose at every point of it a single orientation class depending continuously on the points. A particular choice of such an orientation class for each point is called an *orientation* of the manifold, and a manifold equipped with a particular orientation is said to be *oriented*. If no orientation exists the manifold is *non-orientable*. (Imagine a frame moving continuously along a closed path in the manifold, and returning to the starting point with the opposite orientation.)

2.1.3. Proposition. *Definition* 1.1.3 *is equivalent to the above definition of an orientation on a manifold.*

Proof. If the manifold M is oriented in the sense of Definition 1.1.3, then at each point x of M we may choose as our orienting frame the ordered n-tuple $(e_{1j},\dots,e_{nj})$ consisting of the standard basis vectors tangent to the co-ordinate axes of the local co-ordinate system $x_j^1,\dots,x_j^n$ on the local co-ordinate neighbourhood U_j in which x lies. If x lies in two local co-ordinate neighbourhoods U_j and U_k then we shall have two orienting frames chosen at x; however, since M is oriented in the sense of Definition 1.1.3, the Jacobian of the transition function from the local co-ordinates on U_j to those on U_k is positive, so that (in view of the transformation rule for vectors) the two frames lie in the same orientation class.

Conversely, suppose that M is oriented in the sense of Definition 2.1.2 above, and that there is given at each point x a frame lying in the orientation class of the given orientation of M. Around each point x there is an open neighbourhood (in the Euclidean topology on M, and of size depending on x) sufficiently small for there to exist (new) co-ordinates $x^1, \ldots, x^n$ on the neighbourhood with the property that at each point of it the standard (ordered) basis $(e_1, \ldots, e_n)$ of vectors tangent to the axes of $x^1, \ldots, x^n$ in order, lies in the given orientation class; this is so in view of the continuity of the dependence of the given orientation class on the points of M. If we choose one such neighbourhood (with the new co-ordinates introduced on it) for each point of M, then their totality forms a covering of the manifold by local co-ordinate neighbourhoods; furthermore, the transition functions for the regions of overlap all have positive Jacobians, since at each point of such regions the standard frames lie in the same orientation class (namely the one given beforehand on M). This completes the proof. □

2.1.4. Theorem. *A smooth non-singular surface M^k in n-dimensional space $\mathbb{R}^n$, defined by a system of equations of the form* (1), *is orientable.*

Proof. Let τ denote a point-dependent tangent frame to the surface M^k. Obviously the (ordered) n-tuple $\hat{\tau} = (\tau, \operatorname{grad} f_1, \ldots, \operatorname{grad} f_{n-k})$ of vectors is linearly independent at each point (since the (co)vectors grad f_i are linearly independent among themselves and orthogonal to the surface). Now choose τ at each point of the surface M^k in such a way that the frame $\hat{\tau}$ (for the tangent space of $\mathbb{R}^n$) lies in the same orientation class as the standard frame $(e_1, \ldots, e_n)$. Since this orientation class is certainly continuously dependent on the points of $\mathbb{R}^n$, so also will the orientation class of τ depend continuously on the points of M^k. This completes the proof. □

The simplest example of a non-singular surface in $\mathbb{R}^{n+1}$ is the *n-dimensional sphere S^n*, defined by the equation

$$x_1^2 + \cdots + x_{n+1}^2 = 1;$$

it is a compact n-manifold. Convenient local co-ordinates on the n-sphere are obtained by means of the stereographic projection (see §9 of Part I). Thus let U_N denote the set of all points of the sphere except for the north pole $N = (0, \ldots, 0, 1)$, and similarly let U_S be the whole sphere with the south pole $S = (0, \ldots, 0, -1)$ removed. Local co-ordinates $(u_N^1, \ldots, u_N^n)$ on the region U_N are obtained by stereographic projection, from the north pole, of the sphere onto the hyperplane $x^{n+1} = 0$; similarly, projecting stereographically from

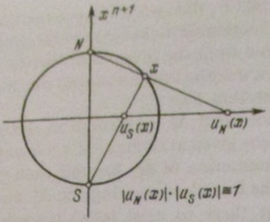

Figure 1. Local co-ordinates on the sphere via stereographic projections.

the south pole onto the same hyperplane yields co-ordinates $(u_S^1, \dots, u_S^n)$ for the region U_S (see Figure 1). It is clear from Figure 1 that the origin and the two points $u_N(x)$ and $u_S(x)$ in the plane x^{n+1} are collinear, and that the product of the distances of $u_N(x)$ and $u_S(x)$ from the origin is unity. From this and a little more it follows easily that the transition function from the co-ordinates $(u_N^1, \dots, u_N^n)$ to the co-ordinates $(u_S^1, \dots, u_S^n)$ is given by (verify it!)

$$(u_S^1, \dots, u_S^n) = \left(\frac{u_N^1}{\sum_{\alpha=1}^{n} (u_N^\alpha)^2}, \dots, \frac{u_N^n}{\sum_{\alpha=1}^{n} (u_N^\alpha)^2} \right), \tag{5}$$

while the transition functions in the other direction are obtained by interchanging the letters N and S in this formula.

The n-sphere bounds a manifold with boundary, denoted by D^{n+1} and called the (closed) $(n+1)$-*dimensional disc* (or *ball*), defined by the inequality

$$f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1 \leq 0.$$

Note finally that the sphere S^n separates the whole space $\mathbb{R}^{n+1}$ into two non-intersecting regions defined by $f(x) < 0$ and $f(x) > 0$.

Finally (before turning to the consideration of the classical transformation groups) we introduce the concept of "two-sidedness".

2.1.5. Definition. A connected $(n-1)$-dimensional submanifold of Euclidean space $\mathbb{R}^n$ is called *two-sided* if a (single-valued) continuous field of unit normals can be defined on it. We shall call such a submanifold a *two-sided hypersurface*. (See the remark below for the justification of this.)

2.1.6. Theorem. *A two-sided hypersurface in* $\mathbb{R}^n$ *is orientable.*

PROOF. Let v be a continuous field of unit normal vectors to a two-sided hypersurface M. At each point of M choose an ordered basis τ for the tangent space in such a way that the frame (τ, v) and the standard tangent frame $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ lie in the same orientation class of $\mathbb{R}^n$. It follows that the orientation class of τ must be continuously dependent on the points of M, yielding the desired conclusion. □

Remark. It will be shown in §7 that any two-sided hypersurface in $\mathbb{R}^n$ is defined by a single non-singular equation $f(x)=0$ (and hence is indeed a hypersurface), whence it follows that such a hypersurface always bounds a manifold-with-boundary. Somewhat later, in Chapter 3, it will also be proved that any closed hypersurface in $\mathbb{R}^n$ is two-sided.

The transformation groups introduced in §14 of Part I constitute important instances of manifolds defined by systems of equations in Euclidean space. Thus in particular:

(1) the general linear group $GL(n, \mathbb{R})$, consisting of all $n \times n$ real matrices with non-zero determinant, is clearly a region of $\mathbb{R}^{n^2}$;

(2) the special linear group $SL(n, \mathbb{R})$ of matrices with determinant $+1$ is the hypersurface in $\mathbb{R}^{n^2}$ defined by the single equation

$$\det A = 1;$$

(3) the orthogonal group $O(n, \mathbb{R})$ is the manifold defined by the system of equations

$$AA^{\mathrm{T}} = 1.$$

(4) the group $U(n)$ of unitary matrices is defined in the space of dimension $2n^2$ of all complex matrices by the equations

$$A\bar{A}^T = 1,$$

where the bar denotes complex conjugation.

In §14 of Part I it was shown that these groups (and others) are smooth non-singular surfaces in $\mathbb{R}^{n^2}$ (or $\mathbb{R}^{2n^2}$); we can now therefore safely call them smooth manifolds.

Note that all of these "group" manifolds G have the following property, linking their manifold and group structures: the maps $\varphi\colon G \to G$, defined by $\varphi(g)=g^{-1}$ (i.e. the taking of inverses), and $\psi\colon G \times G \to G$ defined by $\psi(g, h)=gh$ (i.e. the group multiplication), are smooth maps.

2.1.7. Definition. A manifold G is called a *Lie group* if it has given on it a group operation with the property that the maps φ, ψ defined as above in terms of the group structure, are smooth.

All of the transformation groups considered in Part I are in fact Lie groups.

2.2. Projective Spaces

We define an equivalence relation on the set of all non-zero vectors of $\mathbb{R}^{n+1}$ (regarded as a vector space) by taking two non-zero vectors to be equivalent if they are scalar multiples of one another. The equivalence classes under this relation are then taken to be the points of (*real*) *projective space of dimension* n, denoted by $\mathbb{R}P^n$. (Each projective space comes with a natural manifold structure, which will be precisely defined below.)

We now give an alternative (topologically equivalent) description of $\mathbb{R}P^n$. Consider the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin. Since such a straight line is completely determined by any direction vector, and since any non-zero scalar multiple of any particular direction vector serves equally well, we may take these straight lines as the points of $\mathbb{R}P^n$. Now each of these straight lines intersects the sphere S^n (with equation $(y^0)^2+\cdots+(y^n)^2=1$) at exactly two (diametrically opposite) points. Thus the points of $\mathbb{R}P^n$ are in one-to-one correspondence with the pairs of diametrically opposite points of the n-sphere. We may therefore think of projective space $\mathbb{R}P^n$ as obtained from S^n by "glueing", as they say, that is by identifying, diametrically opposite points. (We note in passing the consequence that functions on $\mathbb{R}P^n$ may be considered as even functions on the sphere S^n: $f(y)=f(-y)$.)

Examples. (a) The *projective line* $\mathbb{R}P^1$ has as its points pairs of diametrically opposite points of the circle S^1. Since every point of the upper semicircle (where $y>0$) has its partner in the lower semicircle, we can obtain (a topologically equivalent space to) $\mathbb{R}P^n$ by taking only the bottom semicircle (together with the points where $x=\pm 1$) and identifying its end points $x=\pm 1$. Clearly the result is again a circle; we have thus constructed a one-to-one correspondence (which is in fact a topological equivalence) between $\mathbb{R}P^1$ and the circle S^1 (see Figure 2).

The analogous construction can be carried out in the general case, i.e. for $\mathbb{R}P^n$. One takes the disc D^n (obtained as the lower half of the sphere S^n) and identifies diametrically opposite points of its boundary. (The case $n=2$ is illustrated in Figure 3.)

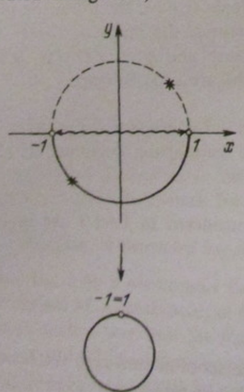

Figure 2

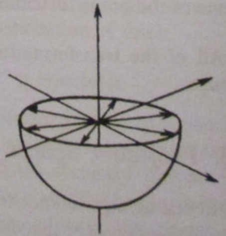

Figure 3

(b) In §14.3 of Part I a homomorphism from the group $SU(2)$ onto the group $SO(3)$ was constructed, under which each matrix A of $SU(2)$ has the same image as $-A$ (i.e. having kernel $\{\pm 1\}$), or, in other words, identifying the points A and $-A$ of the manifold $SU(2)$ in the image manifold $SO(3)$. We saw in §14.1 of Part I that there is a homeomorphism between the manifold $SU(2)$ and the 3-sphere S^3 under which matrices A and $-A$ are sent to diametrically opposite points of S^3. Hence we obtain an identification (in fact topological equivalence) of $SO(3)$ with projective 3-space $\mathbb{R}P^3$.

We now introduce explicitly a (natural) manifold structure on the projective spaces $\mathbb{R}P^n$.

For this purpose we return to our original characterization of $\mathbb{R}P^n$ as consisting of equivalence classes of non-zero vectors in the space $\mathbb{R}^{n+1}$ with co-ordinates $y^0, \dots, y^n$. For each $q = 0, 1, \dots, n$, let U_q denote the set of equivalence classes of vectors $(y^0, \dots, y^n)$ with $y^q \neq 0$. On each such region U_q of $\mathbb{R}P^n$ we introduce the local co-ordinates $x_q^1, \dots, x_q^n$ defined by

$$\begin{gathered} x_q^1 = \frac{y^0}{y^q}, \dots, x_q^q = \frac{y^{q-1}}{y^q}, \\ x_q^{q+1} = \frac{y^{q+1}}{y^q}, \dots, x_q^n = \frac{y^n}{y^q}. \end{gathered} \tag{6}$$

Clearly the regions U_q, $q = 0, 1, \dots, n$, cover the whole of projective n-space.

We next calculate the transition functions. For notational simplicity we do this for the particular pair U_0, U_1: the general formulae for the transition functions on $U_j \cap U_k$ can be obtained from those for $U_0 \cap U_1$ by the appropriate replacement of indices. Now the co-ordinates in U_0 are given by

$$x_0^1 = \frac{y^1}{y^0}, x_0^2 = \frac{y^2}{y^0}, \dots, x_0^n = \frac{y^n}{y^0},$$

and in U_1 by

$$x_1^1 = \frac{y^0}{y^1}, x_1^2 = \frac{y^2}{y^1}, \dots, x_1^n = \frac{y^n}{y^1}.$$

Hence in the region $U_0 \cap U_1$, where both $y^0, y^1 \neq 0$, the transition function from (x_0) to (x_1) is obviously

$$x_1^1 = \frac{1}{x_0^1}, x_1^2 = \frac{x_0^2}{x_0^1}, x_1^3 = \frac{x_0^3}{x_0^1}, \dots, x_1^n = \frac{x_0^n}{x_0^1}. \tag{7}$$

(Note that $x_0^1 = y^1/y^0$ is non-zero on $U_0 \cap U_1$.) The Jacobian of this transition function is given by

$$J_{(x_0)\to(x_1)} = \det \begin{pmatrix} -\frac{1}{(x_0^1)^2} & 0 & \dots & \dots & 0 \\ -\frac{x_0}{(x_0^1)^2} & \frac{1}{x_0^1} & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = -\frac{1}{(x_0^1)^{n+1}} \neq 0.$$

Since, as noted above, the general transition functions on $U_j \cap U_k$ are obtained similarly, it follows that $\mathbb{R}P^n$ with the U_q as local coordinate neighbourhoods is indeed a smooth manifold. The manifold $\mathbb{R}P^2$ (with $n = 2$) is called the *projective plane*; in this case the region U_0 is called the *finite part of the projective plane*.

Finally we note that, as is easily shown, the one-to-one correspondences $S^1 \to \mathbb{R}P^1$ and $SO(3) \to \mathbb{R}P^3$ described in the above examples, are in fact diffeomorphisms.

We define *complex projective space* $\mathbb{C}P^n$ similarly; its points are the equivalence classes of non-zero vectors in $\mathbb{C}^{n+1}$ under the analogous equivalence relation (i.e. scalar multiples are identified), and the local co-ordinate neighbourhoods, with their co-ordinates, are defined as in the real case, making $\mathbb{C}P^n$ a $2n$-dimensional smooth manifold.

By way of an example, we consider in detail the complex projective line $\mathbb{C}P^1$. Its points are the equivalence classes of non-zero pairs (z^0, z^1) of complex numbers, where the equivalence is defined by $(z^0, z^1) \sim (\lambda z^0, \lambda z^1)$ for any non-zero complex number λ. Consider the (complex) function $w_0(z^0, z^1) = z^1/z^0$; this function is defined (and one-to-one) on all of $\mathbb{C}P^1$ except (the equivalence class of) (0, 1): we shall formally define w_0 as taking the value ∞ at this point. Thus via the function w_0 the complex projective line $\mathbb{C}P^1$ becomes identified with the "extended complex plane" (i.e. the ordinary complex plane with an additional "point at infinity").

2.2.1. Theorem. *The complex projective line* $\mathbb{C}P^1$ *is diffeomorphic to the 2-dimensional sphere* S^2.

PROOF. On the region U_0 of the complex projective line consisting of all equivalence classes of non-zero pairs (i.e. non-zero pairs determined only up to scalar multiples) (z^0, z^1) with $z^0 \neq 0$, we introduce local co-ordinates u_0, v_0 defined by $u_0 + iv_0 = w_0 = z^1/z^0$. (These local co-ordinates may be regarded as defining a one-to-one map from U_0 onto the real plane $\mathbb{R}^2$.) Similarly, u_1, v_1, defined by $u_1 + iv_1 = w_1 = z^0/z^1$, will serve as co-ordinates on the region U_1 consisting of pairs (z^0, z^1) (up to scalar multiples) with $z^1 \neq 0$. Clearly the regions U_0 and U_1 cover $\mathbb{C}P^1$. The transition function from (u_0, v_0) to (u_1, v_1) on the region of intersection is given by

$$(u_1, v_1) = \left(\frac{u_0}{u_0^2 + v_0^2}, -\frac{v_0}{u_0^2 + v_0^2} \right),$$

or, in complex notation, by

$$u_1 + iv_1 = w_1 = \frac{1}{w_0} = \frac{u_0 - iv_0}{u_0^2 + v_0^2}.$$

Since this formula coincides with the formula (5) (in the case $n = 2$) for the transition functions for the stereographic co-ordinates on the sphere S^2, the theorem follows. □

It is on account of this result that the extended complex plane is often called the "Riemann sphere". Note that if $w = u + iv$ provides local co-ordinates u, v for the finite part of the extended complex plane (i.e. for the ordinary complex plane), then $1/w$ provides local co-ordinates of a (punctured) neighbourhood of the "point at infinity" ∞.

We now return to the consideration of the general complex projective space $\mathbb{C}P^n$. From each equivalence class of $(n+1)$-vectors we may choose as representative a vector whose tip lies on the unit sphere S^{2n+1}, i.e. satisfying

$$|z^0|^2 + \cdots + |z^n|^2 = 1,$$

by simply multiplying any vector $z = (z^0, \ldots, z^n)$ in the class by the scalar $\lambda = (\sum_{\alpha=0}^{n} |z^\alpha|)^{-1/2}$. The resulting vector (with tip on S^{2n+1}) is then clearly unique only up to multiplication by scalars of the form $e^{i\varphi}$, i.e. by complex numbers of modulus 1. We therefore conclude that:

Complex projective space $\mathbb{C}P^n$ can be obtained from the (unit) sphere $S^{2n+1} = \{z \mid \sum_{\alpha=0}^{n} |z^\alpha|^2 = 1\}$, by identifying all points $e^{i\varphi}z$ on the sphere (φ variable) with z, i.e. by identifying all points differing by a scalar factor of the form $e^{i\varphi}$.

Thus we have a map

$$S^{2n+1} \to \mathbb{C}P^n, \tag{8}$$

such that the pre-image of each point of $\mathbb{C}P^n$ is (topologically equivalent to) the circle $S^1 = \{e^{i\varphi}\}$. In particular, in view of Theorem 2.2.1, we obtain thence a map

$$S^3 \to S^2, \qquad (z^0, z^1) \mapsto w = \frac{z^1}{z^0} \qquad (|z^0|^2 + |z^1|^2 = 1).$$

2.3. Exercises

1. Prove that the odd-dimensional projective spaces $\mathbb{R}P^{2k+1}$ are orientable.
2. Prove that the connected component containing the identity element of a Lie group, is a normal subgroup.
3. Prove that a connected Lie group is generated by an arbitrarily small neighbourhood of the identity element.
4. Prove that every Lie group is orientable.
5. Prove that the projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are compact.
6. *Quaternion projective space* $\mathbb{H}P^n$ is defined as the set of equivalence classes of non-zero quaternion vectors in $\mathbb{H}^{n+1}$, where two $(n+1)$-tuples are equivalent if one is a

multiple of the other (by a non-zero quaternion). Define a manifold structure on $\mathbb{H}P^n$, and verify that $\mathbb{H}P^1$ is diffeomorphic to S^4.

7. Construct a mapping $S^{4n+3} \to \mathbb{H}P^n$, analogous to the mapping (8), and identify the complete inverse image of a point of $\mathbb{H}P^n$ under this map.

§3. Essential Facts from the Theory of Lie Groups

3.1. The Structure of a Neighbourhood of the Identity of a Lie Group. The Lie Algebra of a Lie Group. Semisimplicity

Every Lie group G (see Definition 2.1.6) has a distinguished point $g_0 = 1 \in G$ (the identity element), and, being by definition a smooth manifold, has a tangent space $T = T_{(1)}$ at that point. For each $h \in G$ the transformation $G \to G$, defined by $g \mapsto hgh^{-1}$, is called the *inner automorphism* of G determined by h. Any such transformation of G clearly fixes the identity element $g_0 = 1$ (since $hg_0h^{-1} = g_0$), and therefore the induced linear map of the tangent space T to G at the identity (see §1.2 above) is a linear transformation of T, denoted by

$$\mathrm{Ad}(h)\colon T \to T.$$

From the definitions of the inner automorphism determined by each element h, and the linear map of the tangent space T which it induces, it follows easily that $\mathrm{Ad}(h^{-1}) = [\mathrm{Ad}(h)]^{-1}$ and $\mathrm{Ad}(h_1h_2) = \mathrm{Ad}(h_1)\,\mathrm{Ad}(h_2)$, for all h, h_1, h_2 in G. Hence the map $h \mapsto \mathrm{Ad}(h)$ is a linear representation (i.e. a homomorphism to a group of linear transformations) of the group G:

$$\mathrm{Ad}\colon G \to GL(n, \mathbb{R}),$$

where n is the dimension of G. (Note that for commutative groups G the representation Ad is trivial, i.e. $\mathrm{Ad}(h) = 1$ for all $h \in G$.)

We shall now express the group operation on a Lie group G in a neighbourhood of the identity, in terms of local co-ordinates on such a neighbourhood. We first re-choose co-ordinates in a neighbourhood of the identity element so that the identity element is the origin: $1 = g_0 = (0, \ldots, 0)$. We then express in functional notation the co-ordinates of the product g_1g_2 (if it is still in the neighbourhood) of elements $g_1 = (x^1, \ldots, x^n)$ and $g_2 = (y^1, \ldots, y^n)$ by

$$\psi^\alpha(x, y) = \psi^\alpha(x^1, \ldots, x^n, y^1, \ldots, y^n), \qquad \alpha = 1, \ldots, n,$$

and the co-ordinates of the inverse g^{-1} of an element $g = (x^1, \ldots, x^n)$ by

$$\varphi^\alpha(x) = \varphi^\alpha(x^1, \ldots, x^n), \qquad \alpha = 1, \ldots, n.$$

The functions $\psi(x, y)$ $(=g_1g_2)$ and $\varphi(x)$ $(=g^{-1})$ obviously satisfy the following conditions (arising from the defining properties of a group):

(i) $\psi(x, 0) = \psi(0, x) = x$ (property of the identity);
(ii) $\psi(x, \varphi(x)) = 0$ (property of inverses);
(iii) $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$ (associative property).

Given sufficient smoothness of the function $\psi(x, y)$, it follows from condition (i) (and Taylor's theorem) that

$$\psi^\alpha(x, y) = x^\alpha + y^\alpha + b^\alpha_{\beta\gamma} x^\beta y^\gamma + \text{(terms of order} \geq 3) \tag{1}$$

(where of course $b^\alpha_{\beta\gamma} = \partial^2\psi^\alpha/\partial x^\beta \partial y^\gamma$ evaluated at the origin).

Now let ξ and η be tangent vectors to the group at the identity, i.e. elements of the space T, and as usual denote their components, in terms of our co-ordinates x^α, by ξ^α, η^α respectively. The *commutator* $[\xi, \eta] \in T$ of tangent vectors ξ, η is defined by

$$[\xi, \eta]^\alpha = (b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta})\xi^\beta \eta^\gamma. \tag{2}$$

This commutator operation on T has the following three basic properties:

(a) $[\ ,\ ]$ is a bilinear operation on the n-dimensional linear space T (where n is the dimension of G);
(b) $[\xi, \eta] = -[\eta, \xi]$;
(c) $[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$ ("Jacobi's identity"). (3)

(The first two of these properties are almost immediate from the definition of the commutator operation. Here is a sketch of the proof of (c) as a consequence of the associative law (iii) above: From (1) we obtain that

$$\psi^\alpha(\psi(x, y), z) = \psi^\alpha(x, y) + z^\alpha + b^\alpha_{\beta\gamma}\psi^\beta(x, y)z^\gamma$$
$$+ \text{(terms of degree} \geq 3 \text{ in } \psi^i, z^j).$$

Substitution in this from (1) yields an expansion of $\psi^\alpha(\psi(x, y), z)$ in terms of x^μ, y^ν, z^γ in which the coefficient of $x^\mu y^\nu z^\gamma$ is $b^\alpha_{\beta\gamma} b^\beta_{\mu\nu}$. Repeating this procedure for $\psi^\alpha(x, \psi(y, z))$ and comparing the coefficient of $x^\mu y^\nu z^\gamma$ with that obtained in the case of $\psi^\alpha(\psi(x, y), z)$, we find that

$$b^\alpha_{\beta\gamma} b^\beta_{\mu\nu} = b^\alpha_{\mu\beta} b^\beta_{\nu\gamma} \tag{4}$$

On the other hand from the definition of the commutator we obtain

$$[[\xi, \eta], \zeta]^\alpha = (b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta})[\xi, \eta]^\beta \zeta^\gamma = (b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta})(b^\beta_{\mu\nu} - b^\beta_{\nu\mu})\xi^\mu \eta^\nu \zeta^\gamma.$$

It follows that Jacobi's identity is equivalent to

$$(b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta})(b^\beta_{\mu\nu} - b^\beta_{\nu\mu}) + (b^\alpha_{\beta\nu} - b^\alpha_{\nu\beta})(b^\beta_{\gamma\mu} - b^\beta_{\mu\gamma}) + (b^\alpha_{\beta\mu} - b^\alpha_{\mu\beta})(b^\beta_{\nu\gamma} - b^\beta_{\gamma\nu}) = 0,$$

which is easily seen to be a consequence of (4), as required.)

Thus the tangent space to G at the identity is with respect to the commutator operation a *Lie algebra*; since it arises from G it is called the *Lie algebra of the Lie group G*. (Cf. §24.1 of Part I.)

If $e_1, \ldots, e_n$ are the standard basis vectors of T (in terms of the co-ordinates $x^1, \ldots, x^n$) then since $[e_\beta, e_\gamma]$ is again a vector in T, we may write

$$[e_\beta, e_\gamma] = c^\alpha_{\beta\gamma} e_\alpha,$$

whence by bilinearity

$$[\xi, \eta]^\alpha = c^\alpha_{\beta\gamma} \xi^\beta \eta^\gamma, \tag{5}$$

for all vectors ξ, η in T. The constants $c^\alpha_{\beta\gamma}$, which clearly determine the commutator operation on the Lie algebra, and which are skew-symmetric in the indices β, γ, are called the *structural constants of the Lie algebra.*

A *one-parameter subgroup* of a Lie group G is defined to be a parametrized curve $F(t)$ on the manifold G such that $F(0) = 1$, $F(t_1 + t_2) = F(t_1)F(t_2)$, $F(-t) = F(t)^{-1}$. (Thus the one-parameter subgroup is determined by a homomorphism $t \mapsto F(t)$ from the additive reals to G.)

(Before proceeding we note parenthetically that left multiplication by a fixed element h of an abstract Lie group G defines a diffeomorphism $G \to G$ $(g \mapsto hg)$; the induced map of tangent spaces is defined as before to send the tangent vector $\dot{g}(t)$ to a curve $g(t)$, to the tangent vector $(d/dt)(hg(t))$ to the curve $hg(t)$.) Now if $F(t)$ is a one-parameter subgroup of G, then

$$\frac{dF}{dt} = \frac{dF(t+\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}(F(t)F(\varepsilon))|_{\varepsilon=0} = F(t)\frac{dF(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0},$$

where the last equality follows from the preceding parenthetical definition of the tangent-space map induced by left multiplication on the group by the element $F(t)$. Hence $\dot{F}(t) = F(t)\dot{F}(0)$, or $F(t)^{-1}\dot{F}(t) = \dot{F}(0)$, i.e. the induced action of left multiplication by $F(t)^{-1}$ sends $\dot{F}(t)$ to $\dot{F}(0) = \text{const}$. Conversely, for each particular tangent vector A of T, the equation

$$F^{-1}\dot{F} = A \tag{6}$$

is satisfied by a unique one-parameter subgroup $F(t)$ of G; to see this note first that (6) is (when formulated in terms of the function $\psi(x, y)$ defining the multiplication of points x, y of G) a system of ordinary differential equations, and therefore by the appropriate existence and uniqueness theorem for the solutions of such systems, has, for some sufficiently small $\varepsilon > 0$ a unique solution $F(t)$ for $|t| < \varepsilon$. The values of $F(t)$ for all larger $|t|$ can then be obtained by forming long enough products of elements $F(\delta)$ with $|\delta| < \varepsilon$.

In the case that G is a matrix group it follows from (6) that $F(t) = \exp At$ (see §§14.2, 24.3 of Part I). We shall use this notation also for the one-parameter subgroup arising from A via (6) in the general case of an arbitrary Lie group.

EXERCISE

Let $F_1(t)$ and $F_2(t)$ be two one-parameter subgroups of a Lie group G with $A_1 = \dot{F}_1(0)$, $A_2 = \dot{F}_1(0)$, whence $F_1(t) = \exp A_1 t$, $F_2(t) = \exp A_2 t$. Prove that

$$t^2[A_1, A_2] = F_1(t)F_2(t)F_1^{-1}(t)F_2^{-1}(t) + O(t^3). \tag{7}$$

Let $F(t) = \exp At$ be a one-parameter subgroup of a Lie group G. For each t the inner automorphism $g \mapsto FgF^{-1}$ induces (as we saw above) a linear transformation $\mathrm{Ad}\, F(t)$ of the Lie algebra $T = T_{(1)}$, which, since T is n-dimensional, lies in $GL(n, \mathbb{R})$. It follows that $\mathrm{Ad}\, F(t)$ is a one-parameter subgroup of $GL(n, \mathbb{R})$, whence the vector $(d/dt)\, \mathrm{Ad}\, F(t)|_{t=0}$ lies in the Lie algebra of the group $GL(n, \mathbb{R})$, and so can be regarded as a linear operator.

EXERCISE

Prove that, as operator, $(d/dt)\, \mathrm{Ad}\, F(t)|_{t=0}$ is given by $B \mapsto [A, B]$ for B in the Lie algebra (which is identifiable with $\mathbb{R}^n$). (As in §24.1 of Part I, we denote the map $B \mapsto [A, B]$ by $\mathrm{ad}\, A: \mathbb{R}^n \to \mathbb{R}^n$.)

We next use the one-parameter subgroups to define canonical co-ordinates in a neighbourhood of the identity of a Lie group G. Let $A_1, \ldots, A_n$ form a basis for the Lie algebra T (which we may identify with $\mathbb{R}^n$), the tangent space to G at the point $g_0 = 1$. We saw above that to each vector $A = \sum A_i x^i$ in T there corresponds a one-parameter group $F(t) = \exp At$. To the point $F(1)$ (which it is natural to denote also by $\exp A$) we assign as co-ordinates the coefficients $x^1, \ldots, x^n$; in this way we obtain a system of co-ordinates (by "projecting down from the tangent space" as it were) in a sufficiently small neighbourhood of the identity element of G. (Verify this!) These are called *canonical co-ordinates of the first kind*.

Alternatively, writing $F_i(t) = \exp A_i t$, we have that each point g of a sufficiently small neighbourhood of the identity element can be expressed uniquely as

$$g = F_1(t_1) \ldots F_n(t_n),$$

for small $t_1, \ldots, t_n$. Assigning co-ordinates $t_1 = x_1, \ldots, t_n = x_n$ to the point g, we thus obtain the *co-ordinates of the second kind* in a neighbourhood of the identity.

EXERCISES

1. Given a curve in the form $g(\tau) = F_1(\tau t_1) \ldots F_n(\tau t_n)$, prove that

$$\left.\frac{dg}{d\tau}\right|_{\tau=0} = \sum_{i=1}^{n} t_i A_i.$$

2. Show that the "Euler angles" φ, ψ, θ (see §14.1 of Part I) constitute co-ordinates of the second kind on $SO(3)$.

Co-ordinates of the first kind are exploited in the proof of the following result.

3.1.1. Theorem. *If the functions $\psi^\alpha(x, y)$ defining the multiplication of points x, y of a Lie group G are real analytic (i.e. are representable by power series), then in some neighbourhood of $1 \in G$, the structure of the Lie algebra of G determines the multiplication in G.*

(Here the condition that ψ be real analytic (or, as they say, that G be analytic) is not crucial; however, the proof under weaker assumptions about ψ is more complicated.)

PROOF. Define auxiliary functions $v^\alpha_\beta(x)$ by

$$v^\alpha_\beta(x) = \left.\frac{\partial\psi^\alpha(x, y)}{\partial x^\beta}\right|_{y=\varphi(x)},$$

where as before $\varphi(x)$ is the function defining (in terms of co-ordinates) the inverse of x. It follows from properties (i), (ii) and (iii) of ψ and φ (towards the beginning of this section) that the functions $\psi^\alpha(x, y)$ satisfy the following system of partial differential equations in x:

$$v^\alpha_\beta(\psi(x, y))\frac{\partial\psi^\beta(x, y)}{\partial x^\gamma} = v^\alpha_\gamma(x), \tag{8}$$

with the initial conditions

$$\psi(0, y) = y.$$

(To see (8), note first that the left-hand side is

$$\left.\frac{\partial\psi^\alpha(x, y)}{\partial x^\beta}\right|_{\substack{x=\psi(x, y)\\ y=\varphi(\psi(x, y))}} \cdot \frac{\partial\psi^\beta(x, y)}{\partial x^\gamma},$$

which is the same as

$$\left.\frac{\partial\psi^\alpha(\psi(x, y), z)}{\partial x^\gamma}\right|_{z=\varphi(\psi(x, y))},$$

and then apply properties (i), (ii) and (iii) of φ and ψ.)

It can be shown that the system (8) has a solution precisely if

$$\frac{\partial v^\alpha_\beta}{\partial x^\gamma} - \frac{\partial v^\alpha_\gamma}{\partial x^\beta} = 2c^\alpha_{\mu\nu}v^\mu_\beta v^\nu_\gamma. \tag{9}$$

EXERCISE

Taking the invertibility of the matrix $(v^\alpha_\beta(\psi))$ into account, show that (9) is equivalent to

$$\frac{\partial^2\psi}{\partial x^\alpha\partial x^\gamma} = \frac{\partial^2\psi}{\partial x^\gamma\partial x^\alpha},$$

the condition for solubility of the system of "Pfaffian" equations (8) (cf. (5), (6) in §29.1).

Since (8) does indeed have a solution, namely the ψ defining the multiplication in G, it follows that equation (9) must hold.

On the other hand, if $x = x(t)$ represents the one-parameter subgroup determined by the initial velocity vector $A = (A^i)$, then, putting equation (6) into functional notation, we have

$$A^\alpha = \frac{d}{d\varepsilon} x^\alpha(\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \psi^\alpha(x(\varepsilon+t), x(-t))\Big|_{\varepsilon=0}$$

$$= \frac{\partial \psi^\alpha(x, y)}{\partial x^\beta}\bigg|_{y=\varphi(x)} \frac{dx^\beta(\varepsilon+t)}{d\varepsilon}\bigg|_{\varepsilon=0} = v^\alpha_\beta(x(t)) \frac{dx^\beta(t)}{dt}.$$

If we now take the x^α to be canonical co-ordinates of the first kind (in some neighbourhood of 1) then by definition of such co-ordinates, $x^\alpha(t) = A^\alpha t$, whence by the above

$$A^\alpha = v^\alpha_\beta(At)A^\beta,$$

yielding

$$x^\alpha = v^\alpha_\beta(x)x^\beta. \tag{10}$$

Our aim is to show that the functions $v^\alpha_\beta(x)$ are fully determined (in some neighbourhood of 1) by these canonical co-ordinates. Differentiating the last equation with respect to x^β, we obtain

$$\delta^\alpha_\beta = x^\gamma \frac{\partial v^\alpha_\gamma}{\partial x^\beta} + v^\alpha_\beta. \tag{11}$$

By multiplying equation (9) by x^β (and summing with respect to β), and then substituting from (10) and (11), we obtain

$$x^\beta \frac{\partial v^\alpha_\gamma}{\partial x^\beta} + v^\alpha_\gamma(x) = \delta^\alpha_\gamma + c^\alpha_{\mu\nu} x^\nu v^\mu_\gamma,$$

whence, on replacing x by At,

$$tA^\beta \frac{\partial v^\alpha_\gamma}{\partial x^\beta} + v^\alpha_\gamma(x) = \delta^\alpha_\gamma + c^\alpha_{\mu\nu} A^\nu t v^\mu_\gamma. \tag{12}$$

In terms of the new functions $w^\alpha_\gamma(t) = tv^\alpha_\gamma(At)$ (also dependent on A) the equations (12) take the form

$$\frac{dw^\alpha_\gamma}{dt} = \delta^\alpha_\gamma + c^\alpha_{\mu\nu} A^\nu w^\mu_\gamma, \tag{13}$$

which is a system of ordinary linear differential equations for the functions $w^\alpha_\gamma(t)$, with initial conditions $w^\alpha_\gamma(0) = 0$. Hence for each fixed A the functions $w^\alpha_\gamma(t)$ are uniquely determined by the Lie algebra structure (since the system (13) is determined by the structure constants $c^\alpha_{\mu\nu}$ (as well as A)). The w^α_γ in turn determine the functions $v^\alpha_\gamma(x)$, and thence the multiplication operation $\psi(x, y)$ as the solution of the system (8) with the given initial conditions. (It is here that the assumption of analyticity of the $\psi^\alpha(x, y)$ enters the picture, via for instance the Cauchy–Kovalevskaja theorem on (existence and) uniqueness of solutions of certain systems of partial differential equations.) This completes the proof of the theorem. □

3.1.2. Corollary. *If the Lie algebra of a connected analytic Lie group G is commutative (i.e. $[A, B]=0$), then the group G is commutative (i.e. abelian).*

PROOF. Setting $c^{\alpha}_{\beta\gamma}=0$ in (13), yields that $v^{\alpha}_{\beta}(x)=\delta^{\alpha}_{\beta}$, and then from (8) with the initial condition $\psi(0, y)=y$, that $\psi(x, y)=x+y$ on some neighbourhood. The corollary then follows from the connectedness of G, since this implies that G is generated as a group by the elements in any (arbitrarily small) neighbourhood of the identity. □

3.1.3. Definition. A Lie algebra $L=\{\mathbb{R}^n, c^i_{jk}\}$ is said to be *simple* if it is non-commutative and has no proper ideals (i.e subspaces $I \neq L, 0$, for which $[I, L] \subset I$), and *semisimple* if $L=I_1 \oplus \cdots \oplus I_k$ where the I_j are ideals which are simple as Lie algebras. (It follows that these ideals are pairwise commuting, i.e. $[I_i, I_l]=0$ for $i \neq l$). A Lie group is defined to be *simple* or *semisimple* according as its Lie algebra is respectively simple or semisimple. The *Killing form* on an arbitrary Lie algebra L is defined by

$$\langle A, B\rangle=-\operatorname{tr}(\operatorname{ad} A \operatorname{ad} B), \tag{14}$$

where the operator ad A on L is in turn defined by

$$u \mapsto [A, u], \qquad u \in L. \tag{15}$$

Earlier in this section we defined for each g in a Lie group G an automorphism Ad(g) of the Lie algebra of G (namely, that induced by the inner automorphism of G determined by its element g); it is thus natural to call the automorphism Ad(g) an *inner automorphism* of the Lie algebra.

3.1.4. Theorem. (i) *If the Lie algebra L of a Lie group G is simple, then the linear representation* Ad: $G \rightarrow GL(n, \mathbb{R})$ *is irreducible* (i.e. *L has no proper invariant subspaces under the group of inner automorphisms* Ad(G)).

(ii) *If the Killing form of a Lie algebra is positive definite then the Lie algebra is semisimple.*

PROOF. (i) Suppose on the contrary that the representation Ad is reducible, and let I be a proper invariant subspace of L invariant under Ad G. Let X, Y be any elements of L, I, respectively, and let $x(\tau)$, $y(t)$ be the one-parameter subgroups determined by the tangent vectors X, Y. The invariance of I means in particular that for all τ, the vector

$$\operatorname{Ad}(x(\tau))(Y)=\frac{d}{dt}(x(\tau)y(t)x(\tau)^{-1})|_{t=0}$$

lies again in I. We shall use canonical co-ordinates of the first kind, so that in some neighbourhood of 1 we have $x(\tau)=X\tau$, $y(t)=Yt$. From (1) applied twice in succession it follows easily that the αth component of $x(\tau)y(t)x(\tau)^{-1}$ $=x(\tau)y(t)x(-\tau)$ is given by

$$Y^{\alpha}t+[X, Y]^{\alpha}t\tau-b^{\alpha}_{\beta\gamma}X^{\beta}X^{\gamma}\tau^2-b^{\alpha}_{\beta\gamma}Y^{\beta}Y^{\gamma}t^2+\text{higher-order terms in } t, \tau.$$

(From (1) and the fact that $\psi(x(\tau), x(-\tau)) = 0 = \psi(y(t), y(-t))$, it follows that the two negative terms vanish.) On differentiating with respect to t and then setting $t = 0$, we obtain for all sufficiently small τ that

$$\mathrm{Ad}(x(\tau))(Y) = Y + [X, Y]\tau + O(\tau^2).$$

It follows that

$$[X, Y]\tau + O(\tau^2) \in I.$$

Dividing by τ and letting $\tau \to 0$, we get finally that $[X, Y] \in I$ (since with respect to the Euclidean norm every subspace of a finite-dimensional vector space is closed). Hence $[L, I] \subset I$, so that I is an ideal of L, contradicting the assumed simplicity of L.

(ii) Let I be any ideal of the Lie algebra L, and let J be the orthogonal complement of I in L with respect to the Killing form (i.e. J is the subspace of all vectors in L orthogonal to I). We first prove that J is also an ideal of L.

To this end let X, Y, Z be arbitrary elements of L, J, I, respectively; we wish to show that $[X, Y]$ is orthogonal to Z, i.e. that $\mathrm{tr}(\mathrm{ad}[X, Y]\mathrm{ad}\ Z) = 0$. Now it is easily verified from the Jacobi identity that

$$\mathrm{ad}[A, B] = \mathrm{ad}\ A\ \mathrm{ad}\ B - \mathrm{ad}\ B\ \mathrm{ad}\ A.$$

Hence

$$\mathrm{tr}(\mathrm{ad}[X, Y]\ \mathrm{ad}\ Z) = \mathrm{tr}(\mathrm{ad}\ X\ \mathrm{ad}\ Y\ \mathrm{ad}\ Z - \mathrm{ad}\ Y\ \mathrm{ad}\ X\ \mathrm{ad}\ Z),$$

and since a trace of a matrix product is invariant under cyclic permutations of the factors, it follows that

$$\begin{aligned}\mathrm{tr}(\mathrm{ad}[X, Y]\ \mathrm{ad}\ Z) &= \mathrm{tr}(\mathrm{ad}\ X\ \mathrm{ad}\ Y\ \mathrm{ad}\ Z - \mathrm{ad}\ X\ \mathrm{ad}\ Z\ \mathrm{ad}\ Y)\\ &= \mathrm{tr}(\mathrm{ad}\ X\ \mathrm{ad}[Y, Z]).\end{aligned}$$

Since $[Y, Z] \in I$ and $X \in J$, the final expression above is zero, as required.

The positive definiteness of the Killing form implies both that $L = I \oplus J$, and that no non-zero ideals of L can be commutative (since the restriction of the Killing form to a commutative ideal is zero). This completes the proof. □

Remark. There is a stronger result than (ii), due to Killing and E. Cartan: *A Lie algebra is semisimple if and only if its Killing form is non-degenerate.* In addition to the above argument, the proof of this stronger result uses the fact that the Killing form of a (non-commutative) simple Lie algebra cannot be identically zero. This is in turn a consequence of a theorem of Engel which states that the Killing form of a Lie algebra L is identically zero if and only if the Lie algebra is "nilpotent"; i.e. if there exists a positive integer k such that

$$[[\ldots[A_1, A_2], \ldots], A_k] = 0$$

for all $A_1, \ldots, A_k \in L$.

EXERCISES

1. (i) Prove that the isometries of a connected Riemannian manifold form a Lie group.
   (ii) Prove the analogous result for the group of all conformal transformations of a Riemannian manifold (see §15 of Part I).
2. Decide which of the Lie algebras encountered in Part I (see especially §24) are simple or semisimple.

3.2. The Concept of a Linear Representation. An Example of a Non-matrix Lie Group

We begin with a definition:

3.2.1. Definition. A *(linear) representation* of a group G is a homomorphism $\rho: G \to GL(n, \mathbb{R})$ or $\rho: G \to GL(n, \mathbb{C})$, from G to a group of real or complex matrices. Given a representation ρ of G, the map $\chi_\rho: G \to \mathbb{R}$ (or $G \to \mathbb{C}$) defined by $\chi_\rho(g) = \operatorname{tr} \rho(g)$, $g \in G$, is called the *character* of the representation ρ. As noted above, a representation ρ of G is said to be *irreducible* if the vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$) contains no proper subspaces invariant under the matrix group $\rho(G)$.

3.2.2. Theorem ("Schur's Lemma"). *Let $\rho_i: G \to GL(n_i, \mathbb{R})$, $i = 1, 2$, be two irreducible representations of a group G. If $A: \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is a linear transformation changing ρ_1 into ρ_2 (i.e. satisfying $A\rho_1(g) = \rho_2(g)A$), then either A is the zero transformation or else a bijection (in which case of course $n_1 = n_2$).*

PROOF. If x is an element of the kernel of A, i.e. $Ax = 0$, then for all $g \in G$

$$A\rho_1(g)x = \rho_2(g)Ax = 0,$$

whence the kernel of A is invariant under $\rho_1(G)$ and so by the irreducibility of ρ_1 must be either the whole of $\mathbb{R}^{n_1}$ (in which case $A = 0$) or else the null space. Similarly the image space $A(\mathbb{R}^{n_1}) \subset \mathbb{R}^{n_2}$ is invariant under $\rho_2(G)$, and must therefore either be the null space or the whole of $\mathbb{R}^{n_2}$. This completes the proof. □

Note that if G is a Lie group and we have a representation $\rho: G \to GL(N, \mathbb{R})$ which is a smooth map, then the differential (i.e. induced map) ρ_* is a linear map from the Lie algebra $\mathfrak{g} = T_{(1)}$ to the space of all $N \times N$ matrices:

$$\rho_*: \mathfrak{g} \to M(N, \mathbb{R}).$$

We leave it to the reader to verify that ρ_* is actually a *representation of the Lie algebra* $\mathfrak{g}$, i.e. that it is a Lie algebra homomorphism: as well as being linear, it preserves commutators:

$$\rho_*[\zeta, \eta] = [\rho_*\zeta, \rho_*\eta].$$

(We note that it can be shown that if ρ is continuous, then it will automatically be smooth.)

A representation $\rho: G \to GL(N, \mathbb{R})$ (or $G \to GL(N, \mathbb{C})$) is called *faithful* if it is one-to-one, i.e. if its kernel is trivial: $\rho(g) \neq 1$ unless $g = 1$. A matrix Lie group trivially has a faithful Lie representation (i.e. a representation which is also a topological equivalence). However, as we shall now show (by means of an example) not every Lie group can be realized (i.e. has a faithful Lie representation) as a matrix Lie group. As our example we take the group $G = \widetilde{SL}(2, \mathbb{R})$ consisting of all transformations of the real line of the form

$$x \mapsto x + 2\pi a + \frac{1}{i} \ln \frac{1 - ze^{-ix}}{1 - \bar{z}e^{ix}}, \tag{16}$$

where $x \in \mathbb{R}$, $a \in \mathbb{R}$, $z \in \mathbb{C}$, $|z| < 1$, and ln denotes the main branch of the natural logarithmic function, i.e. the continuous branch determined by $\ln 1 = 0$. (Note that in (16) the argument of the function ln is a fraction whose numerator and denominator are complex conjugates; hence the fraction has modulus 1, so that its natural logarithm is either zero or purely imaginary, and therefore the image of x in (16) is indeed real (in fact between $-\pi$ and π).)

It is not difficult to see that the group $\widetilde{SL}(2, \mathbb{R})$ is a connected 3-dimensional Lie group (with the obvious co-ordinates a and the real and imaginary parts of z). The subgroup isomorphic to $\mathbb{Z}$ consisting of those transformations (16) with $a \in \mathbb{Z}$ and $z = 0$, is easily seen to be central in the whole group $\widetilde{SL}(2, \mathbb{R})$ (i.e. each of its elements commutes with all elements). (We shall see below that in fact it coincides with the centre of $\widetilde{SL}(2, \mathbb{R})$.) Note also for later use that the transformations (16) with $a \in \mathbb{R}$ and $z = 0$ form a one-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$.

Each transformation (16) has the property that if $x \mapsto y$ under the transformation, then $x + 2\pi k \mapsto y + 2\pi k$ for all $k \in \mathbb{Z}$. Hence each such transformation yields a transformation $w = e^{ix} \mapsto e^{iy}$ of the unit circle $|w| = 1$. It is easily verified that the latter transformation has the explicit form

$$w \mapsto \frac{w - z}{1 - \bar{z}w} e^{2\pi i a} \tag{17}$$

If one conjugates this by the linear fractional transformation $z = i[(1 - w)/(1 + w)]$, which maps the unit circle (with one point removed) to the real line (and the interior of the unit circle to the open upper half-plane) then one finds that the group of such transformations is isomorphic to $SL(2, \mathbb{R})/\{\pm 1\}$. (Alternatively one may use the results of §13.2 of Part I to get that the group of transformations (17) is isomorphic to $SU(1, 1)/\{\pm 1\} \simeq SL(2, \mathbb{R})/\{\pm 1\}$.) We thus have a homomorphism from our group $\widetilde{SL}(2, \mathbb{R})$ onto $SL(2, \mathbb{R})$ with kernel the above central subgroup isomorphic to $\mathbb{Z}$. Since $SL(2, \mathbb{R})/\{\pm 1\}$ has trivial centre, it follows that the centre of $\widetilde{SL}(2, \mathbb{R})$ is precisely that infinite cyclic subgroup.

3.2.3. Theorem. *The group $\widetilde{SL}(2, \mathbb{R})$ has no faithful Lie representation.*

PROOF. While the above-mentioned one-parameter subgroup (consisting of the transformations (16) with $a \in \mathbb{R}$ arbitrary and $z = 0$) clearly has infinite intersection with the centre of $\widetilde{SL}(2, \mathbb{R})$ (which consists, as just shown, of the transformations (16) with $a \in Z$ and $z = 0$), it is obviously not contained in the centre. We shall show that this is incompatible with the existence of a faithful Lie representation of $\widetilde{SL}(2, \mathbb{R})$.

Thus suppose that there is a subgroup G of $GL(n, \mathbb{C})$ which is identical with $\widetilde{SL}(2, \mathbb{R})$ as far as its Lie group structure is concerned. Denote by H the one-parameter subgroup of G corresponding to the one-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ just mentioned. It follows from §14.2 of Part I and §3.1 above that H has the form $\{\exp tA \mid t \in \mathbb{R}\}$, where A is some fixed $n \times n$ matrix. By conjugating, if need be, by a suitable matrix from $GL(n, \mathbb{C})$ we may bring A into its Jordan canonical form; hence we may assume that A is in block diagonal form, with blocks each of the type

$$\begin{pmatrix} \lambda & a_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & a_k \\ 0 & & & \lambda \end{pmatrix}, \tag{18}$$

where $a_i = 0$ or 1. (We are supposing here that different blocks correspond to different λ, i.e. that the degree of each block is equal to the multiplicity of the eigenvalue λ.) The matrix $\exp(tA)$ (see §14.2 of Part I) will then also be in block diagonal form with blocks of the same size as those of A, and with the block corresponding to (18) having the form $e^{\lambda t} B_\lambda(t)$ where

$$B_\lambda(t) = \begin{pmatrix} 1 & a_1 t & \frac{1}{2} a_1 a_2 t^2 & \frac{1}{6} a_1 a_2 a_3 t^3 & \cdots & \frac{1}{k!} a_1 \ldots a_k t^k \\ 0 & 1 & a_2 t & \frac{1}{2} a_2 a_3 t^2 & \cdots & \frac{1}{(k-1)!} a_2 \ldots a_k t^{k-1} \\ 0 & 0 & 1 & a_3 t & \cdots & \frac{1}{(k-2)!} a_3 \ldots a_k t^{k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Since for infinitely many t the matrix with blocks $e^{\lambda t} B_\lambda(t)$ lies in the centre of G, it follows that every element of G also has the same block diagonal form as A, i.e. has blocks of the same degree in the same order. The set P of all $n \times n$ matrices (including the singular ones) which commute with every element of G, clearly forms a linear subspace of the vector space $\mathbb{C}^{n^2}$ of all $n \times n$ matrices; the intersection $P \cap G$ is the centre of G. For reasons similar to before, every element of P again has the same block diagonal form as the matrices in G.

The condition that any given $n \times n$ matrix of that block diagonal form lie in P is equivalent to the condition that its image in the quotient space $\mathbb{C}^{n^2}/P$ be zero, and is therefore expressible as a homogeneous system of linear equations in the entries in the diagonal blocks of the given matrix, which can be so arranged that any single equation involves only those entries in a single block. Applying this to the matrix $\exp tA$ with blocks $e^{\lambda t}B_\lambda(t)$, we obtain, on multiplying each equation by $e^{-\lambda t}$ for the appropriate λ, a system of polynomial equations in t. Since such a system is satisfied by either all or else only finitely many values of t, this contradicts the fact that the group $H = \{\exp tA\}$ is not contained in the centre of G but has infinite intersection with it.

□

EXERCISES

1. Calculate the Lie algebra of the Lie group $\widetilde{SL}(2, \mathbb{R})$.

2. Verify that the above-described group homomorphism $\widetilde{SL}(2, \mathbb{R}) \to SL(2, \mathbb{R})/\{\pm 1\}$ is a local isomorphism in some neighbourhood of the identity.

§4. Complex Manifolds

4.1. Definitions and Examples

We now introduce the concept of a complex manifold.

4.1.1. Definition. A *complex analytic manifold* of *complex dimension* n is a manifold M of dimension $2n$, for which the charts $U_q (M = \bigcup_q U_q)$ with their local co-ordinate systems $z_q^\alpha = x_q^\alpha + iy_q^\alpha$, $\alpha = 1, \ldots, n$, are identifiable with regions of n-dimensional complex space $\mathbb{C}^n$. It is further required that on each region of intersection $U_q \cap U_p$, the transition functions from the co-ordinates z_q^α to the co-ordinates z_p^α and in the reverse direction, be complex analytic (see §12.1 of Part I):

$$\frac{\partial z_q^\alpha}{\partial \bar{z}_p^\beta} \equiv 0; \qquad \frac{\partial z_p^\beta}{\partial \bar{z}_q^\alpha} \equiv 0. \tag{1}$$

We define a *holomorphic map* between complex manifolds to be one which is complex analytic (in terms of the given complex local co-ordinates on the manifolds). Holomorphic maps from a complex manifold to the complex line $\mathbb{C}$ will be called *analytic* or *holomorphic functions* on the manifold. A bijection between complex manifolds will be said to be *biholomorphic* if both it and its inverse are holomorphic. If two complex manifolds are such that there exists a biholomorphic map between them, we shall say that they are *biholomorphically equivalent* or *complex diffeomorphic*.

One important property of a complex manifold is that it always comes with an orientation.

4.1.2. Theorem. *A complex analytic manifold is oriented.*

PROOF. Let M be a complex manifold and let $z_q^\alpha = x_q^\alpha + iy_q^\alpha$, $z_p^\alpha = x_p^\alpha + iy_p^\alpha$ be local co-ordinates on charts U_q, U_p, respectively. By Lemma 12.2.2 of Part I, on the region of overlap of each such pair of neighbourhoods, the (real) Jacobian of the transition function from the co-ordinates x_q^α, y_q^α to the co-ordinates x_p^α, y_p^α, satisfies

$$J^{\mathbb{R}} = |J^{\mathbb{C}}|^2 = \left|\det\left(\frac{\partial z_q^\alpha}{\partial z_p^\beta}\right)\right|^2.$$

Since such Jacobians are therefore all positive, the theorem follows. □

The complex projective spaces $\mathbb{C}P^n$, introduced in §2.2 above, provide examples of complex analytic manifolds. Complex local co-ordinates on $\mathbb{C}P^n$ are defined as in the real case; the transition functions, exemplified by formula (7) of §2.2 (with the x_i^j replaced by the complex co-ordinates z_i^j) are clearly complex analytic. The manifolds $\mathbb{C}P^n$ are compact (see Exercise 5 of §2.3). It follows from the discussion in §2.2 that the complex manifold $\mathbb{C}P^1$ is biholomorphic to the extended complex plane with complex local co-ordinates $w = 1/z$ in the neighbourhood of the point at infinity (and with $w = 0$ at ∞).

The simplest examples of complex manifolds are furnished by regions of $\mathbb{C}^n$. Further important examples are provided by the non-singular complex surfaces in $\mathbb{C}^n$. Such a manifold is defined by a system of equations

$$\left.\begin{array}{c} f_1(z^1, \ldots, z^n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-k}(z^1, \ldots, z^n) = 0 \end{array}\right\}, \tag{2}$$

where the functions $f_1, \ldots, f_{n-k}$ are all complex analytic, and at every point the rank of the matrix $(\partial f_i/\partial z^j)$ is largest possible (namely $n-k$). The verification that a non-singular complex surface in $\mathbb{C}^n$ is indeed a complex analytic manifold is carried out in a manner analogous to that of the real case (see the proof of Theorem 2.1.1), with the aid of results from §12 of Part I.

In contrast with the real case (see §9), compact complex analytic manifolds are not realizable as non-singular complex surfaces in some $\mathbb{C}^n$. This is a consequence of the following theorem.

4.1.3. Theorem. *A holomorphic function on a compact, connected complex manifold is necessarily constant.*

PROOF. If $f: M \to \mathbb{C}$ is a holomorphic function on a compact, connected complex manifold M, then it follows by means of a well-known argument (using the continuity of $|f|$ and the compactness of M) that $|f|$ attains a largest value; i.e. there is a point P_0 of M such that $|f(P)| \leq |f(P_0)|$ for all points P of M. The constancy of f on M follows from the connectedness assumption, and the following basic result of complex function theory (to state and prove which, we now interrupt the present proof).

4.1.4. Lemma ("Maximum-Modulus Principle"). *Let f be a function holomorphic on some region U of n-dimensional complex space $\mathbb{C}^n$. If the function $|f|$ has a local maximum at some point P_0 of U, i.e. if $|f(P)| \leq |f(P_0)|$ for all points P of U sufficiently close to P_0, then f is constant in some neighbourhood of P_0.*

PROOF. Since the function $|f|$ will clearly have a local maximum at P_0 on any complex line through P_0, it suffices to prove the lemma for the case $n = 1$. We may also assume without loss of generality that $P_0 = 0$, and that $f(0) \neq 0$ (since if $f(0) = 0$, the assertion of the lemma is trivial). By multiplying the function f, if necessary, by an appropriate complex number we may further suppose that $f(0)$ is a positive real number.

In §26.3 of Part I we gave a proof of the "Residue Theorem" of complex function theory; the well-known "Cauchy integral formula" for holomorphic functions of a complex variable is an almost immediate corollary of that result:

$$f(0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{z},$$

where γ is any circle enclosing the origin. Putting $z = re^{i\varphi}$ where r is any constant small enough to ensure that both γ and its interior are contained in U, this becomes

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\varphi})\,d\varphi, \tag{3}$$

which formula obviously must also hold if in it f is replaced by $\operatorname{Re} f$ (or $\operatorname{Im} f$).

The function $g(z) = \operatorname{Re}(f(0) - f(z))$ is non-negative on some neighbourhood of the origin, since, by hypothesis, for all z sufficiently close to 0 we have $f(0) - |f(z)| \geq 0$, and since also $|\operatorname{Re} f(z)| \leq |f(z)|$. On the other hand since formula (3) continues to hold with f replaced by g, it follows that

$$\int_0^{2\pi} g(re^{i\varphi})\,d\varphi = 0$$

for all sufficiently small r. Hence throughout some neighbourhood of the origin $g(z)$ must be identically zero, i.e. $\operatorname{Re} f(z) \equiv f(0)$. Since $|f(z)| \leq f(0)$ on some neighbourhood of the origin, we deduce that $f(z) = f(0)$ on some such neighbourhood, as required. □

The proof of Theorem 4.1.3 is now completed as follows. Recall that $|f(P_0)|$ is the maximum modulus of $f\colon M \to \mathbb{C}$ on the compact, connected complex manifold M. If we denote by M' the set of all points P of M such that $f(P) = P_0$, then by the maximum-modulus principle, M' is open. The set M' has no boundary, since by continuity any hypothetical boundary point would have to lie in M', contradicting the fact that M' is open. Hence the complement of M' is also open, whence M is the union of two disjoint open sets. Since M is connected this is impossible unless M' is empty (which it certainly is not) or the whole space M. This completes the proof of the theorem. □

4.1.5. Corollary. *Any complex analytic submanifold of* $\mathbb{C}^n$, *of dimension greater than zero, is non-compact.*

Proof. Suppose that M is a compact complex analytic manifold which can be embedded in $\mathbb{C}^n$ for some n and let $f\colon M \to \mathbb{C}^n$ be an holomorphic embedding. Then in view of the above theorem, on each connected component of M, each co-ordinate function f^i of f, being an analytic function on that connected component, is constant. Hence f maps each connected component of M to a single point, which proves the corollary. □

The classical complex transformation groups constitute important examples of non-singular complex surfaces:

(1) $GL(n, \mathbb{C})$, the set of all non-singular, complex, $n \times n$ matrices, is an open region of the space $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ of all complex matrices;
(2) $SL(n, \mathbb{C})$, the surface in $\mathbb{C}^{n^2}$ of all unimodular complex $n \times n$ matrices (i.e. of determinant 1).
(3) $O(n, \mathbb{C})$, the surface in $\mathbb{C}^{n^2}$ whose points comprise all complex orthogonal matrices, i.e. complex matrices A satisfying $AA^{\mathrm{T}} = 1$.

The non-singularity of these surfaces is verified much as it was for their real analogues (see §14.1 of Part I).

Each of these manifolds is a Lie group (see Definition 2.1.6). In fact the maps ψ and φ defining the group structure:

$$\psi\colon G \times G \to G, \qquad \psi(g, h) = gh;$$

$$\varphi\colon G \to G, \qquad \varphi(g) = g^{-1},$$

are everywhere complex analytic (i.e. holomorphic). Thus the above groups are examples of matrix "complex Lie groups".

4.1.6. Definition. A Lie group G which is a complex analytic manifold, is called a *complex Lie group* if the above maps ψ and φ are complex analytic.

4.1.7. Theorem. *Every compact, connected, complex Lie group G is commutative.*

Proof. As usual we
difficult to see that th
analytic map (betwee
ted, Theorem 4.1.3
homomorphism, we
If $g(t)$ is any (smo
$\dot{g}(0) = X$ say, and if
part (i) of Theorem

Since $\mathrm{Ad}(g(t))(Y) =$
by Corollary 3.1.2,

It can be show
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verification th
structure, T^{2n}

Proof. As usual we denote by $\mathfrak{g}$ the Lie algebra of the group G. It is not difficult to see that the adjoint representation $\mathrm{Ad}: G \to GL(n, \mathbf{C})$ is a complex analytic map (between complex manifolds). Since G is compact and connected, Theorem 4.1.3 implies that Ad is a constant map; since Ad is a homomorphism, we must in fact have that $\mathrm{Ad}(g) = 1$ for all $g \in G$.

If $g(t)$ is any (smooth) curve in G passing through the identity element, with $\dot{g}(0) = X$ say, and if Y is any element of $\mathfrak{g}$, then, as we showed in the proof of part (i) of Theorem 3.1.4,

$$\mathrm{Ad}(g(t))(Y) = Y + t[X, Y] + O(t^2).$$

Since $\mathrm{Ad}(g(t))(Y) = Y$, we conclude that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$, whence by Corollary 3.1.2, G is commutative, as required. □

It can be shown that in fact the only compact, connected, complex Lie groups are the "complex tori", which we shall now consider. As usual let $e_1, \ldots, e_{2n}$ denote the standard basis vectors in $\mathbf{R}^{2n} = \mathbf{C}^n$. (In fact for our purposes any basis for $\mathbf{R}^{2n}$ will serve.) The *complex torus* T^{2n} is defined to have as its points the equivalence classes of vectors, where two vectors are equivalent if they differ by an integral linear combination of the given basis vectors:

$$z \sim z + \sum_{\alpha=1}^{2n} n_\alpha e_\alpha, \qquad n_\alpha \in \mathbf{Z}.$$

(Such integral linear combinations form a subgroup Γ of $\mathbf{C}^n$ called the *integral lattice* determined by the given basis $e_1, \ldots, e_{2n}$.) Thus T^{2n} is the quotient group of $\mathbf{C}^n$ by Γ:

$$T^{2n} = \mathbf{C}^n/\Gamma.$$

Obviously two integral lattices Γ and Γ', determined by bases $e_1, \ldots, e_{2n}$ and $f_1, \ldots, f_{2n}$, respectively, coincide precisely if the vectors f_i lie in Γ and the e_j in Γ':

$$f_i = n_i^j e_j, \qquad e_j = m_j^i f_i.$$

Since the matrices (n_i^j) and (m_j^i) have integer entries and are mutual inverses, it follows that their determinants are both ± 1.

Thus far we have defined the group structure of the torus T^{2n}. We now endow it with its manifold structure by taking as local (complex) co-ordinate neighbourhoods the images of appropriately chosen open subsets of $\mathbf{C}^n$ under the natural map

$$\mathbf{C}^n \to T^{2n} = \mathbf{C}^n/\Gamma,$$

where these open subsets are chosen on the one hand sufficiently small for the restriction to each of the natural map to be one-to-one, and on the other hand so that their images cover T^{2n}. We leave to the reader the details of the verification that with this manifold structure and the above abelian group structure, T^{2n} is indeed a complex Lie group.

Functions on T^{2n} may obviously be regarded as $2n$-fold periodic functions on $\mathbb{C}^n$:

$$f\left(z+\sum_{\alpha=1}^{2n} n_\alpha e_\alpha\right)=f(z).$$

It follows from Theorem 4.1.3 that: *A holomorphic $2n$-fold periodic function on $\mathbb{C}^n$ is constant.*

By way of an interesting example we consider the special case $n=1$. A complex torus T^2 is determined by a basis for $\mathbb{R}^2=\mathbb{C}$, i.e. by a pair of non-zero complex numbers z_1, z_2 such that $z_1 \notin \mathbb{R}z_2$. Multiplying by z_1^{-1} we obtain a pair of the form $(1, \tau)$ where $\tau(=z_2/z_1 \in \mathbb{C})$ is non-real (since 1 and τ are linearly independent over $\mathbb{R}$). Since, as it is easy to see, the multiplications of $\mathbb{C}$ by z_1 and z_1^{-1} induce holomorphic maps between the tori determined by the pairs (z_1, z_2) and $(1, \tau)$, it follows that those tori are biholomorphically equivalent. Hence each one-dimensional torus is determined, at least up to biholomorphic equivalence, by a complex number τ with non-zero imaginary part.

4.1.8. Lemma. *If τ and τ' are two non-real complex numbers related by a linear-fractional transformation of the form*

$$\tau'=\frac{m\tau+n}{p\tau+q},$$

where the matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ *is integral and has determinant* ± 1, *then the tori determined by τ and τ' are biholomorphically equivalent.*

PROOF. In view of the conditions on the coefficients m, n, p, q, the integral lattices determined by the pairs of vectors $(1, \tau)$ and $(p\tau+q\cdot 1,\ m\tau+n\cdot 1)$ coincide. The lemma now follows from the fact that the second pair defines a torus which, by the remark preceding the lemma, is biholomorphically equivalent to that determined by $(1, \tau')$. ☐

Remark. It can be shown (using the theory of elliptic functions) that tori determined by complex numbers τ, τ' sufficiently close to one another, are not biholomorphically equivalent.

Regarded as merely a (2-dimensional) real manifold, the torus T^2 is diffeomorphic to the familiar 2-dimensional real torus $S^1 \times S^1$, where one of the two circles is obtained by identifying points on the straight line determined by 0 and z_1 which differ by an integral multiple of z_1, and the other circle is obtained by carrying out a similar identification of points on the line through 0 and z_2. Similarly, the torus T^{2n} is diffeomorphic to the $2n$-dimensional real torus $S^1 \times \cdots \times S^1$ ($2n$ factors).

Returning to the complex case, suppose we have a torus T^{2n} determined by a basis $e_1, \ldots, e_{2n}$ (not necessarily standard) for $\mathbb{R}^{2n}=\mathbb{C}^n$. Among these $2n$

vectors there will be n which are linearly independent over $\mathbb{C}$; by re-indexing if necessary, we may suppose that $e_1, \ldots, e_n$ are linearly independent over $\mathbb{C}$. If we express the remaining vectors $e_{n+1}, \ldots, e_{2n}$ in terms of the first n, say

$$e_{n+k} = \sum_{j=1}^{n} b_{kj} e_j, \qquad k = 1, \ldots, n,$$

we obtain a complex matrix $B = (b_{kj})$, which (as in the particular case $n = 1$ examined above) determines the torus up to a biholomorphic equivalence. It is easy to see that the imaginary part of the matrix B must be non-singular, since otherwise the vectors $e_1, \ldots, e_{2n}$ would be linearly dependent over $\mathbb{R}$.

4.1.9. Definition. A complex torus T^{2n} is said to be *abelian* if for some basis $e_1, \ldots, e_{2n}$ of its integral lattice, the above-defined matrix $B = (b_{kj})$ is symmetric and its imaginary part $H = (h_{kj}) = (\operatorname{Im} b_{kj})$ is positive definite; i.e. if

$$b_{jk} = b_{kj} \quad \text{and} \quad h_{kj} \xi^k \xi^j > 0,$$

for all non-zero real vectors $(\xi^1, \ldots, \xi^n)$.

For example, the one-dimensional complex torus determined (up to a biholomorphic equivalence—see above) by a complex number τ with $\operatorname{Im} \tau > 0$, is abelian; since the tori determined by τ and $-\tau$ clearly coincide, it follows that in fact all one-dimensional complex tori are abelian. However even for $n = 2$ non-abelian tori exist.

EXERCISE

Show that almost all 2-dimensional complex tori T^4 are non-abelian.

On an abelian torus the *Jacobi-Riemann θ-function* $\theta(z_1, \ldots, z_n)$ of n complex variables, is defined by

$$\theta(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n} \exp i \left\{ \frac{1}{2} \sum_{j,k} b_{kj} m_k m_j + \sum_k m_k z_k \right\}, \tag{4}$$

where the summation is over all n-tuples $(m_1, \ldots, m_n)$ of integers. The condition that the imaginary part of the matrix $B = (b_{kj})$ be positive definite, guarantees convergence of the series.

4.2. Riemann Surfaces as Manifolds

A *Riemann surface* is defined (cf. §12.3 of Part I) as a non-singular surface in $\mathbb{C}^2$ given by an equation of the form

$$f(z, w) = 0, \tag{5}$$

where $f(z, w)$ is an analytic function of z and w (for instance a polynomial in z and w). The condition for non-singularity, which makes the surface a one-dimensional complex manifold (i.e. complex curve), is as follows (see §12.3 of Part I, especially Theorem 12.3.1):

$$\operatorname{grad}_{\mathbb{C}} f = \left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right) \neq 0.$$

If we solve equation (5) for w it may happen that we obtain a multi-valued function; for instance:

(i) if $f(z, w) = w^2 - P_n(z)$ where $P_n(z)$ is a polynomial without multiple roots (see 12.3.2 of Part I), we obtain the two-valued function $w = \sqrt{P_n(z)}$ (a "hyperelliptic" Riemann surface);

(ii) if $f(z, w) = e^w - z$, we obtain $w = \ln z = \ln|z| + i \arg z + 2\pi i n$.

(The geometric meaning of multi-valuedness of $w(z)$ is that (some of) the surfaces $z = \text{const.}$ meet the surface $f(z, w) = 0$ in more than one point.)

Consider the case where $f(z, w)$ is a polynomial of degree n in the variables z, w. On making the substitution $z = y^1/y^0$, $w = y^2/y^0$, we obtain

$$f(z, w) = \frac{1}{(y^0)^n} Q_n(y^0, y^1, y^2),$$

where Q_n is a homogeneous polynomial in three variables. This furnishes a device for re-realizing our surface $f(z, w) = 0$ in $\mathbb{C}^2$, as the surface in the projective space $\mathbb{C}P^2$ given by the equation

$$Q_n(y^0, y^1, y^2) = 0, \tag{6}$$

except that the points of the latter surface for which $y^0 = 0$ correspond to "points at infinity" on the original Riemann surface (5). The adjunction of these points at infinity has compactified our surface:

4.2.1. Lemma. *The Riemann surface in complex projective space $\mathbb{C}P^2$ defined by equation (6), is compact.*

Proof. The set of zeros of Q_n is clearly a closed set in $\mathbb{C}P^2$. Since $\mathbb{C}P^2$ is compact, and any closed subset of a compact space is compact, the lemma follows. □

Thus the original Riemann surface $f(z, w) = 0$ gives rise to a compact 2-dimensional real manifold. What do these manifolds actually look like in the case where $f(z, w) = w^2 - P_n(z)$, i.e. when $f(z, w)$ is as in (a) above? We first examine cases of low degree, and from these infer a general result. (It turns out that the points at infinity on such a surface are singular points, so that they may not all appear on the manifolds which we are about to construct (as realizations of such surfaces), while those which do appear should strictly speaking be removed from the manifolds.)

Examples. (a) Let $f(z, w) = w^2 - z$; then $Q_2(y^0, y^1, y^2) = (y^2)^2 - y^1 y^0$. We join the points $z = 0$, $z = \infty$ in the domain of the (extended) multiple-valued function $w = \sqrt{z}$ by a line segment α (or to put it more vividly, we "cut" the Riemann sphere S^2, diffeomorphic to $\mathbb{C}P^1$, to obtain the sphere with the segment α removed, depicted in Figure 4). It is not difficult to see intuitively that the restriction of the (extended) surface $f(z, w) = 0$ to those z of S^2 off the slit α, consists of two disjoint connected components, each of which can be seen, by projecting onto the extended z-plane, to be diffeomorphic to the sphere S^2 with the line segment α removed (see Figure 5). (These connected pieces are called the "branches" of the multi-valued function.) At the points 0 and ∞ the (extended) function $w = \sqrt{z}$ is single-valued. The desired surface is obtained by identifying the boundary segment α_1 of the component denoted I in Figure 5, with the boundary segment β_2 of the region II, and the boundary segment β_1 of region I with the boundary segment α_2 of region II of the surface (as indicated in Figure 5). It is intuitively plausible that as a result of this cutting and pasting, we obtain a manifold diffeomorphic to S^2.

(b) We next consider the case $f(z, w) = w^2 - P_2(z)$, where $P_2(z)$ is a polynomial of degree 2 with simple roots $z = z_1$, $z = z_2$, $z_1 \neq z_2$. Join the points z_1, z_2 by a straight line segment on the z-plane. For z outside that line segment the surface $f(z, w) = 0$ falls into two disjoint connected parts. If we adjoin a point at infinity to each of these connected parts, they will be as shown in Example (a), with the difference that here $z_1 \neq \infty$ (see Figure 6). As in that example, on identifying the appropriate boundary segments ($\alpha_1 \sim \beta_2$ and $\beta_1 \sim \alpha_2$), we see that the Riemannian manifold is in this case also diffeomorphic to S^2 (with two points removed).

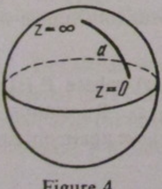
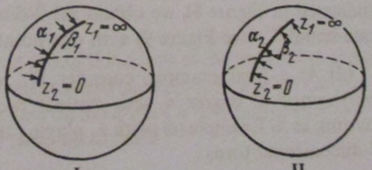


Figure 4



I

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Figure 5

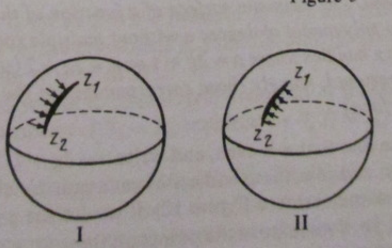

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Figure 6

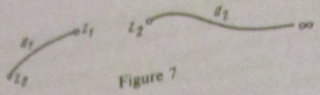

Figure 7

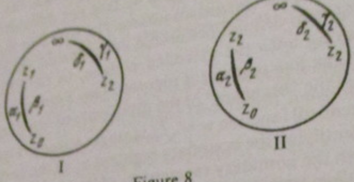

Figure 8

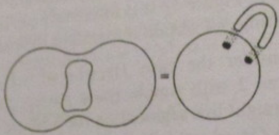

Figure 9

(c) Consider $f(z, w) = w^2 - P_3(z)$ where $P_3(z)$ is a polynomial of degree 3 with distinct roots z_0, z_1, z_2. Make cuts on S^2 as indicated in Figure 7; for z off these slits the extended surface $f(z, w) = 0$ again falls into two disjoint connected pieces, as shown in Figure 8. On identifying the appropriate edges of the slits on these two pieces (α_1 with β_2, α_2 with β_1, γ_1 with δ_2, γ_2 with δ_1, as indicated in Figure 8), we obtain the 2-dimensional torus (or "sphere-with-one-handle"—see Figure 9) with one point removed.

(d) As a final example, consider $f(z, w) = w^2 - P_4(z)$, where $P_4(z)$ is a polynomial of degree 4 with distinct roots z_0, z_1, z_2, z_3. By cutting and pasting as in Example (c) (with z_3 playing the role of ∞), we again obtain the 2-dimensional torus.

4.2.2. Proposition. *The Riemann surface of a function of the form* $w = \sqrt{P_n(z)}$, *where* $P_n(z)$ *is a polynomial of degree* n *without multiple roots, is diffeomorphic to a sphere with* g *handles where* $n = 2g + 1$ *or* $n = 2g + 2$ *(strictly speaking with certain points removed, namely those corresponding to the points at infinity of the original surface.)*

PROOF. Suppose first that n is even, and write $n = 2g + 2$. Pair off the roots of $P_n(z)$ arbitrarily, and join the members of each pair by an arc in such a way that no two arcs intersect (see Figure 10). If we cut the z-plane along each of these $g + 1$ arcs, i.e. if we remove the points on these arcs, then the surface falls into two disjoint connected parts U_1 and U_2. (If we move around any pair of

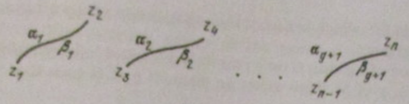

Figure 10

roots on the original surface, we stay on the same branch.) The edges α_i, β_i of the ith cut lie on (or rather are boundary segments of) different connected pieces U_1, U_2. We now glue these edges back together as follows:

$$(U_1, \alpha_i) \sim (U_2, \beta_i), \qquad (U_1, \beta_i) \sim (U_2, \alpha_i).$$

(This is justified by the fact that if on the original surface we move along the piece U_1 approaching the edge α_i, then on crossing it we pass smoothly over onto the branch U_2 (with corresponding edge β_i), and similarly if we approach on U_1 the edge β_i, we cross over onto U_2 (with corresponding edge α_i).) □

For odd n the construction is similar, with $z_{n+1} = \infty$ taken as the $(n+1)$st branch point.

§5. The Simplest Homogeneous Spaces

5.1. Action of a Group on a Manifold

We begin with the definition of such an action.

5.1.1. Definition. We say that a Lie group G (e.g. one of the matrix Lie groups considered in §14 of Part I) is *represented as a (Lie) group of transformations of a manifold M* (or *has a left (Lie)-action* on M) if there is associated with each of its elements g a diffeomorphism from M to itself

$$x \mapsto T_g(x), \qquad x \in M,$$

such that $T_{gh} = T_g T_h$ for all $g, h \in G$ (whence $T_1 = 1$), and if furthermore $T_g(x)$ depends smoothly on the arguments g, x (i.e. the map $(g, x) \mapsto T_g(x)$ should be a smooth map from $G \times M$ to M).

The Lie group G is said to have a *right action* on M if the above definition is valid with the property $T_g T_h = T_{gh}$ replaced by $T_g T_h = T_{hg}$.

If G is any of the Lie groups $GL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $O(p, q)$, or $GL(n, \mathbb{C})$, $U(n)$, $U(p, q)$ (where $p + q = n$), then G acts in the obvious way on the manifold $\mathbb{R}^n$ or $\mathbb{R}^{2n} = \mathbb{C}^n$; moreover, in these cases the elements of G act as linear transformations. (Note that if, more generally, a Lie group has a Lie action on the manifold $\mathbb{R}^n$, which is linear, then that action yields a Lie representation of the Lie group.)

The action of a group G on a manifold M is said to be *transitive* if for every two points x, y of M there exists an element g of G such that $T_g(x) = y$.

5.1.2. Definition. A manifold on which a Lie group acts transitively is called a *homogeneous space* of the Lie group.

In particular, any Lie group G is a homogeneous space for itself under the action of left multiplication: $T_g(h) = gh$; in this context G is called the *principal (left) homogeneous space* (of itself). (The action $T_g(h) = hg^{-1}$ makes G into its own *principal right homogeneous space*.)

Let x be any point of a homogeneous space of a Lie group G. The *isotropy group* (or *stationary group*) H_x of the point x is the stabilizer of x under the action of G:

$$H_x = \{g \mid T_g(x) = x\}.$$

5.1.3. Lemma. *All isotropy groups H_x of points x of a homogeneous space, are isomorphic.*

Proof. Let x, y be any two points of the homogeneous space and g be an element of the Lie group such that $T_g(x) = y$. It is then easy to check that the map $H_x \to H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism (assuming a left action). ☐

5.1.4. Theorem. *There is a one-to-one correspondence between the points of a homogeneous space M of a group G, and the left cosets gH of H in G, where H is the isotropy group (and G is assumed to act on the left).*

Proof. Let x_0 be any point of the manifold M. Then with each left coset gH_{x_0} we let correspond the point $T_g(x_0)$ of M. It is straightforward to verify that this correspondence is well defined (i.e. independent of the choice of representative of the coset), one-to-one, and onto. ☐

For right actions the analogous result holds with right cosets instead of left cosets.

Remark. It can be shown under certain general conditions that the isotropy group H is a closed subgroup of G, and that the set G/H of left cosets of H with the natural quotient topology can be given a unique (real) analytic manifold structure such that G is a Lie transformation group of G/H.

5.2. Examples of Homogeneous Spaces

(a) The group $O(n+1)$ clearly acts (in the natural way) of the sphere S^n (defined as the surface in Euclidean space $\mathbb{R}^{n+1}$ given by the equation $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$). It is easy to see that this action is transitive, so that S^n is a homogeneous space for the Lie group $O(n+1)$ of orthogonal transformations of $\mathbb{R}^{n+1}$. The isotropy group of the point $x=(1,0,\ldots,0)\in S^n$ is comprised of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \qquad A \in O(n).$$

Hence by Theorem 5.1.4 above S^n can be identified with $O(n+1)/O(n)$ (where the quotient denotes merely the set of left cosets of the isotropy group, which is not normal in $O(n+1)$). In fact $S^n \cong O(n+1)/O(n)$, where $\cong$ denotes diffeomorphism (cf. the above remark).

The group $SO(n+1)$ is also transitive on S^n, and, analogously to the above, the isotropy group is isomorphic to $SO(n)$, so that we may identify S^n with $SO(n+1)/SO(n)$, and again $S^n \cong SO(n+1)/SO(n)$, as quotient space of $SO(n+1)$.

(b) From the definition of real projective space $\mathbb{R}P^n$ as consisting of the straight lines through the origin in $\mathbb{R}^{n+1}$, we obtain a transitive action of $O(n+1)$ on the manifold $\mathbb{R}P^n$. The subgroup of orthogonal transformations fixing the straight line through O with direction vector $(1,0,\ldots,0)$ is comprised of all matrices of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix}, \qquad A \in O(n).$$

Hence the isotropy group is isomorphic to the direct product $O(1)\times O(n)$, and, again essentially by Theorem 5.1.4, we have

$$\mathbb{R}P^n \cong O(n+1)/O(1)\times O(n).$$

(c) The additive group of reals $\mathbb{R}$ acts (transitively) on the circle $S^1=\{e^{2\pi i\varphi}\}$ in the following way:

$$T_t(e^{2\pi i\varphi}) = e^{2\pi i(\varphi+t)}, \qquad t\in\mathbb{R}.$$

From the equality $e^{2\pi i}=1$ it follows that the isotropy group is exactly the group of integers.

More generally the group of all translations of $\mathbb{R}^n$ (which group it is natural to denote also by $\mathbb{R}^n$) acts transitively on the n-dimensional torus $T^n = (S^1)^n$, in the following way: if $y=(t_1,\ldots,t_n)\in\mathbb{R}^n$, and $z=(e^{2\pi i\varphi_1},\ldots,e^{2\pi i\varphi_n})$ is a point of the n-dimensional torus, define

$$T_y(z) = (e^{2\pi i(\varphi_1+t_1)},\ldots,e^{2\pi i(\varphi_n+t_n)}).$$

Clearly the isotropy group consists of all vectors y with integer components, i.e. the isotropy group of this homogeneous space is the integral lattice Γ of $\mathbb{R}^n$. Hence (cf. §4.1)

$$T^n \cong \mathbb{R}^n/\Gamma.$$

(d) *Stiefel manifolds.* For each n, k $(k \leq n)$ the Stiefel manifold $V_{n,k}$ has as its points all orthonormal frames $x = (e_1, \ldots, e_k)$ of k vectors in Euclidean n-space (i.e. ordered sequences of k orthonormal vectors in Euclidean $\mathbb{R}^n$). Any orthogonal matrix A of degree n sends any such orthonormal frame x to another, namely $Ax = (Ae_1, \ldots, Ae_k)$; this defines an action of $O(n)$ on $V_{n,k}$ which is transitive. (Verify this!)

Each Stiefel manifold $V_{n,k}$ can be realized as a non-singular surface in the Euclidean space $\mathbb{R}^{nk}$ in the following way. Fix on an orthonormal basis for $\mathbb{R}^n$ (e.g. the standard basis), and introduce the following notation for the components with respect to this basis of any orthonormal k-frame $(e_1, \ldots, e_k)$ (i.e. point of $V_{n,k}$):

$$e_i = (x_{i1}, \ldots, x_{in}), \qquad i = 1, \ldots, k.$$

The nk quantities x_{ij}, $i = 1, \ldots, k$; $j = 1, \ldots, n$, (in lexicographic order, say) are now to be regarded as the co-ordinates of a point in nk-dimensional Euclidean space $\mathbb{R}^{nk}$, related by the following $k(k+1)/2$ equations:

$$\langle e_i, e_j \rangle = \delta_{ij} \Leftrightarrow \sum_{s=1}^{n} x_{is} x_{js} = \delta_{ij}, \qquad i, j = 1, \ldots, k, \quad i \leq j. \tag{1}$$

5.2.1. Lemma. *The Stiefel manifold $V_{n,k}$ is (embeddable as) a non-singular surface of dimension $nk - k(k+1)/2$ in $\mathbb{R}^{nk}$.*

Proof. In view of the transitive action of the group $O(n)$ on $V_{n,k}$, it suffices to establish the non-singularity at any particular point. For convenience we choose the point $x_0 = (x_{ij})$ where $x_{ij} = \delta_{ij}$, $i = 1, \ldots, k$; $j = 1, \ldots, n$. Thus we wish to show that at x_0 the rank of the Jacobian matrix of the system of equations (1) is largest possible, namely $k(k+1)/2$, or, equivalently, that the tangent space at the point x_0, to the surface defined by that system, has dimension $nk - k(k+1)/2$.

To this end let $x_{ij} = x_{ij}(t)$ be a curve on the Stiefel manifold (as defined by (1)), passing through x_0 when $t = 0$:

$$\sum_{s=1}^{n} x_{is}(t) x_{js}(t) = \delta_{ij}, \qquad i, j = 1, \ldots, k;$$

$$x_{ij}(0) = \delta_{ij}, \qquad i = 1, \ldots, k; \quad j = 1, \ldots, n.$$

It follows that the components

$$\xi_{ij} = \frac{d}{dt} x_{ij}(t) \bigg|_{t=0}$$

of the velocity vector at the point x_0, satisfy

$$0 = \frac{d}{dt} \left(\sum_{s=1}^{n} x_{is}(t) x_{js}(t) \right)_{t=0} = \xi_{ij} + \xi_{ji}, \qquad i, j = 1, \ldots, k.$$

Hence the tangent space at the point x_0 to the surface $V_{n,k}$ consists of all nk-component vectors (ξ_{ij}), $i = 1, \ldots, k$; $j = 1, \ldots, n$, satisfying $\xi_{ij} = -\xi_{ji}$,

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$i, j = 1, \ldots, k$. Since the dimension of this space is clearly $nk - k(k+1)/2$, the lemma is proved. □

Thus $V_{n,k}$ is indeed a smooth manifold. We now investigate the isotropy group of this homogeneous space. Take any orthonormal k-frame $e_1, \ldots, e_k$ and enlarge it to an orthonormal basis $e_1, \ldots, e_n$ for the whole of Euclidean n-space. Any orthogonal transformation fixing the vectors $e_1, \ldots, e_k$ must (relative to the above basis for $\mathbb{R}^n$) have the form

$$\begin{array}{r}k\left\{\vphantom{\begin{matrix}1\\ \ddots\\ 0\end{matrix}}\right.\\ n-k\{\end{array}\begin{pmatrix}1 & & 0 & \\ & \ddots & & 0\\ 0 & & 1 & \\ & 0 & & A\end{pmatrix}, \qquad A \in O(n-k),$$

whence the isotropy group is isomorphic to $O(n, k)$, and $V_{n,k}$ can be identified with $O(n)/O(n-k)$. (In fact $V_{n,k} \cong O(n)/O(n-k)$.)

The Stiefel manifolds $V_{n,k}$ for $k < n$ are also homogeneous spaces for the group $SO(n)$. From this point of view the isotropy group is clearly (isomorphic to) $SO(n-k)$, and therefore also

$$V_{n,k} \cong SO(n)/SO(n-k).$$

In particular, we have

$$V_{n,n} \cong O(n), \qquad V_{n,n-1} \cong SO(n), \qquad V_{n,1} \cong S^{n-1}.$$

(e) *Grassmannian manifolds*. The points of the Grassmannian manifold $G_{n,k}$ are by definition the k-dimensional planes passing through the origin of n-dimensional Euclidean space. The usual action of the group $O(n)$ on $\mathbb{R}^n$ yields a transitive action of that group on the set of all k-dimensional planes through 0, i.e. on $G_{n,k}$. To find the (isomomorphism class of the) isotropy group, choose any k-dimensional plane π through 0, and then choose an orthonormal frame for $\mathbb{R}^n$ with its first k vectors in the plane π (whence the remaining $n-k$, however chosen, will be perpendicular to it). In terms of such a basis an orthogonal matrix fixing π (as a whole) will necessarily have the form

$$\begin{array}{r}k\{\\ n-k\{\end{array}\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}, \qquad A \in O(k), \qquad B \in O(n-k).$$

It follows that

$$G_{n,k} \cong O(n)/(O(k) \times O(n-k)).$$

Note finally that there is an obvious identification of the manifolds $G_{n,k}$ and $G_{n,n-k}$, and that, by its very definition $G_{n,1}$ is the same manifold as $\mathbb{R}P^{n-1}$.

(f) The following manifolds are homogeneous spaces for the unitary group $U(n)$:

(i) The odd-dimensional sphere S^{2n-1}, defined in n-dimensional complex space $\mathbb{C}^n$ by the equation

$$|z^1|^2+\cdots+|z^n|^2=1.$$

It is not difficult to show that in this case

$$S^{2n-1}\cong U(n)/U(n-1)\cong SU(n)/SU(n-1).$$

(ii) The complex projective space $\mathbb{C}P^{n-1}$. In this case we have

$$\mathbb{C}P^{n-1}\cong U(n)/(U(1)\times U(n-1)).$$

(iii) The complex Grassmannian manifold $G^{\mathbb{C}}_{n,k}$ consisting of the k-dimensional complex planes in $\mathbb{C}^n$ passing through the origin. Here

$$G^{\mathbb{C}}_{n,k}\cong U(n)/(U(k)\times U(n-k)).$$

5.3. Exercises

1. Let M be a homogeneous space of a Lie group G, and let H be the isotropy group. Prove that the dimension of the manifold M is the difference in the dimensions of G and H:

$$\dim M=\dim G-\dim H.$$

Compute the dimension of the Grassmannian manifold $G_{n,k}$.

2. Prove the compactness of the manifolds $V_{n,k}$ and $G_{n,k}$.

3. Let $m=(m_1,\ldots,m_k)$ be a partition of the integer n, i.e.

$$n=m_1+\cdots+m_k,\qquad m_i\geq 0.$$

A collection of linear subspaces $\pi_0,\pi_1,\ldots,\pi_k$ of the space $\mathbb{R}^n$ is called an *m-flag* if:

(i) $\dim\pi_i-\dim\pi_{i-1}=m_i$;
(ii) $\pi_0=0,\ \pi_k=\mathbb{R}^n$;
(iii) $\pi_{i-1}\subset\pi_i$.

Show how the totality of all *m-flags* $F(n,m)$ can be made to serve as a homogeneous space for the group $O(n)$, and calculate its isotropy group.

§6. Spaces of Constant Curvature (Symmetric Spaces)

6.1. The Concept of a Symmetric Space

Of great interest are those manifolds endowed with a metric g_{ab} whose curvature tensor (defined in §30.1 of Part I in terms of the connexion compatible with the metric) has identically zero covariant derivative (see §§28, 29 of Part I):

$$\nabla_s(R_{abcd}) = 0. \tag{1}$$

In any metrized space the components of the covariant derivative of the curvature tensor satisfy certain relations (namely Bianchi's identities—see Exercise 7, §30.5, Part I). However condition (1) places further severe restrictions on those components; thus, in particular, it follows from (1) that the scalar characteristics of the curvature are constant:

$$R = R^a_a = \text{const.}; \qquad R_{abcd}R^{abcd} = \text{const.}$$

It turns out also that under certain global conditions on a manifold, condition (1) implies the homogeneity of the metric g_{ab}; this is the case if the manifold is "simply connected" (see §17 below). Any manifold satisfying (1) is obtainable from some simply connected such manifold M as the quotient space (i.e. by identification) under some discrete group of motions. In this construction it can happen that the discrete group Γ in question is not central in the full isometry group of M, in which case the space M/Γ will not be homogeneous; such spaces are sometimes called "locally homogeneous" or "locally symmetric".

However our approach to symmetric spaces will be via the following definition.

6.1.1. Definition. A simply connected manifold M with a metric g_{ab} defined on it, is called a *symmetric space* (or *symmetric manifold*) if for every point x of M there exists an isometry (motion) $s_x\colon M \to M$ with the properties that x is an isolated fixed point of it, and that the induced map on the tangent space at x reflects (i.e. reverses) every tangent vector at x, i.e. $\xi \mapsto -\xi$. Such an isometry is called a *symmetry of M at the point x*.

The significance of the requirement in this definition that the manifold be simply connected will appear below (in §§17, 18). In the present section we shall not make use of the properties of simply connected manifolds; the reader not familiar with these properties may wish to attempt the appropriate exercises in §6.6 below, after he has studied the relevant sections of Chapter 4.

6.1.2. Lemma. *Every symmetric space satisfies condition* (1).

Proof. Let x be any particular point of the symmetric space M and let s_x be a symmetry of M at the point x. We can choose co-ordinates in some neighbourhood of x such that at x itself we have (see §29.3 of Part I)

$$x^a = 0, \qquad g_{ab} = \delta_{ab}, \qquad \frac{\partial g_{ab}}{\partial x^a} = 0.$$

(Here we are making the (inessential) assumption that the metric is Riemannian.)

Under the symmetry s_x, at the point x the tensor $\nabla_s R_{abcd}$ goes into its negative (and the tensors g_{ab} and R_{abcd} go into themselves) simply by virtue of the symmetry property of s_x and the transformation rule for tensors. On the other hand, since s_x is actually an isometry it is clear that $\nabla_s R_{abcd}$ must go into itself at x. The only way out is for $\nabla_s R_{abcd}$ to vanish at x, as required. □

Remark. The converse of this statement (under the assumption of simple-connectedness) is also true; however, in view of the greater technical complexity of its proof, we shall not prove it here. The following discussion may, however, be illuminating. In some neighbourhood of each point x of a Riemannian manifold we can define a *local symmetry* s_x as follows: Consider the pencil of geodesics on M passing through the point x (and assumed parametrized so that for each geodesic γ we have $\gamma(0)=x$); then for sufficiently small τ set

$$s_x(\gamma(\tau)) = \gamma(-\tau).$$

(Recall (e.g. from §29.2 of Part I) that in a sufficiently small neighbourhood of x the geodesics through x will intersect nowhere else.)

EXERCISE

Prove that such local transformations s_x are for all x isometries of the manifold precisely if condition (1) holds on the manifold. (*Hint*. The simplest case is $n=2$, when the curvature tensor is determined by a single scalar R. For the general case it is easiest to deduce the preservation of the curvature tensor by the transformations s_x, from the preservation of Jacobi's equation along a geodesic.)

The existence of the "symmetries" s_x for all points x of a manifold M guarantees a sufficiently large supply of isometries for M to be (locally at least) homogeneous.

6.1.3. Lemma. *A symmetric space M is locally homogeneous; i.e. around each point $x \in M$, there is a neighbourhood such that for each point $\bar{x}$ in that neighbourhood (i.e. for each $\bar{x}$ sufficiently close to x) there exists an isometry of M sending x to $\bar{x}$. For each pair of points x, $y \in M$ which can be joined by a geodesic, there exists an isometry g of M such that $g(x)=y$.*

PROOF. The first statement of the lemma follows from the second since around every point x there is a neighbourhood with the property that every point $\bar{x}$ in that neighbourhood is joined to x by a geodesic, i.e. the geodesics through x sweep out the whole neighbourhood.

For the second statement, let γ be a geodesic arc joining the points x and y, and parametrized by the natural (length) parameter τ, with $0 \le \tau \le T$, $\gamma(0)=x$, $\gamma(T)=y$. Let s_z be a symmetry of the manifold M at the point $z=\gamma(T/2)$. It follows from its symmetry property, together with the fact that it is an isometry of M, that s_z must send γ to γ, and therefore interchange x and y. (If the metric is pseudo-Riemannian of type $(1, n-1)$, and γ is an isotropic geodesic, then we may take as τ the "affine" (called also "natural" in Chapter 5 of Part I) parameter yielded in solving the equation for the geodesics (see §29.2 of Part I). ☐

Remark. Since any two points of a (connected) Riemannian manifold can be joined by a broken geodesic, it follows almost immediately that a symmetric Riemannian manifold is always homogeneous.

6.2. The Isometry Group of a Manifold. Properties of Its Lie Algebra

Henceforth in this section we shall consider only homogeneous, symmetric manifolds M (hence satisfying (1)), with metric g_{ab}. The Lie group of all isometries of M will be denoted by G, and the isotropy group by H, so that we may identify M with the set of left cosets of H in G (i.e. $M=G/H$, in the notation of the preceding section).

Let $\gamma=\gamma(\tau)$ be a geodesic on M parametrized by a natural parameter τ, and write $x_0=\gamma(0)$. For each appropriate real T we define a map $f_{T,\gamma}\colon M\to M$ by setting

$$f_{T,\gamma}=s_{x_0}^{-1}s_x,$$

where $x=\gamma(-T/2)$ (see Figure 11). This map has the following three important properties:

(i) $f_{T,\gamma}$ moves each point of γ through a time-interval T along the geodesic (as indicated in Figure 11):

$$\gamma(T)\mapsto\gamma(\tau+T);$$

(ii) $f_{T,\gamma}$ parallel transports vectors along the geodesic;

(iii) for any fixed geodesic γ, the transformations $f_{T,\gamma}$ with T variable, form a one-parameter subgroup of the isometry group G:

$$f_{T_1+T_2,\gamma}=f_{T_1,\gamma}f_{T_2,\gamma},$$
$$f_{-T,\gamma}=(f_{T,\gamma})^{-1}.$$

From the last of these properties and §3.1 above, it follows that for each geodesic γ the one-parameter subgroup $f_{T,\gamma}$ of G has the form

$$f_{T,\gamma}=\exp(TB_\gamma),$$

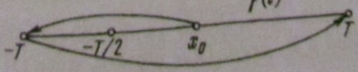

Figure 11

where B_γ is a certain vector in the Lie algebra $\mathfrak{g}$ of G (namely the tangent vector to the curve $f_{T,\gamma}$ at $T=0$). We denote by L^1 the linear subspace of the algebra $\mathfrak{g}$ spanned by the vectors $B_\gamma \in \mathfrak{g}$, where γ ranges over all geodesics through x_0, and by L^0 the Lie algebra of the isotropy group H_{x_0} of the point x_0. It follows (essentially from the fact that corresponding to each direction on the tangent plane to M at x_0 there is a geodesic through x_0 with that direction (see §29.2 of Part I)) that

$$\mathfrak{g} = L^0 + L^1 \quad \text{(direct sum of subspaces).} \tag{2}$$

If γ_1, γ_2 are two geodesics through the point x_0 then it can be shown that for small ε the product

$$f_{\varepsilon,\gamma_1} f_{\varepsilon,\gamma_2} f_{-\varepsilon,\gamma_1} f_{-\varepsilon,\gamma_2}$$

sends x_0 to a point whose distance from x_0 is of order ε^3. (This follows without difficulty from the properties of the Riemann curvature tensor. Verify!) We deduce from this (e.g. using formula (7) of §3.1) that we must have $[B_{\gamma_1}, B_{\gamma_2}] \in L_0$, whence $[L^1, L^1] \subset L^0$.

Suppose now that $g_T = \exp(TA)$ is a one-parameter subgroup of G leaving x_0 fixed (so that $A \in L_0$). Let γ be any geodesic through x_0, and denote by $\tilde{\gamma}$ the image of γ under the map g_ε. It then follows, again from the isometric property of the maps involved, that for small enough ε the map $g_\varepsilon f_{T,\gamma} g_{-\varepsilon}$ translates the points of the geodesic $\tilde{\gamma}$ along that geodesic (which of course also passes through x_0). Hence the tangent vector to the one-parameter subgroup $g_\varepsilon f_{T,\gamma} g_{-\varepsilon}$ (with parameter T), is in L_1. We now look for this tangent vector. From the basic facts about Lie algebras described in §3.1 it follows that for any two elements X, Y of the Lie algebra of a Lie group we have

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X, Y] + \text{higher-order terms}\right).$$

(This is a weak form of the "Campbell–Baker–Hausdorff formula".) It is an easy consequence of this that

$$\exp(tX)\exp(tY)\exp(-tX) = \exp(tY + t^2[X, Y] + \text{higher-order terms}).$$

Putting $t=1$, $X=\varepsilon A$, $Y=TB_\gamma$, we deduce that the desired tangent vector is $B_\gamma + \varepsilon[A, B_\gamma]$. Since $B_\gamma \in L^1$, it follows that $[A, B_\gamma] \in L^1$, whence $[L^0, L^1] \subset L^1$.

We include these facts in the following

6.2.1. Lemma. *With G and $\mathfrak{g} = L^0 + L^1$ as above, we have*

$$[L^0, L^0] \subset L^0, \qquad [L^1, L^0] \subset L^1, \qquad [L^1, L^1] \subset L^0. \tag{3}$$

A Lie algebra which decomposes as the direct sum of two subspaces satisfying (3) is called a *$\mathbb{Z}_2$-graded Lie algebra*, since (3) can be rewritten as

$$[L^i, L^j] \subset L^{(i+j) \bmod 2}. \tag{4}$$

6.2.2. Corollary. *With G as above, and $\mathfrak{g} = L^0 + L^1$ (direct) its Lie algebra, the linear operator*

$$\sigma: \mathfrak{g} \to \mathfrak{g}$$

whose restriction to L^0 is the identity map 1, and whose restriction to L^1 is the reflection -1, is a Lie algebra automorphism (i.e. also preserves commutators). (The map σ is an "involution", i.e. $\sigma^2 = 1$.)

(The converse is also true. To each involuntary automorphism σ of a Lie algebra $\mathfrak{g}$, there corresponds a $\mathbb{Z}_2$-grading $\mathfrak{g} = L^0 + L^1$ (direct sum of spaces) of the Lie algebra, where L^0 is the set of elements fixed by σ and L^1 the set of elements which σ negates.)

In view of the homogeneity of the manifold M, the local geometry around any point x_0 is determined by a scalar product on the tangent space $\mathbb{R}^n_{x_0}$ at the point. (Here $n = \dim M$.) We now elicit a certain property (familiar from Part I) which this scalar product must have.

Note first that the tangent space $\mathbb{R}^n_{x_0}$ can be identified naturally with the space $L^1 \subset \mathfrak{g}$. Let A be any element of L^0, and consider the one-parameter subgroup $g_T = \exp(TA)$. For each T we have the map

$$\mathrm{Ad}(g_T): \xi \mapsto \xi_T, \qquad \xi \in L^1.$$

As was shown in the course of proving part (i) of Theorem 3.1.4, we have

$$\xi_T = \mathrm{Ad}(g_T)(\xi) = \xi + T[A, \xi] + O(T^2),$$

whence

$$\left.\frac{d\xi_T}{dT}\right|_{T=0} = [A, \xi] = (\mathrm{ad}\, A)(\xi). \tag{5}$$

In view of the fact that g_T is an isometry of M the inner product on L^1 should be invariant under $\mathrm{Ad}(g_T)$, i.e.

$$\langle \xi_T, \eta_T \rangle = \langle \xi, \eta \rangle,$$

whence on differentiating with respect to T at $T = 0$, and using (5), we obtain

$$\langle [A, \xi], \eta \rangle + \langle \xi, [A, \eta] \rangle = 0. \tag{6}$$

This is the condition on the metric (i.e. scalar product) on $L^1 = \mathbb{R}^n_{x_0}$, that we were seeking. (Cf. the definition of a Killing metric in §24.4 of Part I.)

6.3. Symmetric Spaces of the First and Second Types

In the preceding subsection we obtained what might be called the algebraic model of a symmetric space. In principle all symmetric spaces can be classified (in the framework of the classification of compact Lie groups). In the present subsection we consider the most important examples of such spaces.

The simplest examples of (simply-connected) symmetric spaces are provided of course by the Euclidean spaces $\mathbb{R}^n$ and pseudo-Euclidean spaces $\mathbb{R}^n_{p,q}$ (where the curvature is zero). Here the group G is the full isometry group of $\mathbb{R}^n$ (or $\mathbb{R}^n_{p,q}$). (For the structure of some of these groups see §4 of Part I.) The isotropy group H is $O(n)$ (or $O(p,q)$), and the space $L^1=\mathbb{R}^n$ consists of the translations. We have, as in the general case, $\mathfrak{g}=L^0+L^1$, with the usual commutator relations between L^0 and L^1 (as in Lemma 6.2.1), and with the additional relation $[L^1, L^1]=0$. Non-simply-connected symmetric spaces (earlier called "locally symmetric") can be obtained from these simply-connected ones by identification under various discrete groups Γ consisting of translations (and possibly also involving certain reflections, as for instance in the case of the Klein bottle—see §18 below).

In the remainder of our examples the group G will be semisimple. Recall from §3.1 that semisimplicity is equivalent to non-degeneracy of the Killing form on the Lie algebra $\mathfrak{g}$. Recall also that the Killing form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is defined by:

$$\langle A, B\rangle = -\operatorname{tr}(\operatorname{ad} A \operatorname{ad} B),$$

where $(\operatorname{ad} X)(\xi)=[X,\xi]$. (We shall also restrict G to being the connected component of the identity of the full isometry group.)

There are two distinct important types of simply-connected symmetric spaces with such G (even assuming the metric positive definite, i.e. Riemannian), namely:

Type I: the group G is compact and the Killing form on $\mathfrak{g}$ is positive definite. (We note here the result that a Lie group is compact if and only if it is a closed subgroup of some $O(m)$.)

Type II: the group G is non-compact and the Killing form on the Lie algebra $\mathfrak{g}$ is indefinite.

We consider the simplest (non-trivial) example of each type.

(a) The sphere S^2 is of type I. Here $G=SO(3)$, which is compact, and $H=SO(2)$.

(b) The Lobachevskian plane L^2 is of type II. Here G is the connected component of the identity of $SO(1,2)$ (shown in §13.2 of Part I to be isomorphic to $SL(2,\mathbb{R})$), and $H=SO(2)$. The Lie algebra $\mathfrak{g}$ consists of all 2×2 matrices with zero trace, and the Killing form is given by

$$\langle A, B\rangle = -\operatorname{tr}(AB).$$

As a basis for the Lie algebra we may take

$$A_1=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad A_2=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad A_3=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

We then have

$$A_1^2=A_2^2=0,\qquad A_3^2=1,$$

and

$$\langle A_1, A_2\rangle = -1, \qquad \langle A_1, A_3\rangle = \langle A_2, A_3\rangle = 0,$$
$$\langle A_3, A_3\rangle = -2, \qquad \langle A_1, A_1\rangle = \langle A_2, A_2\rangle = 0.$$

It follows essentially from the fact that the matrices in $H = SO(2)$ have the form

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

that the subalgebra $L^0 \subset \mathfrak{g}$ is comprised of the matrices of the form $\lambda(A_1 - A_2)$. The subspace L^1 of $\mathfrak{g}$ is spanned by the vectors $A_1 + A_2, A_3$. It is easy to check that the inclusions (3) hold.

The restriction of the Killing form to the subspace L^1 is positive definite; this reflects the positive definiteness of the metric on the Lobachevskian plane.

EXERCISE

Investigate the general cases S^n and L^n.

6.4. Lie Groups as Symmetric Spaces

A Lie group Q endowed with a Riemannian metric invariant under left and right multiplications by group elements, can itself be regarded as a symmetric space. The isometry group of Q has a subgroup isomorphic to $Q \times Q$, whose action on Q is defined by

$$(g_1, g_2)\colon q \mapsto g_1 q g_2^{-1}.$$

The isotropy group of this action is clearly the diagonal subgroup $H = \{(q, q) | q \in Q\}$, which is isomorphic to Q: clearly $H(1) = 1$. For each q in Q the corresponding symmetry is defined by

$$s_q\colon x \mapsto qx^{-1}q.$$

(Verify that this does indeed define a symmetry.) In particular, $s_1(x) = x^{-1}$.

We shall examine in detail the case where Q is a compact connected subgroup of $SO(m)$, with the Euclidean metric

$$\langle A, B\rangle = \operatorname{tr}(AB^{\mathrm{T}}), \tag{7}$$

where B^{T} denotes the transpose of the matrix B. (Recall that it can be shown that a Lie group is compact if and only if it is a closed subgroup of some $O(m)$.)

EXERCISE

Show that the scalar product (7) is the Killing metric on $SO(m)$ determined by the Killing form on its Lie algebra (cf. §24.4 of Part I).

(The formula for the curvature of the Killing metric was derived in §30.3 of Part I. It follows from that formula that the Ricci tensor R_{ab} is positive definite. In the same subsection it was shown that the geodesics through the identity are precisely the one-parameter subgroups of Q.)

As noted in §24.4 of Part I, the group $SO(m)$ lies on the sphere of radius $\sqrt{m}$ (in the Euclidean space $\mathbb{R}^{m^2}$ of all $m \times n$ matrices with the metric (7)), since for $A \in SO(m)$, we have $AA^{\mathrm{T}} = 1$, whence $\langle A, A \rangle = m$; thus

$$SO(m) \subset S^{m^2-1}$$

6.4.1. Lemma. *The (Euclidean) scalar product (7) is invariant under right and left translations (i.e. multiplications) by elements of $SO(m)$.*

Proof. Let $g \in SO(m)$, and let A, B be any $m \times m$ matrices. Then

$$\langle gA, gB \rangle = \operatorname{tr}(gAB^{\mathrm{T}}g^{\mathrm{T}}) = \operatorname{tr}(gAB^{\mathrm{T}}g^{-1}) = \operatorname{tr}(AB^{\mathrm{T}}) = \langle A, B \rangle,$$

and

$$\langle Ag, Bg \rangle = \operatorname{tr}(Agg^{\mathrm{T}}B^{\mathrm{T}}) = \operatorname{tr}(AB^{\mathrm{T}}) = \langle A, B \rangle,$$

whence the desired conclusion. □

6.4.2. Corollary. *The metric (7) restricted to any subgroup Q of $SO(m)$ is invariant under right and left multiplications $q \mapsto q_1 q q_2$.*

We call such a metric *bi-invariant* or *two-sided invariant*.

6.4.3. Lemma. *Every bi-invariant metric on a simple Lie group is proportional (with constant proportionality factor) to the Killing metric.*

Proof. Let Q be a simple Lie group with bi-invariant metric $\langle \, , \rangle$. The bi-invariance implies that for all elements A, B, C of the Lie algebra L of Q, and all $g_T = \exp(AT)$, we have

$$\langle \operatorname{Ad}(g_T)(B), \operatorname{Ad}(g_T)(C) \rangle = \langle B, C \rangle, \tag{8}$$

whence it follows, just as in the derivation of (6) above, that

$$\langle [A, B], C \rangle + \langle B, [A, C] \rangle = 0. \tag{9}$$

Now let $g_{ab}, \bar{g}_{ab}$ be two metrics on Q satisfying (8), (9). Then any linear combination $g_{ab} - \lambda \bar{g}_{ab}$ will also be Ad-invariant (or equivalently skew ad-invariant). Let λ_1 be any eigenvalue of the pair of quadratic forms $g_{ab}, \bar{g}_{ab}$, i.e. $\det(g_{ab} - \lambda_1 \bar{g}_{ab}) = 0$. (The symmetry of $g_{ab}, \bar{g}_{ab}$ implies that λ_1 is real.) The subspace R_{λ_1} of all eigenvectors corresponding to the eigenvalue λ_1 is easily seen (from (9)) to be a (non-zero) ideal of L, whence by the assumed simplicity of L, we must have $R_{\lambda_1} = L$. Hence $g_{ab} = \lambda_1 \bar{g}_{ab}$. Since the Killing metric satisfies (9), the desired result follows. □

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6.5. Constructing

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6.4.4. Corollary. *Every simple subgroup Q of the group $SO(m)$ endowed with the Killing metric, can be isometrically embedded in the sphere S^{m^2-1} endowed with a metric proportional to the usual metric on the sphere.*

6.4.5. Corollary. *Since the Ricci tensor R_{ab} (determined by the Killing metric g_{ab} on a group) also satisfies* (8), (9), *it follows that for simple groups, $R_{ab}=\lambda g_{ab}$ where $\lambda=$* const.

Note finally that, as remarked above, it follows essentially from §30.3 of Part I that R_{ab} is positive definite for compact connected Lie groups. This is true also for semisimple groups, since each such group is (locally) a direct product $G_1\times\cdots\times G_k$ of simples, and the sign of λ_i is easily determined for each of the simple factors G_i.

6.5. Constructing Symmetric Spaces. Examples

We now return to general symmetric spaces. In the notation of §6.2 above write

$$M=G/H,\qquad \mathfrak{g}=L^0+L^1\quad\text{(direct sum of subspaces)},$$

where M is a given symmetric space, G is its isometry group, L^0 is the Lie algebra of the isotropy group H, and L^1 is identifiable with the tangent space $\mathbb{R}^n_{x_0}$ to M at the point x_0 (fixed by H, i.e. $H=Hx_0$). Recall also that by virtue of the homogeneity of M, its metric is determined locally by a metric on L^1 satisfying (6); in what follows we shall assume the metric on M to be obtained from the Killing form on $\mathfrak{g}$ (see below).

6.5.1. Lemma. *The subspaces L^0 and L^1 of the Lie algebra $\mathfrak{g}$ are orthogonal with respect to the Killing form.*

PROOF. From Lemma 6.2.1, it is immediate that for all $A\in L^0$, $B\in L^1$ we have

$$\operatorname{ad}A(L^0)\subset L^0,\qquad \operatorname{ad}A(L^1)\subset L^1,$$
$$\operatorname{ad}B(L^1)\subset L^0,\qquad \operatorname{ad}B(L^0)\subset L^1.$$

It follows readily (using a basis of $\mathfrak{g}$ which is the union of bases for L^0 and L^1) that $\operatorname{tr}(\operatorname{ad}A\operatorname{ad}B)=0$, as required. ☐

We deduce at once from this that in terms of a basis for $\mathfrak{g}$ of the kind just mentioned, the Killing form on $\mathfrak{g}$ has the form

$$(g_{ab})=\begin{pmatrix} g^{(0)}_{\alpha\beta} & 0\\ 0 & g^{(1)}_{\gamma\delta}\end{pmatrix},\tag{10}$$

where α,β range over the indices of the basis for L^0, and γ,δ over the indices of the basis for L^1. (The form $g^{(1)}_{\gamma\delta}$ is often called the *Killing form of the symmetric space M*.)

Since the Killing form (10) on $\mathfrak{g}$ satisfies (9), so also does the form $g^{(0)}_{\alpha\beta}$ on L^0. Hence by the proof of Lemma 6.4.3, if L^0 is simple then the form $g^{(0)}_{\alpha\beta}$ will be a constant multiple of the Killing form on the algebra L^0. However, in the important examples the algebra L^0 is not simple, but rather semisimple of the form $L^0 = L^0_1 \oplus L^0_2$ where L^0_1 and L^0_2 are simple. From Lemma 6.5.1 (with now L^0 in the role of $\mathfrak{g}$), we see that in this situation the restrictions of the form $g^{(0)}_{\alpha\beta}$ to the factors L^0_1, L^0_2 will be constant multiples (by λ_1, λ_2 say) of the Killing forms on those factors.

It can be seen that if H is compact and the metric $g^{(1)}_{\gamma\delta}$ is positive definite, then with respect to suitable bases for L^0 and L^1 the matrices ad A, $A \in L^0$, are skew-symmetric. Hence $\langle A, A\rangle = -\mathrm{tr}(\mathrm{ad}\, A)^2$ is positive, and therefore in view of

$$-\mathrm{tr}(\mathrm{ad}\, A)^2 = -[\mathrm{tr}(\mathrm{ad}\, A)^2_{L^0} + \mathrm{tr}(\mathrm{ad}\, A)^2_{L^1}],$$

it follows that

$$\langle A, A\rangle_{\mathfrak{g}} > \langle A, A\rangle_{L^0}. \tag{11}$$

Hence for compact H (and positive-definite metric on the symmetric space M) the restriction to L^0 (namely $g^0_{\alpha\beta}$) of the Killing form on the Lie algebra $\mathfrak{g}$, is positive definite. (Cf. the fact that the Killing form on the Lie algebra of a compact Lie group (e.g. on the Lie algebra L^0 of H) is non-negative.)

We see that in order to construct a symmetric space it essentially suffices to choose a suitable subalgebra L^0 of $\mathfrak{g}$ on which the restriction of the Killing form of the enveloping algebra $\mathfrak{g}$ is non-degenerate; then L^1 is defined as the orthogonal complement of L^0 in $\mathfrak{g}$. However the inequality (11) greatly restricts the choice of L^0: If the Killing form on $\mathfrak{g}$ is indefinite (type II) then for symmetric spaces with Riemannian metric the subalgebra $L^0 \subset \mathfrak{g}$ must be such that the restriction of the Killing form to its orthogonal complement is either positive or negative definite, and at the same time L^0 must be the Lie algebra of a compact group, and therefore of a subgroup of $SO(m)$.

Remark. A given symmetric space can be realized as a submanifold of the group G in such a way that the geodesics of M are geodesics also in the manifold G. This embedding can be obtained in any one of the following three (equivalent) ways:

(i) by considering all one-parameter subgroups of G emanating from the identity in the direction of vectors $B \in L^1$ (show that these geodesics sweep out a submanifold of G diffeomorphic to M);

(ii) via the map $\varphi: M \to G$, defined by $\varphi(x) = s_{x_0}^{-1} s_x$ (where s_{x_0}, s_x are the appropriate symmetries);

(iii) by means of an "involution" $\bar{\sigma}: G \to G$ (by which we mean an anti-automorphism of the group ($\bar{\sigma}(g_1 g_2) = \bar{\sigma}(g_2)\bar{\sigma}(g_1)$)) such that the map

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induced on $\mathfrak{g}$ is
map on L^1); M

EXERCISE
Show the equivalence

What follows is a
of type I. (As an
decomposition $\mathfrak{g} =$

(1) $SO(2n)/U(n)$.
(2) $SU(n)/SO(n)$.
(3) $SU(2n)/Sp(n)$.
(4) $Sp(n)/U(n)$.
(5) $SO(p+q)/(SO$
(6) $SU(p+q)/(SU$
(7) $Sp(p+q)/(Sp$

The following
definite metric). (
have the topolog

(1) $SO(p, q)/(SO$

(2) $SU(p, q)/(U$

(3) $Sp(p, q)/(Sp$
(4) $SL(n, \mathbb{R})/SO$
(5) $SL(n, \mathbb{C})/SU$
(6) $SO(n, \mathbb{C})/SO$

We conclude
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induced on $\mathfrak{g}$ is the identity map on L^0 and the negative of the identity map on L^1); $M \subset G$ is then the image under the map $g \mapsto g\tilde{\sigma}(g^{-1})$.

EXERCISE

Show the equivalence of these embeddings.

What follows is a list of the basic examples of connected symmetric spaces of type I. (As an exercise, find in each case the corresponding direct decomposition $\mathfrak{g} = L^0 + L^1$.)

(1) $SO(2n)/U(n)$.
(2) $SU(n)/SO(n)$.
(3) $SU(2n)/Sp(n)$.
(4) $Sp(n)/U(n)$.

$$\left.\begin{array}{l}(5)\ \ SO(p+q)/(SO(p)\times SO(q)).\\ (6)\ \ SU(p+q)/(SU(p)\times U(q)).\\ (7)\ \ Sp(p+q)/(Sp(p)\times Sp(q)).\end{array}\right\}\ \text{Grassmannian manifolds (including the projective spaces and spheres).}$$

The following are examples of symmetric spaces of type II (with positive-definite metric). (The simply-connected ones among such spaces turn out to have the topology of Euclidean space $\mathbb{R}^n$.)

(1) $SO(p, q)/(SO(p) \times SO(q))$. (For $q = 1$ this is the Lobachevsky space L^p.)
(2) $SU(p, q)/(U(p) \times SU(q))$. (For $q = 1$ this is the unit ball in $\mathbb{C}^p$, as a complex manifold; if also $p = 1$, this manifold is identifiable with $L^2 \cong SU(1, 1)/U(1)$.)
(3) $Sp(p, q)/(Sp(p) \times Sp(q))$.
(4) $SL(n, \mathbb{R})/SO(n)$.
(5) $SL(n, \mathbb{C})/SU(n)$.
(6) $SO(n, \mathbb{C})/SO(n, \mathbb{R})$.

We conclude with a list of symmetric spaces of dimension 4 with metric of signature $(+ - - -)$. (These spaces are of potential importance for the general theory of relativity since (by Corollary 6.4.5) the metric g_{ab} satisfies the equation $R_{ab} - \lambda g_{ab} = 0$ (see §37.4 of Part I).

I. Spaces of constant curvature with isotropy group $H = SO(1, 3)$:

(1) Minkowski space $\mathbb{R}^4_{1,3}$.
(2) The de Sitter space $S_+ = SO(1, 4)/SO(1, 3)$; note that S_+ is homeomorphic to $\mathbb{R} \times S^3$. Here the curvature tensor R is the identity operator on the space of bivectors $\Lambda^2(\mathbb{R}^4)$: $R = 1$.
(3) The de Sitter space $S_- = SO(2, 3)/SO(1, 3)$; this space is homeomorphic to $S^1 \times \mathbb{R}^3$, and its "universal covering space" $\tilde{S}_- = \widetilde{SO}(2, 3)/\widetilde{SO}(1, 3)$ (see §18) is homeomorphic to $\mathbb{R}^4$. Here the curvature tensor $R = -1$.

II. Reducible spaces (products of spaces of constant curvature):

(1) $H=SO(3)$; $M=\mathbb{R}_+\times M^3_{---}$, where M^3_{---} is a space of constant curvature, with signature $(---)$.

(2) $H=SO(1,2)$; $M=\mathbb{R}_-\times M^3_{+--}$, where M^3_{+--} is a space of constant curvature with signature $(+--)$.

(3) $H=SO(2)\times SO(1,1)$; $M=M^2_{--}\times M^2_{+-}$, the product of two 2-dimensional spaces of constant curvatures.

III. The symmetric spaces M_t of plane waves. (For these the isotropy group is abelian, and the isometry group is soluble.) In terms of a certain system of global co-ordinates the metric has the form

$$dl^2=2\,dx_1\,dx_4+\underbrace{[(\cos t)x_2^2+(\sin t)x_3^2]}_{K}\,dx_4^2+dx_2^2+dx_3^2,$$

$$\cos t\geq \sin t.$$

In terms of the tetrad (see §30.1 of Part I) given by the 1-forms

$$p=dx_1,\qquad q=dx_1+K\,dx_4,\qquad x=dx_3,\qquad y=dx_4,$$

the curvature tensor is constant, of the form

$$R=-4[\cos t(p\wedge x)\otimes(p\wedge x)+\sin t(p\wedge y)\otimes(p\wedge y)].$$

Remarks. 1. A simply connected symmetric space is uniquely determined by its curvature tensor at a point. To see this let $R\colon \Lambda^2(V)\to\Lambda^2(V)$ be the curvature tensor, and denote by $\mathfrak{h}$ the Lie algebra of skew-symmetric linear operators on the space V generated by those operators of the form $R(x,y)$, $x,y\in V$. (Then $\mathfrak{h}$ is the Lie algebra of the isotropy group (previously denoted by L^0).) Let $\mathfrak{g}$ denote the Lie algebra $V+\mathfrak{h}$, where the commutator operation on this direct sum of spaces is defined by

$$[(u,a),(v,b)]=(av-bu,[a,b]+R(u,v)).$$

Then in terms of the pair $\mathfrak{g}$, $\mathfrak{h}$ the structure of the original symmetric space is naturally reproduced on the symmetric space $M=G/H$.

2. The problem of classifying all curvature tensors of symmetric spaces with a given isotropy group H reduces to that of finding the H-invariant tensors R of the type of the curvature tensor, for which $R(x,y)$ belongs to the Lie algebra of H for all x,y in V.

6.6. Exercises

1. Show that for symmetric spaces of type II with positive-definite metric, the dimension of the subalgebra L^0 of $\mathfrak{g}$ is equal to the number of positive squares in the Killing form on $\mathfrak{g}$.

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2. Show that in the comp negative and positive dim $L^0=\frac{1}{2}\dim\mathfrak{g}$. Fi $G=SL(n,\mathbb{C})$.

3. Show that for symme has $M\cong G/H$, wher particular cases $SL($

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6. Show that for spa from this that a si same as $\mathbb{R}^n$ (assun

7. Decide which of above have non-v of type I, and the II.

8. Prove that in di with positive-def $H\subset G$ must be S sectional curvatu

9. Prove that a simp (where the G_i ar

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§7. Vector

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2. Show that in the complex case (e.g. where $G = SL(n, \mathbb{C})$ or $SO(n, \mathbb{C})$) the numbers of negative and positive square terms in the complex Lie algebra $\mathfrak{g}$ are equal, and $\dim L^0 = \frac{1}{2} \dim \mathfrak{g}$. Find the subalgebra L^0 of the Lie algebra $\mathfrak{g}$ of the group $G = SL(n, \mathbb{C})$.

3. Show that for symmetric spaces of type II with positive-definite metric one always has $M \cong G/H$, where H is a maximal compact subgroup of G. Investigate the particular cases $SL(n, \mathbb{R})/SO(n)$, $SL(n, \mathbb{C})/SU(n)$.

4. Show that a simply-connected, symmetric space of type II always has the topology of Euclidean $\mathbb{R}^n$.

For the next few exercises, note that, as for Lie groups (see §30.3 of Part I), so also for symmetric spaces do we have

$$\langle R(\xi, \eta)\zeta, \tau\rangle|_{x_0} = \tfrac{1}{4}\langle[\xi, \eta], [\zeta, \tau]\rangle_{L^0}, \qquad \xi, \eta, \zeta, \tau \in \mathbb{R}^n_{x_0} = L^1.$$

5. Show that for spaces of type I, the Ricci tensor R_{ab} is positive definite, and the "sectional curvature" $\langle R(\xi, \eta)\xi, \eta\rangle$ (where ξ, η span a parallelogram of unit area) is non-negative.

6. Show that for spaces of type II the sectional curvature is non-positive. Deduce from this that a simply-connected, symmetric space of type II is topologically the same as $\mathbb{R}^n$ (assuming the metric Riemannian).

7. Decide which of the 7 symmetric spaces of type I and 6 spaces of type II listed above have non-vanishing sectional curvature. Investigate the spaces S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ of type I, and the spaces L^n, $SU(n, 1)/U(n)$, $SL(n, \mathbb{R})/SO(n)$, $SL(n, \mathbb{C})/SU(n)$ of type II.

8. Prove that in dimensions $n = 2, 3$ the only simply-connected, symmetric spaces with positive-definite metric are L^n, S^n, $\mathbb{R}^n$. (*Hint*. Show that the isotropy group $H \subset G$ must be $SO(n)$ $(n = 2, 3)$, and thence deduce (for $n = 3$) the constancy of all sectional curvatures.)

9. Prove that a simply-connected symmetric space M with semisimple $G = G_1 \times \cdots \times G_k$ (where the G_i are simple) has the form

$$M = (G_1/H_1) \times \cdots \times (G_k/H_k),$$

with the metric decomposing as a direct product of metrics on the factors $M_i = G_i/H_i$, each of which is proportional to the Killing metric on the subspace L_i^1 of the corresponding Lie algebra $\mathfrak{g}_i = L_i^0 + L_i^1$.

§7. Vector Bundles on a Manifold

7.1. Constructions Involving Tangent Vectors

From any n-dimensional manifold M we can construct a $2n$-dimensional manifold, called the *tangent bundle* $L(M)$ of M as follows. The points of the manifold $L(M)$ are defined to be the pairs (x, ξ) where x ranges over the points of M and ξ ranges over the tangent space to M at x. Local co-ordinates are introduced on $L(M)$ in the following way. Let U_q be a chart of M with local co-ordinates x_q^α. Then in terms of the usual standard basis $e_\alpha = \partial/\partial x_q^\alpha$ (of operators on real-valued functions on M), any vector ξ in the tangent space to M at a point x of U_q can be written in terms of components as $\xi = \xi_q^\alpha e_\alpha$. As a typical chart U_q^L of $L(M)$ we take the set of all pairs (x, ξ) where x ranges over U_q, with local co-ordinates

$$(y_q^1, \ldots, y_q^{2n}) = (x_q^1, \ldots, x_q^n, \xi_q^1, \ldots, \xi_q^n).$$

The transition functions on the region of intersection of two charts U_q^L and U_p^L (with co-ordinates x_p^β) are then of the form

$$(y_p^1, \ldots, y_p^{2n}) = (x_p^\beta, \xi_p^\beta) = \left(x_p^\beta(x_q^1, \ldots, x_q^n), \frac{\partial x_p^\beta}{\partial x_q^\alpha} \xi_q^\alpha \right).$$

The Jacobian matrix of such a transition function is then clearly

$$\left(\frac{\partial y_p^j}{\partial y_q^i} \right) = \begin{pmatrix} A & 0 \\ H & A \end{pmatrix}, \qquad A = \left(\frac{\partial x_p^\beta}{\partial x_q^\alpha} \right), \qquad H = \left(\frac{\partial^2 x_p^\beta}{\partial x_q^\alpha \partial x_q^\gamma} \xi_q^\alpha \right),$$

whence the Jacobian is $(\det A)^2 > 0$. This gives immediately the

7.1.1. Proposition. *The tangent bundle $L(M)$ is a smooth, oriented $2n$-dimensional manifold.*

Note by way of an example that the tangent bundle on a region U of Euclidean space $\mathbb{R}^n$ is diffeomorphic to the direct product $U \times \mathbb{R}^n$.

If the manifold M comes with a Riemannian metric, then we can delineate in $L(M)$ a submanifold, the *unit tangent bundle* $L_1(M)$, consisting of those points (x, ξ) with $|\xi| = 1$. The dimension of L_1 is $2n - 1$. (It is defined in $L(M)$ by the single non-singular equation $f(x, \xi) = g_{\alpha\beta}\xi^\alpha\xi^\beta = 1$.)

Example. Each tangent vector ξ at a point of the n-sphere S^n (defined in Euclidean space $\mathbb{R}^{n+1}$ by the equation $\sum_{\alpha=0}^n (x^\alpha)^2 = 1$) is perpendicular to the radius vector x from the origin to the point x. Hence in the case $M = S^n$, the unit tangent bundle $L_1(M)$ of pairs (x, ξ) with $|\xi| = 1$, is (intuitively) identifiable with the Stiefel manifold $V_{n+1,2}$ (see §5.2). In particular for $n = 2$, the unit tangent bundle $L_1(S^2)$ is identifiable with $V_{3,2} \cong SO(3)$ (which is in turn diffeomorphic to $\mathbb{R}P^3$—see §2.2).

A smooth map $f: M \to N$ from a manifold M to a manifold N, determines a smooth map of the corresponding tangent bundles:

$$L(M) \to L(N), \qquad (x, \xi) \mapsto (f(x), f_*\xi),$$

where f_* is the induced map of the tangent spaces (see §1.2).

We note briefly a few other constructions similar to that of the tangent bundle.

(i) One often meets with the manifold $L_p(M)$ whose points are the pairs (x, τ) where τ ranges over the straight lines through the origin, in the tangent space $\mathbb{R}^n$ to M at the point $x \in M$.

(ii) Given any n-dimensional manifold M, we may construct from it the *tangent n-frame bundle* $E = E(M)$ having as points the pairs (x, τ) with $x \in M$ and $\tau = (\xi_1, \ldots, \xi_n)$ any ordered basis (i.e. frame) for the tangent space to M at x.

(iii) If M is oriented then $\tilde{E} = \tilde{E}(M)$ is defined as in (ii) except that the frames τ are required to be in the orientation class determining the orientation of M.

(iv) If M is a Riemannian manifold, then $E_0 = E_0(M)$ is defined as in (ii) with the frames τ restricted to being orthogonal.

Further examples of such constructions will be considered in Chapter 6.

We now define the *cotangent bundle* $L^*(M)$ on a manifold M. The points of $L^*(M)$ are taken to be the pairs (x, p) where p is a covector (i.e. 1-form on M) at the point x. Local co-ordinates x_p^α on a chart U_p of M determine the local co-ordinates $(x_p^\alpha, p_{p\alpha})$ on the corresponding chart of $L^*(M)$, where the $p_{p\alpha}$ are defined by

$$p = p_{p\alpha} dx_p^\alpha$$

(i.e. they are the components of p with respect to the standard dual basis of 1-forms on U_p).

The transition functions from co-ordinates $(x_p^\alpha, p_{p\alpha})$ to co-ordinates $(x_q^\beta, p_{q\beta})$ on $U_p \cap U_q$ are as follows:

$$(x_q^\beta, p_{q\beta}) = \left(x_q^\beta(x_p^1, \ldots, x_p^n), \frac{\partial x_p^\alpha}{\partial x_q^\beta} p_{p\alpha} \right). \tag{1}$$

The Jacobian matrix is then

$$\begin{pmatrix} A^{-1} & 0 \\ \tilde{H} & A \end{pmatrix}, \qquad A = \left(\frac{\partial x_p^\alpha}{\partial x_q^\beta} \right), \qquad \tilde{H} = \left(\frac{\partial^2 x_p^\alpha}{\partial x_q^\beta \partial x_q^\gamma} p_{p\alpha} \right),$$

whence the Jacobian is 1, and the manifold $L^*(M)$ is oriented.

The existence of a metric $g_{\alpha\beta}$ on the manifold M gives rise to a map $L(M) \to L^*(M)$, defined by

$$(x^\alpha, \xi^\alpha) \mapsto (x^\alpha, g_{\alpha\beta}(x)\xi^\beta),$$

i.e. by means of the tensor operation of lowering of indices (see §19.1 of Part I).

Since the expression $\omega = p_\alpha dx^\alpha$ (a differential form on M) is invariant under transformations of the form (1), it can be regarded as a differential form on $L^*(M)$. Its differential $\Omega = d\omega = \sum_{\alpha=1}^n dp_\alpha \wedge dx^\alpha$ (see §25.2 of Part I) is a non-degenerate (skew-symmetric) 2-form on $L^*(M)$, which is, obviously, closed, i.e. $d\Omega = 0$. We conclude that: *The manifold* $L^*(M)$ *is symplectic.* (Recall that in Part I we defined a symplectic space to be one equipped with a closed (skew-symmetric) 2-form.)

7.2. The Normal Vector Bundle on a Submanifold

Let M be an n-dimensional Riemannian manifold with metric $g_{\alpha\beta}$, and let N be a smooth k-dimensional submanifold of M. The *normal (vector) bundle* $\nu_M(N)$ on the submanifold N in M, is defined to have as its points the pairs (x, v) where x ranges over the points of N, and v is a vector tangent to M at the point x, and orthogonal to N at x (i.e. orthogonal to the tangent space to N at x, which is a subspace of the tangent space to M at x). Assuming (as always—see §1.3) that the submanifold N is defined by a non-singular system of $(n-k)$ equations, then (as noted in §1.3) in terms of suitable local co-ordinates $y^1, \dots, y^n$ on M these equations take the simple form $y^{k+1}=0, \dots, y^n=0$, and $y^1, \dots, y^k$ serve as local co-ordinates on N. In terms of such local co-ordinates $y^1, \dots, y^n$ on M, the normal bundle $\nu_M(N)$ is then determined as a submanifold of $L(M)$ by the system of equations

$$y^{k+1}=0, \dots, y^n=0, \qquad g_{\alpha\beta}(y)v^\beta=0, \qquad \alpha=1, \dots, k.$$

Since this system is non-singular (verify it!), it follows that $\nu_M(N)$ is an n-dimensional submanifold of $L(M)$.

Examples. (a) Let M be Euclidean n-space $\mathbb{R}^n$, and suppose N is defined by the non-singular system of $(n-k)$ equations

$$f_1(y)=0, \dots, f_{n-k}(y)=0, \qquad y=(y^1, \dots, y^n),$$

where $y^1, \dots, y^n$ are Euclidean co-ordinates on $\mathbb{R}^n$. Then the vectors $\operatorname{grad} f_1, \dots, \operatorname{grad} f_{n-k}$ are at each point of N perpendicular to the surface N and linearly independent (see §7.2 of Part I). Hence we see that $\nu_{\mathbb{R}^n}(N)$ has the structure of a direct product:

$$\nu_{\mathbb{R}^n}(N) \cong N \times \mathbb{R}^{n-k}.$$

More generally if N is defined as a submanifold of a manifold M by a non-singular system of equations

$$f_1(x)=0, \dots, f_{n-k}(x)=0,$$

then at each point x of N the vector fields

$$e_i(x) = \operatorname{grad} f_i(x) = g_{ij}\frac{\partial f_i}{\partial x^j}, \qquad i=1, \dots, n-k,$$

are orthogonal to N and linearly independent, whence any vector normal to N at $x \in N$ has the form $v = v^i e_i(x)$. The correspondence $(x, v) \leftrightarrow (x, v^1, \dots, v^k)$ is then a diffeomorphism:

$$\nu_M(N) \cong N \times \mathbb{R}^{n-k}.$$

An important special case arises from the consideration of a manifold A with boundary, defined by an inequality $f(x) \leq 0$ in M. Here N is the boundary ∂A of A defined by the single equation $f(x)=0$, and of dimension

$n-1$. The normal bundle to the boundary then decomposes as a direct product:

$$\nu_M(\partial A) = \partial A \times \mathbb{R}.$$

(b) Suppose $M = N \times N$ where N is a Riemannian manifold. A typical tangent vector to M at a point is then a pair (ξ, η) of tangent vectors to N. Define a Riemannian metric on M by setting

$$\langle(\xi_1, \eta_1), (\xi_2, \eta_2)\rangle = \langle\xi_1, \xi_2\rangle + \langle\eta_1, \eta_2\rangle.$$

Consider the diagonal $\Delta = \{(x, x) | x \in N\}$ of M; this is a submanifold of M manifestly identifiable with N. The tangent vectors to Δ at any point will have the form (ζ, ζ); hence a tangent vector $v = (\xi, \eta)$ will be perpendicular to Δ precisely if

$$0 = \langle(\zeta, \zeta), (\xi, \eta)\rangle = \langle\zeta, \xi + \eta\rangle$$

for all tangent vectors ζ to N. since this is possible if and only if $\xi = -\eta$, it follows that the vectors normal to the diagonal $\Delta \cong N$ have the form $v = (\xi, -\xi)$. Hence we conclude that:

$$\nu_{N \times N}(\Delta) \cong L(N).$$

(c) Let $\nu_M(N)$ be the normal bundle on the submanifold N of the Riemannian manifold M. We define a map h, the *geodesic map* from $\nu_M(N)$ to M as follows. Let (x, v) be any point of $\nu_M(N)$ and let $\gamma(t)$ be the geodesic of M emanating from x with initial velocity vector v; thus $\dot{\gamma}(0) = v$. Then define h by $h(x, v) = \gamma(1)$.

7.2.1. Lemma. *The Jacobian of the map h is non-zero at every point of $\nu_M(N)$ of the form $(x, 0)$.*

PROOF. We give the proof only for the case when M is the space $\mathbb{R}^n$ with the usual Euclidean metric, and N is a hypersurface in $\mathbb{R}^n$ given (locally) by parametric equations $x^i = x^i(u^1, \ldots, u^{n-1})$, $i = 1, \ldots, n$. Then as local co-ordinates for the points $(x, v) \in \nu_{\mathbb{R}^n}(N)$ we may take the n-tuples $(u^1, \ldots, u^{n-1}, t)$, where $x = x(u)$, $v = tn(u)$; here $n(u)$ is the unit normal to the surface N at the point $x(u)$. In terms of these co-ordinates the geodesic map h is clearly given by

$$h(u^1, \ldots, u^{n-1}, t) = x(u) + tn(u).$$

Hence its partial derivatives are as follows:

$$\frac{\partial h}{\partial u^i} = \frac{\partial x}{\partial u^i} + t\frac{\partial n}{\partial u^i}, \qquad \frac{\partial h}{\partial t} = n.$$

On putting $t = 0$ we obtain the non-singular Jacobian matrix $(\partial h/\partial u, \partial h/\partial t) = (\partial x/\partial u, n)$, whence the lemma. □

7.2.2. Corollary. *Suppose that the submanifold N is compact. Then the geodesic map h maps the region*

$$v^{\varepsilon}_M = \{(x, v) \mid |v| < \varepsilon\}$$

diffeomorphically onto some neighbourhood $U_\varepsilon(N)$ of N in M.

Proof. In view of the preceding lemma the map h is a diffeomorphism on some neighbourhood of any point of $v_M(N)$ of the form $(x, 0)$. Since N is compact, some finitely many of these neighbourhoods cover the subset $(N, 0)$ of $v_M(N)$. Then the union of these finitely many neighbourhoods contains some ε-neighbourhood $v^{\varepsilon}_M(N)$ of $(N, 0)$, and on this neighbourhood h is diffeomorphic. ☐

Remark. Let $U_\varepsilon(N)$ be, as in the corollary, the (diffeomorphic) image of $v^{\varepsilon}_M(N)$ under h. Then emanating from each point x in $U_\varepsilon(N)$ there is a (locally unique) "perpendicular geodesic" arc γ to N. We shall call the length of this "perpendicular" the *distance* from $x \in U_\varepsilon(N)$ to the submanifold N, and denote it by $\rho(x, N)$. Clearly the function $\rho(x, N)$ depends smoothly on the points x of the region $U_\varepsilon(N)$ of M.

7.2.3. Theorem. *If M is a compact, two-sided hypersurface in Euclidean $\mathbb{R}^n$ (see §2.1), then M is given by a single non-singular equation $f(x) = 0$.*

Proof. Let $\varphi(t)$ be a smooth function with graph something like that shown in Figure 12. Define a function $f: \mathbb{R}^n \to \mathbb{R}$ by:

$$f(x) = \begin{cases} \pm\varepsilon & \text{if } x \notin U_\varepsilon(M), \\ \varphi(\pm\rho(x, M)) & \text{if } x \in U_\varepsilon(M), \end{cases}$$

where $U_\varepsilon(M)$ is the region of M appearing in the corollary, and where the plus sign is taken if x lies in a particular one of the two disjoint connected regions comprising $\mathbb{R}^n - M$, and the minus sign if x is in the other. (It is here that we are using the two-sidedness of M in $\mathbb{R}^n$.) Then M is defined in $\mathbb{R}^n$ by the equation $f(x) = 0$. ☐

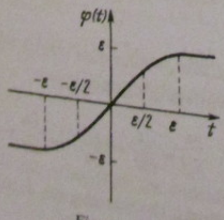

Figure 12

CHAPTER 2

Foundation
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