Graduate Texts in Mathematics

B.A. Dubrovin
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S.P. Novikov

Modern Geometry— Methods and Applications

Part III. Introduction to Homology Theory

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Translated by Robert G. Burns

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Preface

In expositions of the elements of topology it is customary for homology to be given a fundamental role. Since Poincaré, who laid the foundations of topology, homology theory has been regarded as the appropriate primary basis for an introduction to the methods of algebraic topology. From homotopy theory, on the other hand, only the fundamental group and covering-space theory have traditionally been included among the basic initial concepts. Essentially all elementary classical textbooks of topology (the best of which is, in the opinion of the present authors, Seifert and Threlfall's *A Textbook of Topology*) begin with the homology theory of one or another class of complexes. Only at a later stage (and then still from a homological point of view) do fibre-space theory and the general problem of classifying homotopy classes of maps (homotopy theory) come in for consideration. However, methods developed in investigating the topology of differentiable manifolds, and intensively elaborated from the 1930s onwards (by Whitney and others), now permit a wholesale reorganization of the standard exposition of the fundamentals of modern topology. In this new approach, which resembles more that of classical analysis, these fundamentals turn out to consist primarily of the elementary theory of smooth manifolds,† homotopy theory based on these, and smooth fibre spaces. Furthermore, over the decade of the 1970s it became clear that exactly this complex of topological ideas and methods were proving to be fundamentally applicable in various areas of modern physics. It was for these reasons that the present authors regarded as absolutely

† Evidently the beginning ideas of topology, which can be traced back to Gauss, Riemann and Poincaré, actually arose, historically speaking, in this order. However, at the time of Gauss and Riemann, a correspondingly organized conceptual basis for a theory of topology was unrealizable. It was Poincaré who, in creating the homology theory of simplicial complexes, was able to provide a quite different, precise foundation for algebraic topology.

essential material for a training in topology, in the first place precisely the theory of smooth manifolds, homotopy theory, and fibre spaces, and incorporated this subject matter in Part II of their textbook *Modern Geometry*. It is assumed in the present text that the reader is acquainted with that material.

On the other hand, the solution of the more complex problems arising both within topology itself (the computation of homotopy groups, the classification of smooth manifolds, etc.) and in the numerous applications of the algebro-topological machinery to algebraic geometry and complex analysis, requires a very extensive elaboration of the methods of homology theory. There is in the contemporary topological literature a complete lack of books from which one might assimilate the complex of methods of homology theory useful in applications within topology. It is part of the aim of the present book to remedy this deficiency.

In expounding homology theory we have, wherever possible, striven to avoid using the abstract terminology of homological algebra, in order that the reader continually remain cognizant of the fact that cycles and boundaries, and homologies between them, are after all concrete geometrical objects. In a few places, for instance in the section devoted to spectral sequences, this self-imposed restriction has inevitably led to certain defects of exposition. However, it is our experience that the usual expositions of the machinery of modern homological algebra lead to worse defects in the reader's understanding, essentially because the geometric significance of the material is lost from view. Certain fundamental methods of modern algebraic topology (notably those associated with spectral sequences and cohomology operations) are described without full justification, since this would have required a substantial increase in the volume of material. It must be remembered that those methods are based exclusively on the formal algebraic properties of the algebraic entities with which they are concerned, and in no way involve their explicit geometric prototypes whence they derive their *raison d'être*. In the final chapter of the book the methods of algebraic topology are applied to the investigation of deep properties of characteristic classes and smooth structures on manifilds. It is the intention of the authors that the present monograph provide a path for the reader giving access to the contemporary topological literature.

A large contribution to the final version of this book was made by the editor, Victor Matveevich Bukhshtaber. Under his guidance several sections were rewritten, and many of the proofs improved upon. We thank him for carrying out this very considerable task.

Translator's acknowledgements. Thanks are due to G. C. Burns and Abe Shenitzer for much encouragement, to several of my colleagues (especially Stan Kochman) for technical help, and to Eadie Henry for her advice, superb typing, and forbearance.

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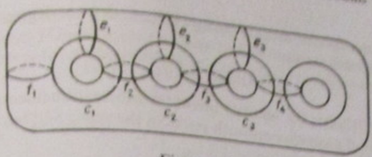

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connected 3-manifolds (with repetitions) can in effect be obtained by enumerating, for each $r \geq 0$, all isotopy classes of self-diffeomorphisms of the surface of genus r, and therefore by enumerating (in any order) all possible finite products of the form

$$\prod_j T_{s_j}^{\lambda_j}, \qquad s_j \in \{c_i, e_i, f_i\}.$$

§25. Unitary Bott Periodicity and Higher-Dimensional Variational Problems

In this section we shall prove the important topological result usually referred to as "Bott periodicity". For the sake of simplicity we shall, for the most part, concentrate our attention on the unitary groups: although "orthogonal Bott periodicity" is established along the same general lines as "unitary Bott periodicity", there are certain rather substantial technical difficulties to overcome in the former case. (We shall nonetheless outline a proof of "orthogonal periodicity" in the final subsection (§25.3).)

25.1. The Theorem on Unitary Periodicity

We shall prove this theorem in its "classical" form, namely as a result on the periodicity of the homotopy groups of the unitary groups, without considering its role as a "periodic" theorem for vector bundles.

25.1. Theorem (On Unitary Periodicity). *The following isomorphisms hold between the homotopy groups of the special unitary groups:*

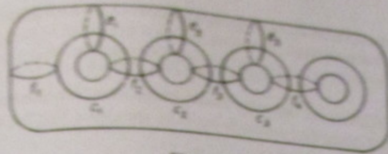
$$\pi_{i-1}(SU(2m)) \simeq \pi_{i+1}(SU(2m)) \qquad \text{for} \quad 1 \leq i \leq 2m.$$

It follows that for the "stable" unitary group $U = \lim_{m\to\infty} U(m)$, the direct limit of the unitary groups $U(m)$ with respect to the standard embeddings $U(m) \subset U(m+1)$, we have

$$\pi_{i-1}(U) \simeq \pi_{i-1}(U) \qquad \text{for} \quad i \geq 1,$$

whence

$$\pi_{2n}(U) = 0, \qquad \pi_{2n+1}(U) \simeq \mathbb{Z} \qquad \text{for} \quad n \geq 0.$$



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Consider the special unitary group $SU(2m)$ of even degree, regarded as a Lie group, and denote up

$$\Omega = \Omega(SU(2m); I_{2m}, -I_{2m}),$$

where $I_{2m} \in SU(2m)$ is the identity linear transformation (identity matrix), the function space of piecewise-smooth paths in the space $SU(2m)$ from the point I_{2m} to $-I_{2m}$, and by

$$\Omega^* = \Omega^*(SU(2m); I_{2m}, -I_{2m})$$

the full space of all continuous paths in $SU(2m)$ from I_{2m} to $-I_{2m}$. (Recall from Lemma 22.4 that the inclusion $\Omega \to \Omega^*$ is a homotopy equivalence.) We shall be particularly concerned with the subspace

$$\tilde{\Omega} = \tilde{\Omega}(SU(2m); I_{2m}, -I_{2m})$$

of Ω consisting of all minimal geodesics γ (i.e. geodesics of least length among all piecewise-smooth paths) joining the points I_{2m} and $-I_{2m}$, with respect to the invariant metric on $SU(2m)$ determined by the Killing form $\langle X, Y\rangle = \operatorname{Re} \operatorname{tr}(X\bar{Y}^T)$, $X, Y \in su(2m)$, on the Lie algebra $su(2m)$ (see Part I, §24.4). (See [44] for general conditions under which such minimal geodesics exist.)

25.2. Lemma. *The space $\tilde{\Omega}$ is homeomorphic to the complex Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$, i.e. the manifold whose points are the m-dimensional complex planes through the origin in complex 2m-dimensional space $\mathbb{C}^{2m}$ (see Part II, §5.2).*

Proof. By Theorem 30.3.7 of Part I, in the Lie group $SU(2m)$ the geodesics with respect to the above-mentioned Killing metric are precisely the one-parameter subgroups (and their translates (i.e. multiples) by elements of $SU(2m)$). Hence in order to characterize the geodesics in $SU(2m)$ joining the points I_{2m} and $-I_{2m}$, it suffices to describe all one-parameter subgroups emanating from the identity I_{2m} of $SU(2m)$ and reaching the point $-I_{2m}$. By Part I, Theorem 24.3.1 (and the definition of an invariant metric on a Lie group given in Part I, §24.4), the one-parameter subgroups $\gamma(t)$ passing through the identity I_{2m} have the form $\gamma(t) = \exp(tX)$ for some skew-Hermitian matrix X with zero trace (i.e. element of the Lie algebra $su(2m)$; see Part I, §14.1). Since the parameter can always be chosen to vary from 0 to 1 along the geodesic arc from I_{2m} to $-I_{2m}$, we infer the conditions $\gamma(0) = I_{2m}$ (as required), and $\gamma(1) = \exp X = -I_{2m}$, from which we can ascertain X. To this end, recall the well-known result (a consequence of the classical process for bringing a matrix into its Jordan canonical form, or, if you like, of an orthogonalization process exploiting the operator Ad applied to the unitary case) to the effect that X is conjugate, by means of a matrix in $SU(2m)$, to a diagonal matrix, i.e. there is an element $g_0 \in SU(2m)$ such that $g_0 X g_0^{-1} = X_0$ with X_0 of the form

$$X_0 = \begin{pmatrix} i\varphi_1 & & 0 \\ & \ddots & \\ 0 & & i\varphi_{2m} \end{pmatrix}, \quad \text{where} \quad \varphi_1 + \cdots + \varphi_{2m} = 0.$$

(Thus X_0 belongs to a "Cartan subalgebra", i.e. maximal commutative subalgebra, of $su(2m)$.) Under the transformation $\mathrm{Ad}(g_0)$ (see Part II, §3.1), applied to (the terminal point of) the geodesic $\gamma(t)$, the above condition $\exp(X) = -I_{2m}$ becomes

$$g_0(\exp X)g_0^{-1} = \exp(g_0 X g_0^{-1}) = \begin{pmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_{2m}} \end{pmatrix} = g_0(-I_{2m})g_0^{-1} = -I_{2m},$$

whence we have, for $i = 1, \ldots, 2m$, that $\varphi_i = \pi k_i$ where the k_i are odd integers satisfying $k_1 + \cdots + k_{2m} = 0$.

Having thus completely described the geodesics in $SU(2m)$ from I_{2m} to $-I_{2m}$, it remains to pick out those of least length. Since, in view of the definition of the metric on $SU(2m)$ in terms of the Killing form on $su(2m)$, the exponential map sends the line segment $\{tX \mid 0 \le t \le 1\}$ in the tangent space $su(2m)$ isometrically onto the geodesic arc $\gamma(t) = \exp(tX)$, the length of this latter arc is equal to that of the line segment in $su(2m)$. Now, as noted above, the Killing form on $su(2m)$ is given by

$$\langle A, B\rangle = \mathrm{Re}\ \mathrm{tr}(A\bar{B}^{\mathrm{T}}),$$

so that the length of the latter line segment is

$$\sqrt{\langle X, X\rangle} = \sqrt{\mathrm{tr}\, X\bar{X}^{\mathrm{T}}} = \pi \sqrt{\sum_{i=1}^{2m} (k_i)^2}.$$

Hence the least length of a geodesic from I_{2m} to $-I_{2m}$ is $\pi\sqrt{2m}$, attained when $k_i = \pm 1$ for all $i = 1, \ldots, 2m$. Since $k_1 + \cdots + k_{2m} = 0$, it follows that the corresponding matrices X_0 must have the same number of i's as $(-i)$'s on the diagonal. The upshot of our argument is, therefore, that the minimal geodesics is $SU(2m)$ joining the points I_{2m} and $-I_{2m}$ are precisely those of the form $\gamma(t) = \exp(tX)$ where X is conjugate in $SU(2m)$ to the matrix

$$X_0 = \pi \begin{pmatrix} i & & & & & 0 \\ & i & & & & \\ & & \ddots & & & \\ & & & i & & \\ & & & & -i & \\ & & & & & \ddots \\ 0 & & & & & & -i \end{pmatrix} = \pi \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}, \qquad (1)$$

i.e. $X = gX_0g^{-1}$ for some $g \in SU(2m)$.

We have thus established a one-to-one correspondence between the space $\tilde{\Omega}$ of all minimal geodesics from I_{2m} to $-I_{2m}$, and the conjugacy class of the matrix X_0 (see (1)) in $SU(2m)$, and it is intuitively clear that in fact this correspondence is a homeomorphism. Since the conjugacy class of X_0 in $SU(2m)$ is in turn homeomorphic to the coset space $SU(2m)/C(X_0)$, where $C(X_0)$ is the centralizer of X_0 in $SU(2m)$, and since clearly $C(X_0) = S(U(m) \times U(m))$ (with the obvious interpretation), we finally conclude that $\tilde{\Omega}$

is homeomorphic to

$$SU(2m)/S((U(m) \times U(m)),$$

which (as indicated in Part II, §5.2) is identifiable with the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$. This completes the proof of the lemma. □

25.3. Lemma. *Each minimal geodesic arc $\gamma(t)$, $0 \le t \le 1$, in $SU(2m)$, from the identity I_{2m} to $-I_{2m}$, is uniquely determined by its midpoint, i.e. by the point $\gamma(\frac{1}{2})$, so that the space $\tilde{\Omega}$ of such minimal geodesics (shown in the preceding lemma to be homeomorphic to the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$) may be identified with the subspace $\{\gamma(\frac{1}{2}) | \gamma \in \tilde{\Omega}\}$ of $SU(2m)$. The latter subspace coincides with the intersection of $SU(2m)$ with its Lie algebra $su(2m)$ (both considered as subspaces of the space ($\simeq \mathbb{R}^{8m^2}$) of all $2m \times 2m$ complex matrices).*

Proof. The first assertion follows easily from the form of the minimal geodesics $\gamma(t)$, established in the proof of Lemma 25.2 (see (1)):

$$\gamma(t) = \exp(tX) = (\cos \pi t) I_{2m} + (\sin \pi t) X;$$

putting $t = 0, 1$ we obtain $\gamma(0) = I_{2m}$, $\gamma(1) = -I_{2m}$, while at $t = \frac{1}{2}$, we have $\gamma(\frac{1}{2}) = X$, i.e. the midpoint of the geodesic arc is just X.

The remainder of the lemma then follows from the observation that those unitary $2m \times 2m$ complex matrices X which are at the same time skew-Hermitian, i.e. satisfy both $X\bar{X}^{\mathrm{T}} = 1$, and $X + \bar{X}^{\mathrm{T}} = 0$, are precisely the solutions of the matrix equation $X^2 = -I_{2m}$, and in $SU(2m)$ these are just X_0 (as in (1)) and its conjugates. (Note incidentally the consequence of this, that the elements of Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$ correspond one-to-one to the possible "complex structures" in $SU(2m)$.) □

25.4. Lemma. *Every non-minimal geodesic γ in $SU(2m)$ from I_{2m} to $-I_{2m}$ has index at least $2m + 2$.*

Proof. By the Index Theorem (21.7) the index of a geodesic arc γ in Ω is equal to the number of points in the interior of γ (counted according to their multiplicities) conjugate to the initial point I_{2m}. Since by definition two points on a geodesic arc are conjugate along that arc if there is a non-zero Jacobi field along the arc vanishing at those points, we are led to consider Jacobi's differential equation (see §21 (8)).

By the proof of Lemma 25.2, we may assume that $\gamma(t) = \exp(tX)$, $0 \le t \le 1$, where

$$X = \begin{pmatrix} i\pi k_1 & & 0 \\ & \ddots & \\ 0 & & i\pi k_{2m} \end{pmatrix}, k_1 + \cdots + k_{2m} = 0, \qquad (2)$$
$$k_1 \ge k_2 \ge \cdots \ge k_{2m}, \quad k_i \text{ odd}.$$

In particular $\dot{\gamma}(0) = X$. Consider the (real) linear transformation

$$K_X \colon su(2m) \to su(2m),$$

defined in terms of the curvature by

$$K_X(Y) = R(X, Y)X \quad (= \tfrac{1}{4}[[X, Y], X] \text{ by Part I, Corollary 30.3.6}).$$

It follows from the symmetry relation

$$\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$$

(see Part I, Corollary 30.3.6 or Theorem 30.2.1 (iv)) that K_X is a self-adjoint linear transformation:

$$\langle K_X(Y), W\rangle = \langle Y, K_X(W)\rangle.$$

We may therefore choose an orthonormal basis $e_1, \ldots, e_k$ for $su(2m)$ such that

$$K_X(e_i) = \mu_i e_i, \tag{3}$$

where the μ_i are the eigenvalues of K_X. If we extend the e_i to vector fields $e_i(t)$ along γ by parallel transport, then an arbitrary vector field $v(t)$ along γ can be expressed uniquely in the form

$$v(t) = \sum_{i=1}^{k} v_i(t)e_i(t),$$

and the Jacobi differential equation takes the form of the system (see §21 (9))

$$\frac{d^2 v_i}{dt^2} + \sum_{j=1}^{k} \langle R(\dot\gamma, e_j)\dot\gamma, e_i\rangle v_j = 0, \qquad i = 1, \ldots, k. \tag{4}$$

Since there is an isometry ("symmetry") of the Lie group $SU(2m)$ which interchanges $\gamma(0)$ with any particular point $\gamma(t)$ and preserves the geodesic γ as a whole (see §1 (19) above, or Part II, §6.4), it follows that (3) holds at every point of γ:

$$K_{\dot\gamma(t)}(e_i(t)) = \mu_i e_i(t),$$

so that the system (4) becomes

$$\frac{d^2 v_i}{dt^2} + \lambda_i v_i = 0, \qquad i = 1, \ldots, k.$$

We are interested in solutions of this system vanishing at $t = 0$. If $\mu_i > 0$, then $v_i(t) = c_i \sin\sqrt{\mu_i}\,t$ for some constant c_i, whence the zeros of $v_i(t)$ are at the integer multiples of $t = \pi/\sqrt{\mu_i}$. If $\mu_i = 0$, then $v_i(t) = c_i t$, and if $\mu_i < 0$, then $v_i(t)$ has the form $c_i \sinh\sqrt{|\mu_i|}\,t$; hence if $\mu_i \le 0$ then $v_i(t)$ is either identically zero or vanishes only at $t = 0$. We conclude that the points on the geodesic $\exp(tX)$ conjugate to the initial point $\gamma(0)$, are determined by the positive eigenvalues μ_i of the linear transformation $K_X: su(2m) \to su(2m)$, given by

$$K_X(Y) = \tfrac{1}{4}[[X, Y], X], \tag{5}$$

occurring at the integer multiples of $t = \pi/\sqrt{\mu_i}$ in the open interval $(0, 1)$.

With X as in (2) it is easy to show that

$$[X, Y] = (i\pi(k_i - k_j)y_{ij}), \qquad Y = (y_{ij}),$$

whence

$$[X, [X, Y]] = (-\pi^2(k_i - k_j)^2 y_{ij}),$$

or (see (5))

$$K_X(Y) = \left(\frac{\pi^2}{4}(k_i - k_j)^2 y_{ij}\right).$$

Now it is easy to check that for each pair i, j with $i < j$ the $2m \times 2m$ matrices E_{ij} with entry $+1$ in the (i, j)th place and -1 in the (j, i)th place and zeros elsewhere, and also the matrices E'_{ij} with (i, j)th and (j, i)th entries both i (and zeros elsewhere), are eigenvectors of the linear transformation K_X, corresponding to the eigenvalue $(\pi^2/4)(k_i - k_j)^2$. Furthermore, each diagonal matrix in $su(2m)$ is clearly also an eigenvector of K_X (corresponding to the eigenvalue 0). Since a basis for $su(2m)$ can be chosen from among these eigenvectors, it follows that the non-zero eigenvalues of K_X are just those numbers $(\pi^2/4)(k_i - k_j)^2$ for which $k_i > k_j$. By the first part of the proof each of these positive eigenvalues μ gives rise to a sequence of conjugate points of γ corresponding to the parameter values $t = n\pi/\sqrt{\mu}$, $n = 1, 2, 3, \ldots$, i.e.

$$t = \frac{2}{k_i - k_j}, \frac{4}{k_i - k_j}, \frac{6}{k_i - k_j}, \ldots.$$

Since the number of such values of t in the open interval $(0, 1)$ is $(k_i - k_j)/2 - 1$, and since each eigenvalue $(\pi^2/4)(k_i - k_j)^2 > 0$ has multiplicity 2, the Index Theorem tells us that the index λ of γ is given by the formula

$$\lambda = \sum_{k_i > k_j} (k_i - k_j - 2). \tag{6}$$

(Note that if γ were minimal, then by Lemma 25.2 (see (1)), we should have $k_i = \pm 1$ for all i, so that the index would be zero (as might be expected).) Now if, on the one hand, at least $(m + 1)$ of the k_i have the same sign (negative say), then at least one of the positive k_i must be ≥ 3, whence

$$\lambda \geq \sum_{1}^{m+1} (3 - (-1) - 2) = 2m + 2.$$

On the other hand, if m of the k_i are negative and m positive, then, since they cannot all be ± 1 (γ being non-minimal), there must be one ≤ -3 and one ≥ 3, whence

$$\begin{aligned}\lambda &\geq \sum_{1}^{m-1} [(3 - (-1) - 2) + (1 - (-3) - 2)] + (3 - (-3) - 2) \\ &= 4m \geq 2m + 2.\end{aligned}$$

This completes the proof of the lemma. □

The theorem on unitary periodicity can now be deduced relatively easily in two steps (the first of which involves the Fundamental Theorem of Morse Theory (22.5)), as follows.

25.5. Lemma. *The inclusion of the space $\tilde{\Omega}$ of minimal geodesics (homeomorphic to the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$) in the path space $\Omega = \Omega(SU(2m); I_{2m}, -I_{2m})$, induces an isomorphism between the corresponding homotopy groups in all dimensions up to and including $2m$. In view of the isomorphism $\pi_i(\Omega M) \simeq \pi_{i+1}(M)$ (see Part II, Corollary 22.2.3) it then follows that*

$$\pi_i(G^{\mathbb{C}}_{2m,m}) \simeq \pi_{i+1}(SU(2m)), \qquad 0 \leq i \leq 2m.$$

PROOF. By Theorem 21.3 the action functional E on the path space Ω has as its critical points the geodesic arcs joining I_{2m} and $-I_{2m}$. By the preceding lemmas the critical points of index 0 (the minimal geodesics) form a subspace homeomorphic to $G^{\mathbb{C}}_{2m,m}$, while the index of all other critical points is at least $2m+2$. It follows via Theorem 22.5 that $\Omega(SU(2m); I_{2m}, -I_{2m})$ has the homotopy type of a space obtained by attaching cells of dimension at least $2m+2$ to the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$. Since attachment of cells of dimensions $\geq 2m+2$ has no effect on the homotopy groups of dimensions $\leq 2m$, we conclude that

$$\pi_i(\Omega) \simeq \pi_i(G^{\mathbb{C}}_{2m,m}) \qquad \text{for} \quad 0 \leq i \leq 2m,$$

as claimed. □

25.6. Lemma. *There is an isomorphism*

$$\pi_{i-1}(U(m)) \simeq \pi_i(G^{\mathbb{C}}_{2m,m}) \qquad \text{for} \quad 1 \leq i \leq 2m.$$

PROOF. As noted in Part II, §24.3(c), from the homotopy exact sequence of the standard fibration $U(m+1) \to S^{2m+1}$ with fibre $U(m)$:

$$\cdots \to \pi_j(S^{2m+1}) \xrightarrow{\partial} \pi_{j-1}U(m) \xrightarrow{i_*} \pi_{j-1}U(m+1) \to \pi_{j-1}(S^{2m+1}) \to \cdots,$$

it follows readily that the inclusion $i\colon U(m) \to U(m+1)$ induces an isomorphism

$$i_*\colon \pi_{j-1}U(m) \simeq \pi_{j-1}U(m+1) \qquad \text{for} \quad j \leq 2m$$

(and also that the homomorphism $i_*\colon \pi_{2m}U(m) \to \pi_{2m}U(m+1)$ is onto). Since the maps

$$\pi_{j-1}U(m) \to \pi_{j-1}U(m+1) \to \pi_{j-1}U(m+2) \to \cdots \tag{7}$$

induced by the inclusions are all isomorphisms, these groups are all isomorphic to $\pi_{j-1}(U)$, the $(j-1)$st "stable homotopy group" of the unitary group.

From (7) and the homotopy exact sequence of the fibre bundle with total space $U(2m)$, fibre $U(m) \subset U(2m)$, and base the coset space $U(2m)/U(m)$ (the "complex Stiefel manifold"; cf. Part II, §5.2(d)):

$$\cdots \to \pi_j U(m) \xrightarrow{i_*} \pi_j U(2m) \to \pi_j(U(2m)/U(m)) \xrightarrow{\partial} \pi_{j-1}U(m) \to \cdots,$$

it follows that (8)

$$\pi_j(U(2m)/U(m)) = 0 \qquad \text{for} \quad j \leq 2m.$$

Considering, finally, the homotopy exact sequene of the fibration

$$U(2m)/U(m) \to G^{\mathbb{C}}_{2m,m} = U(2m)/(U(m) \times U(m)),$$

with fibre $U(m)$, namely

$$\cdots \to \pi_j(U(2m)/U(m)) \to \pi_j(G^{\mathbb{C}}_{2m,m}) \xrightarrow{\partial} \pi_{j-1}(U(m)) \xrightarrow{i_*} \pi_{j-1}(U(2m)/U(m)) \to \cdots$$

and invoking (8), we obtain

$$\partial\colon \pi_j(G^{\mathbb{C}}_{2m,m}) \cong \pi_{j-1}(U(m)) \qquad \text{for} \quad j \leq 2m, \tag{9}$$

as required. □

The theorem is now immediate from this lemma and the preceding one, together with (7) and the isomorphism

$$\pi_j(SU(m)) \cong \pi_j(U(m)) \qquad \text{for} \quad j \neq 1,$$

which follows easily by considering the homotopy exact sequence of the bundle $U(m) \to S^1$ with fibre $SU(m)$ (see Part II, §24.4). □

We wish now to exhibit explicitly the isomorphism $\pi_{j-1}(U(m)) \cong \pi_{j+1}(U(2m))$, $j \leq 2m$, obtained above as the composite of the chain of isomorphisms

$$\pi_{j-1}(U(m)) \underset{\cong}{\xleftarrow{\partial}} \pi_j(G^{\mathbb{C}}_{2m,m}) \underset{\cong}{\to} \pi_j(\Omega(SU(2m); I_{2m}, -I_{2m})) \cong \pi_{j+1}(SU(2m))$$
$$\cong \pi_{j+1}(U(2m)). \tag{10}$$

To this end, let $f_{j-1}\colon S^{j-1} \to U(m)$ be a continuous map (representing the homotopy class $[f_{j-1}] \in \pi_{j-1}(U(m))$); starting with f_{j-1}, we wish to construct a corresponding map $f_{j+1}\colon S^{j+1} \to SU(2m)$. We first distinguish in $SU(2)$, considered in its matrix guise as the group of complex 2×2 matrices of the form

$$x(\alpha, \beta) = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \qquad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1, \tag{11}$$

the subspace D^2, homeomorphic to a 2-dimensional closed disc, consisting of those matrices (11) with β non-negative real. We embed $SU(2)$ (and therefore D^2) into $SU(2m)$ by means of the map

$$\varphi\colon x \to x \otimes I_m = \begin{bmatrix} \alpha I_m & \beta I_m \\ -\bar{\beta} I_m & \bar{\alpha} I_m \end{bmatrix}. \tag{12}$$

We next consider the smooth curve $\hat{\gamma}$ in D^2, defined by

$$\hat{\gamma}(\beta, \tau) = \{x(\alpha, \beta) \mid \alpha = i\tau, \tau \text{ real}\},$$

and write $\gamma = \varphi(\hat{\gamma})$. By Lemma 25.3 the complex Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$ may be identified with the set of solutions in $SU(2m)$ of the matrix equation $X^2 = -I_{2m}$; since the points of $\hat{\gamma}$ all satisfy $(\hat{\gamma}(\beta, \tau))^2 = -I_{2m}$, it follows that, assuming this identification made, we have

$$\hat{\gamma}(\beta, \tau) \in G^{\mathbb{C}}_{2m,m} \subset SU(2m) \qquad \text{for} \quad 0 \leq \beta \leq 1.$$

We now distinguish in $SU(2m)$ the subset consisting of all conjugates of the matrices in (12), of the form

$$g(y,\alpha,\beta)=\begin{bmatrix} I_m & 0 \\ 0 & f_{j-1}(y)^{-1}\end{bmatrix}\begin{bmatrix} \alpha I_m & \beta I_m \\ -\beta I_m & \bar{\alpha} I_m\end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & f_{j-1}(y)\end{bmatrix} = \begin{bmatrix} \alpha I_m & \beta f_{j-1}(y) \\ -\beta f_{j-1}(y)^{-1} & \bar{\alpha} I_m\end{bmatrix}, \tag{13}$$

where $\beta \geq 1$, $|\alpha|^2+\beta^2=1$ and $y \in S^{j-1}$. This defines a map $g: S^{j-1}\times D^2 \to SU(2m)$, and since

$$g(y,\alpha,0)=\begin{bmatrix} \alpha I_m & 0 \\ 0 & \bar{\alpha} I_m\end{bmatrix},$$

for all y, it is not difficult to see that, in fact, the image in $SU(2m)$ is a $(j+1)$-dimensional sphre $S^{j+1}\subset SU(2m)$; thus g determines a map

$$f_{j+1}: S^{j+1}\to SU(2m). \tag{14}$$

Restricting β to be real, $0\leq \beta \leq 1$, and α to have the form $i\tau$, τ real, we obtain from (13) a map

$$S^{j-1}\times I \to G^{\mathbb{C}}_{2m,m}\subset SU(2m),$$

whose image is a j-dimensional sphere

$$S^j \subset G^{\mathbb{C}}_{2m,m}\subset SU(2m), \tag{15}$$

obtained essentially as the quotient space of $S^{j-1}\times \hat{\gamma}$ with, on the one hand, all points of the form $(y, \hat{\gamma}(1,0))$ (i.e. with $\alpha=1$, $\beta=0$, forming the lid of the cylinder) identified, and, on the other hand, all points $(y,\hat{\gamma}(-1,0))$ (constituting the base of the cylinder) identified. Putting $\beta=1$ (whence $\alpha=0$) in (13) we obtain the map

$$h: y\mapsto \begin{bmatrix} 0 & f_{j-1}(y) \\ -f_{j-1}(y)^{-1} & 0\end{bmatrix}, \qquad S^{j-1}\to G^{\mathbb{C}}_{2m,m}\subset SU(2m); \tag{16}$$

it can be shown that in fact $\partial[S^j]=[h]$, where $S^j\subset G^{\mathbb{C}}_{2m,m}$ is as in (15), and ∂ is (essentially) the boundary map determining the isomorphism (9). The upshot is that with the element $[f_{j-1}]\in \pi_{j-1}(U(m))$ (which may be identified with $[h]$), we have associated the element $[f_{j+1}]\in\pi_{j+1}(SU(2m))$ given by (14), i.e. by the formula

$$\{g(y,\alpha,\beta)\}=f_{j+1}(S^{j+1})=\left\{\begin{bmatrix} \alpha I_m & \beta f_{j-1}(y) \\ -\beta f_{j-1}(y)^{-1} & \bar{\alpha} I_m\end{bmatrix}\right\}.$$

In fact, the map $[f_{j-1}]\mapsto[f_{j+1}]$ so defined coincides with the isomorphism of unitary periodicity given by (10): The first step, whereby $[f_{j-1}]$ is associated (via ∂^{-1}) with the S^j of (15), has already been noted. The next step is achieved by means of the embedding of $G^{\mathbb{C}}_{2m,m}$ in $SU(2m)$ as the set of mid-points of the minimal geodesics from I_{2m} to $-I_{2m}$ (see Lemma 25.3); this embedding associates the sphere S^j (or, more properly, "spheroid", i.e. map of the sphere) in $G^{\mathbb{C}}_{2m,m}$ with the spheroid in $\Omega(SU(2m))$ consisting of all the minimal geodesics whose mid-points are in S^j, which in turn, via the standard isomorphism of Part II, Corollary 22.2.3, is associated with $S^{j+1} \subset SU(2m)$. We have thus outlined the proof of the following

25.7. Theorem (Fomenko). *Let $f_{j-1}: S^{j-1} \to U(m)$ represent an arbitrary element of the homotopy group $\pi_{j-1}(U(m))$. The isomorphism of unitary periodicity (10) is given by the explicit formula $[f_{j-1}] \to [f_{j+1}]$, where $f_{j+1}: S^{j+1} \to SU(2m)$ is defined by*

$$f_{j+1}: S^{j+1} \to \{g(y, \alpha, \beta) \mid y \in S^{j-1}, \beta \geq 1, |\alpha|^2 + \beta^2 = 1\} \subset SU(2m),$$

with $g(y, \alpha, \beta)$ as in (13).

Thus, to repeat somewhat, the isomorphism of unitary periodicity (10) is, from a visual, geometric point of view, constructed in rather a simple way in the following two steps:

Step 1. Starting with an arbitrary spheroid f_{j-1} in $U(m)$ one transforms this spheroid by means of the boundary homomorphism $\partial: \pi_j(G^{\mathbb{C}}_{2m,m}) \to \pi_{j-1}(U(m))$ (or rather its inverse; here it is in fact an isomorphism—see (9)), into a spheroid S^j of one higher dimension in the Grassmannian manifold (see (16) for the explicit formula).

Step 2. By identifying the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$ with the intersection of the group $SU(2m)$ with its Lie algebra $su(2m)$ (both regarded as subspaces of the space of all complex $2m \times 2m$ matrices), which intersection consists precisely of the mid-points of the minimal geodesics in $SU(2m)$ from I_{2m} to $-I_{2m}$ (see Lemma 25.3), one can associate the spheroid S^j in $G^{\mathbb{C}}_{2m,m}$ obtained in Step 1, with the spheroid $S^{j+1} \subset SU(2m)$ obtained as the union of those minimal geodesics whose mid-points are points of the spheroid $S^j \subset G^{\mathbb{C}}_{2m,m}$. This spheroid, now in $SU(2m)$ and of dimension $j + 1$, is then the final image of the initial spheroid f_{j-1} under the isomorphism of unitary periodicity (the explicit formula for which is given in the above theorem).

In the case $m = 2, j = 4$, if the initial spheroid $f_{j-1} = f_3: S^3 \to SU(2) \subset U(2)$ is taken to be the map identifying the sphere S^3 with $SU(2)$:

$$f_3(y) = \begin{bmatrix} \alpha & \beta \\ -\bar{\alpha} & \bar{\beta} \end{bmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1,$$

then the homotopy class $[f_3]$ is a generator of $\pi_3(SU(2)) \simeq \mathbb{Z}$. Applying repeatedly the above explicit formula for the isomorphism of unitary periodicity, starting with this $f_{j-1} = f_3$, one obtains in succession generators $[f_5]$, $[f_7]$, ..., in the cases $m = 2^2, 2^3, \ldots$, i.e. for each $k = 1, 2, \ldots$, a map

$$f_{2k+1}: S^{2k+1} \to SU(2^k), \tag{17}$$

representing a generator of $\pi_{2k+1}(SU(2^k)) \simeq \mathbb{Z}$. For each k the map f_{2k+1} so obtained is in fact the complex analogue $\alpha^{\mathbb{C}}_{2k+1}$ of the (real) "duality" map α_{2k+1} occurring in the theory of "Clifford algebras" and spinor representations of the orthogonal group (cf. [5], where the "isomorphism of orthogonal periodicity" is related to the structure of certain Clifford algebras).

We conclude this subsection by providing some of the details of this analogy, since for the above pairs of values $m = 2^k$, $j = 2k + 2$ ($k = 1, 2, \ldots$) it furnishes another explicit formula for the isomorphism of unitary periodicity, simplifying still further the geometric picture. Thus the map $\alpha^{\mathbb{C}}_{2k+1}$ may be defined as follows: Let

$$f: S^{n-1} \to GL(N, \mathbb{C}), \qquad g: S^{m-1} \to GL(M, \mathbb{C}),$$

be any two continuous maps. Regarding S^{n-1} and S^{m-1} as embedded in the standard way in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we may clearly extend these maps to homogeneous maps of $\mathbb{R}^n$ and $\mathbb{R}^m$ (to the respective general linear groups). One then defines a map

$$f * g = \omega: \mathbb{R}^{n+m} \setminus \{0\} \to GL(2MN, \mathbb{C}),$$

by setting

$$(f * g)(x, y) = \begin{bmatrix} f(x) \otimes I_M & I_N \otimes g(y) \\ -I_N \otimes g^*(y) & f^*(x) \otimes I_M \end{bmatrix},$$

where $f^*(x)$ and $g^*(y)$ are the conjugate transposes of $f(x)$ and $g(y)$, and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $(x, y) \neq (0, 0)$. The restriction of this map to the standard unit sphere $S^{n+m-1} \subset \mathbb{R}^{n+m}$, then yields a map $S^{n+m-1} \to GL(2MN, \mathbb{C})$. Taking, in particular, the map $\alpha: S^1 \to GL(1, \mathbb{C}) = \mathbb{C}\setminus\{0\}$ to the unit circle: $\alpha(z) = z$, $|z| = 1$, one then defines $\alpha^{\mathbb{C}}_{2k+1}$ to be the restriction to $S^{2k+1} \subset \mathbb{R}^{2k+2}$ of the map

$$\alpha * \cdots * \alpha: \mathbb{R}^{2k+2} \setminus \{0\} \to GL(2^k, \mathbb{C}).$$

It is now not difficult to verify directly that $\alpha^{\mathbb{C}}_{2k+1} \equiv f_{2k+1}$, where f_{2k+1} is as defined above (see (17) *et seqq.*).

25.2. Unitary Periodicity via the Two-Dimensional Calculus of Variations

The proof of unitary periodicity given in the preceding subsection is based on the one-dimensional calculus of variations applied to the action functional defined on (piecewise-smooth) paths in the unitary group. It turns out that the isomorphism of unitary periodicity can be established more naturally by considering instead an appropriate 2-dimensional variational problem.

In the above, "classical", approach to unitary periodicity, the proof is carried out in two quite distinct steps, in each of which the dimension of the homotopy groups under consideration increases by 1. The fact that the proof breaks up in this way into two parts (the required increase of 2 in the dimension being achieved by means of two successive increases of 1) is a natural consequence of the method used, inasmuch as that method involves the one-dimensional calculus of variations of the action (and length) functionals on paths, i.e. maps of the one-dimensional disc D^1 (a line segment). We now recapitulate the details of that method, in order to divine, as it were, the appropriate 2-dimensional analogue: Thus we take as the disc D^1 the interval $[0, 1]$, with boundary $\partial D^1 = S^0$ (the zero-dimensional sphere, consisting of the points 0, 1), and now denote by Π_1^* the space $\Omega^*(SU(2m); I_{2m}, -I_{2m})$ of all continuous maps f of D^1 to $SU(2m)$ satisfying $f|_{S^0} = i_0|_{S^0}$, where $i_0(S^0) = \{I_{2m}, -I_{2m}\}$, i.e. every f sends the end-points 0, 1 of D^1 to the same points $I_{2m}, -I_{2m}$ (respectively) of $SU(2m)$. The action functional E on Π_1, the subspace of Π_1^* consisting of all piecewise-smooth paths in Π_1^* (and homotopically equivalent to Π_1^*) is given by

$$E(\omega) = E_0^1(\omega) = \int_0^1 \left|\frac{d\omega}{dt}\right|^2 dt, \qquad \omega \in \Pi_1.$$

(The associated length functional

$$L_0^1(\omega) = \int_0^1 \left|\frac{d\omega}{dt}\right| dt,$$

was shown (essentially) in Part I, §31.2, to have the same extremal paths as the action functional E_0^1 in the class of all (piecewise-) smooth paths.) The set of points (i.e. paths) on which the action E_0^1 (and therefore also L_0^1) attains an absolute minimum, forms a subspace $W = \tilde{\Pi}_1$, shown in Lemma 25.2 to be homeomorphic to the Grassmannian manifold $G_{2m,m}^{\mathbb{C}}$, whence it was deduced (using Morse theory; see in particular Lemma 25.5) that the spaces Π_1 (and hence also Π_1^*) and $G_{2m,m}^{\mathbb{C}}$ have the homotopy type of cell complexes with identical $(2m+1)$-dimensional skeletons. Thus one might say that the "analytic part" of the isomorphism of unitary periodicity is given by the consequent isomorphism

$$\pi_i(G_{2m,m}^{\mathbb{C}}) \simeq \pi_i(\tilde{\Pi}_1) \simeq \pi_i(\Pi_1) \simeq \pi_{i+1} SU(2m), \qquad i \leq 2m,$$

since the subsequent step, $\pi_i(G_{2m,m}^{\mathbb{C}}) \simeq \pi_{i-1}(U(m))$, does not involve the action functional E_1^0, being of a purely homotopy-theoretic character (see Lemma 25.6).

The above-described geometrical procedure for obtaining the isomorphism of unitary periodicity prompts the idea of trying to construct that isomorphism in a single step, rather than two distinct steps, by utilizing some 2-dimensional functional rather than the one-dimensional action functional on paths. It turns out that there is a feasible such method of obtaining the isomorphism of the unitary periodicity, which moreover leads to a further simplification of the "geometrical picture" of that isomorphism. We shall in the present subsection expound this "one-step" procedure. (Note, however, that the construction we give ultimately invokes the explicit formula (13), so that it does not constitute an independent proof of unitary periodicity.)

We first need to describe the appropriate 2-dimensional problem; thus we seek to define a suitable functional on a suitable specially chosen class of maps of a 2-dimensional disc. To begin with consider in $SU(2m)$ the embedded circle (a one-parameter subgroup)

$$S_0^1 = \left\{ \begin{bmatrix} \alpha I_m & 0 \\ 0 & \bar{\alpha} I_m \end{bmatrix} \middle| \, |\alpha| = 1 \right\}. \tag{18}$$

Let $D^2 \subset \mathbb{R}^2$ be the disc with centre (0, 0) and radius 1, and let $j_0: S^1 \to SU(2m)$ be the (fixed) map identifying the boundary $S^1 = \partial D^2$ of the disc D^2 with the circle S_0^1 in the obvious way. Denote by Π_2^* the topological space (topologized by means of an appropriate distance function, i.e. metric) consisting of all continuous maps $f: D^2 \to SU(2m)$ satisfying $f|_{S^1} = j_0$. (Thus $S_0^1 \subset SU(2m)$ plays, in the present 2-dimensional context, the role analogous to that played previously by the zero-dimensional sphere consisting of the two points I_{2m}, $-I_{2m}$.) It can be shown that Π_2^* has the homotopy type of a cell complex.

We next consider the subspace $\Pi_2 \subset \Pi_2^*$ consisting of all maps f in Π_2^*, belonging to a certain function space $H_1^2(D^2)$ which we shall now define: Given a region X of Euclidean space $\mathbb{R}^n(x^1, \ldots, x^n)$, we shall say that a function $u: X \to \mathbb{R}$ *belongs to the class of functions* $H_m^p(X)$, if it satisfies the following two conditions:

(i) $u \in L_p(X)$, i.e. $|u|^p$ is integrable;
(ii) corresponding to each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers with $0 \le |\alpha| \le m$ if $m > 1$ and $|\alpha| = 1$ if $m = 1$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, there should exist a "generalized derivative" $D^\alpha u$ of u, i.e. a function $r_\alpha \in L_p(X)$ with the property that for every function $g: X \to \mathbb{R}$ of class C^∞

$$\int_X g(x) r_\alpha(x)\, dx = (-1)^{|\alpha|} \int_X |\bar{D}^\alpha g(x)| u(x)\, dx,$$

where $\bar{D}^\alpha g = \partial^{|\alpha|} g / (\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}$.

(Thus, in particular, $H_1^2(D^2)$ consists of those square integrable functions $u: D^2 \to \mathbb{R}$ for which for each $i = 1, 2$ there exists a "generalized derivative" $D_i(u)$, i.e. a square integrable function $\bar{r}_i$ such that for every C^∞-function $g: D^2 \to \mathbb{R}$

$$\int_{D^2} g(x) \bar{r}_i(x)\, dx = \int_{D^2} \left| \frac{\partial g}{\partial x^i} \right| u(x)\, dx,$$

where x_1, x_2 are the standard co-ordinates on $\mathbb{R}^2$.) To say that a map $f: D^2 \to SU(2m)$ is in $H_1^2(D^2)$ is then understood to mean that each of the $8m^2$ real co-ordinate functions is in $H_1^2(D^2)$. Thus the condition that $f: D^2 \to SU(2m)$ belong to $H_1^2(D^2)$ (as well as to Π_2^*) to qualify as a member of Π_2, replaces, in the 2-dimensional context, the previous condition that paths $D^1 \to SU(2m)$ be piecewise-smooth to qualify for membership in $\Pi_1 (= \Omega)$, the appropriate condition for developing a "one-dimensional" Morse theory applicable to path spaces.

Having specified Π_γ, we now choose a suitable functional on Π_γ; the Dirichlet functional $D[f]$, which associates with each $f \in \Pi_\gamma$ the "Dirichlet integral" of f (see the definition below), turns out to be appropriate for our purpose. As was noted in Part I, §37.5, the 2-dimensional Dirichlet functional is the analogue of the one-dimensional action functional in much the same sense that the area functional A is the 2-dimensional analogue of the length functional. In particular, the value of the functional D at f, like the action E, depends also on the parameters, i.e. on the co-ordinates chosen on the domain of f, whereas the values taken on by A and L are parameter-independent. We now give the definition of the general n-dimensional Dirichlet functional (since the 8-dimensional version will be needed in the following subsection). Thus let M and V be Riemannian manifolds with metric tensors $g_{ij}(y)$, $y \in M$, and $\hat{g}_{kl}(v)$, $v \in V$. With each map $f: V \to M$, $f \in H_1^p(V, M)$ (see above) we associate the "mixed" tensor $y_k^i = D^{\alpha_k}(y^i)$, where the y^i are the local co-ordinates on M of the image under f of each $v \in V$: $(y^i) = y = f(v)$, and where $\alpha_k = (0, \ldots, 1, \ldots, 0)$ is the n-tuple ($n = \dim V$) with kth component 1 and all other components 0, and D^{α_k} is the corresponding generalized differential operator defined above. (Thus the upper index i of the tensor y_k^i ranges from 1 to $\dim M$, and the lower, k, from 1 to $\dim V = n$.) Given two such mixed tensors y_k^i, $\bar{y}_l^j$, arising from maps $f, \bar{f}: V \to M$, we define their (mixed) *scalar product* by

$$(y_k^i, \bar{y}_l^j) = \hat{g}^{kl} g_{ij} y_k^i \bar{y}_l^j,$$

where $(\hat{g}^{kl}) = (\hat{g}_{kl})^{-1}$. Then at each $f \in H_1^p(V, M)$ we define the *Dirichlet functional* D on $H_1^p(V, M)$ by

$$D[f] = \int_V \left[\frac{1}{n}(y_k^i, y_l^j)\right]^{n/2} dV, \tag{19}$$

where dV denotes the element of volume of the Riemannian manifold V. A map $f \in H_1^p(V, M)$ is then said to be *harmonic* if $\delta D[f; \eta] = 0$ for every vector field η on $f(V)$ belonging to H_1^p, i.e. if the first-order variation of $D[f]$ resulting from a variation of f by means of any such field η is zero (see, e.g. Part I, §37, and cf. in particular Proposition 37.5.3 there). Thus the harmonic maps are the n-dimensional analogues of the geodesics. It can be shown by direct calculation that the resulting Euler–Lagrange equations for the harmonic maps are of the form $\nabla^k \nabla_k y^i = 0$ (see Part I, Theorem 37.1.2), where ∇_k denotes the operation of covariant differentiation with respect to the connexion on V compatible with its Riemannian metric.

In the particular case of present interest, the manifold V is the 2-dimensional unit disc in its standard position in Euclidean $\mathbb{R}^2$, so that $\hat{g}_{kl} = \delta_{kl}$, and the Dirichlet functional (19) takes the form

$$D[f] = \frac{1}{2}\int_{D^2} [(y_1^i, y_1^j) + (y_2^i, y_2^j)]\, dV = \frac{1}{2}\int_{D^2} g_{ij}(y_1^i y_1^j + y_2^i y_2^j)\, dV. \tag{20}$$

where $f \in \Pi_2$, and g_{ij} is the metric on $M = SU(2m)$ induced from the Euclidean metric on some $\mathbb{R}^N$ with $SU(2m)$ embedded in a standard way in $S^{N-1} \subset \mathbb{R}^N$ (see below). If the Euclidan co-ordinates on the disc $D^2 \subset \mathbb{R}^2$ are u, v, then (20) becomes (cf. Part I, end of §37.5)

$$D[f] = \frac{1}{2}\int_{D^2} [(y_u, y_u) + (y_v, y_v)]\, du\, dv, \qquad y = (y^i) \in SU(2m), \tag{21}$$

and the first variation δD of the functional D corresponding to the addition to f of a vector field $\eta \in H_1^2(D^2)$ on $f(D^2)$, is readily seen to be given by

$$\delta D[f; \eta] = \int_{D^2} \left[\left(\frac{\partial \eta}{\partial u}, y_u\right) + \left(\frac{\partial \eta}{\partial v}, y_v\right)\right] du\, dv. \tag{22}$$

The related functional A on the space Π_2 assigns to each map $f \in \Pi_2$, $f(u, v) = (y^i(u, v))$, the integral (cf. Part I, §37.5)

$$\int_{D^2} \left[\det\begin{pmatrix} (y_u, y_u) & (y_u, y_v) \\ (y_u, y_v) & (y_v, y_v) \end{pmatrix}\right]^{1/2} du\, dv;$$

thus $A[f]$ is the 2-dimensional surface-area functional. As was noted in Part I, §37.5, for all f one has $A[f] \leq D[f]$, with equality occurring precisely if the map f is (generalized) conformal, i.e. if u, v furnish conformal co-ordinates on the surface $f(D^2) \subset SU(2m)$ (so that in terms of these co-ordinates the induced Riemannian metric on $f(D^2)$ has diagonal form). It was also shown (see loc. cit.) that under this condition on u, v as co-ordinates of $f(D^2)$, the Euler–Lagrange equations for a critical (in particular area-minimizing) surface $f(D^2)$ of the functional A, are equivalent to harmonicity (in the usual sense) of the radius vector $y = (y^i(u, v))$. This is analogous to the one-dimensional situation (of paths γ in $SU(2m)$ from I_{2m} to $-I_{2m}$) where the action and length functionals E and L satisfy $L^2(\gamma) \leq E(\gamma)$, with equality holding precisely when γ is parametrized by means of a natural parameter, in which case the critical paths for both functionals are the geodesic arcs from I_{2m} to $-I_{2m}$.

Thus the use of the Dirichlet functonal D (rather than the area functional A) serves the purpose, analogous to that served by E, of enabling us to ignore those maps f which, though not harmonic, can be made harmonic by means of a continuous change of co-ordinates on the disc D^2: although such a change of parameters has of course no effect on the area functional, it will in general cause the value of the Dirichlet functional to change.

Before stating the theorem we are leading up to, we mention the following auxiliary facts:

(i) there is a (natural) isomorphism

$$\beta_2 \colon \pi_j(\Pi_2^*) \simeq \Pi_{j+2}(SU(2m)); \tag{23}$$

(ii) the space Π_2^* is homotopically equivalent to the space $\hat{\Pi}_2$ consisting of all continuous maps $S^2 \to SU(2m)$ with the north pole (say) mapped always to the same point of $SU(2m)$.

(The isomorphism (23) is established by means of two applications of Corollary 22.2.3 of Part II: if $\hat{\Omega}$ denotes the loop space of $SU(2m)$, i.e. the space of all loops beginning and ending at I_{2m}, then that corollary gives immediately $\pi_{j+1}(\hat{\Omega}) \simeq \pi_{j+2}(SU(2m))$; on the other hand, by applying it to the path space in $\hat{\Omega}$ consisting of all trajectories in $\hat{\Omega}$ with initial "point" the circle S_0^1 of (18) (such trajectories being given by maps $D^2 \to SU(2m)$; cf. Definition 21.1 et seqq.), on noting that the fibre above S_0^1 itself is precisely Π_2^*, one obtains $\pi_j(\Pi_2^*) \simeq \pi_{j+1}(\hat{\Omega})$.)

25.8. Theorem (Fomenko). *Let Π_2^*, Π_2 be the spaces of maps $D^2 \to SU(2m)$ defined above, and let W be the subspace of Π_2 consisting of all points (i.e. maps) $f \in \Pi_2$ at which the Dirichlet functional $D[f]$ attains its absolute minimum value. The following assertions hold:*

(i) *the subspace $W \subset \Pi_2$ is homeomorphic to the group $U(m)$;*
(ii) *the inclusion $i: W \to \Pi_2 \to \Pi_2^*$ induces an isomorphism*

$$i'_*: \pi_j(U(m)) \simeq \pi_j(\Pi_2^*) \qquad \text{for} \quad j \leq 2m,$$

(whence it follows that the 2m-dimensional skeletons of $U(m)$ and Π_2^*, or rather of some realizations of these spaces as cell complexes, are homotopically equivalent).

The composite of the isomorphism i'_* with the isomorphism β_2 of (23) coincides with the isomorphism of unitary periodicity (see (10)):

$$\beta_2 \circ i'_*: \pi_j(U(m)) \simeq \pi_{j+2}(SU(2m)), \qquad j \leq 2m. \tag{24}$$

(Thus by considering the set W of absolute minimum points of the 2-dimensional Dirichlet functional, the isomorphism of unitary periodicity can be achieved in a single step (involving an increase of 2 in the dimension of the homotopy groups), instead of the two steps required in the proof using the one-dimensional functionals of action and length.)

We give the proof in the form of a sequence of lemmas. First consider the 2-dimensional sphere

$$S_0^2 = \left\{ \begin{bmatrix} \alpha I_m & \beta I_m \\ -\beta I_m & \bar{\alpha} I_m \end{bmatrix} \middle| \, \beta \in \mathbb{R}, |\alpha|^2 + |\beta|^2 = 1 \right\} \subset SU(2m), \tag{25}$$

whose equator, defined by $\beta = 0$, coincides with the circle S_0^1 of (18). The upper hemisphere, $\beta \geq 0$, which we shall now denote by D_0^2, played a significant role in the earlier explicit construction of the isomorphism of unitary periodicity (see (11) *et seqq.*). Using the fact that the inclusion $S_0^2 \to SU(2m)$ extends to a natural embedding of $SU(2)$ into $SU(2m)$ (see (12)), it can be shown that the sphere S_0^2 is a "totally geodesic" submanifold of $SU(2m)$, and therefore a minimal submanifold (i.e. area-minimizing) (see the discussion following Lemma 25.10 below). (A submanifold of a manifold M is said to be *totally geodesic* if every geodesic in M tangent to the submanifold at any point of it lies wholly in the submanifold or, equivalently, if every geodesic of the submanifold is a geodesic in M. That such submanifolds are locally volume-minimizing follows from the explicit form taken by the Riemannian curvature tensor on M when restricted to the submanifold; in particular, in the (present) case where M is a Lie group, this restriction is a direct summand of the full Riemannian curvature tensor on the ambient group M.)

Thus we may take it that the disc D_0^2, the upper hemisphere of S_0^2, is a totally geodesic submanifold of $SU(2m)$. Let $\hat{W}$ denote the set of (totally geodesic) discs $D^2(x) \subset SU(2m)$ of the form $D^2(x) = xD_0^2x^{-1}$, where $x \in SU(2m)$ and $xsx^{-1} = s$ for all $s \in S_0^1$.

25.9. Lemma. *The subspace $\hat{W} \subset \Pi_2$ is homeomorphic to $U(m)$.*

Proof. Let x be any element of $SU(2m)$ satisfying $xs = sx$ for all $s \in S_0^1$. Since $S_0^1 = \{\alpha I_m \oplus \bar{\alpha} I_m \,|\, |\alpha| = 1\}$ (see (18)), it follows easily that x must also have block diagonal form:

$$x = A \oplus D, \qquad A, D \in U(m),$$

whence

$$x = (I_m \oplus DA^{-1})(A \oplus A) = x_1(A \oplus A) \qquad \text{where} \quad x_1 = I_m \oplus DA^{-1}.$$

From the fact that $(A \oplus A)d = d(A \oplus A)$ for all $d \in D_0^2$, $A \in U(m)$, we infer that

$$D^2(x) = D^2(x_1) = \left\{ \begin{bmatrix} \alpha I_m & \beta C^{-1} \\ -\beta C & \bar{\alpha} I_m \end{bmatrix} \middle| \beta \geq 1, |\alpha|^2 + \beta^2 = 1 \right\} \text{ where } C = DA^{-1}.$$

Since $\beta \geq 0$, the matrix $C \in U(m)$ is uniquely determined by the disc $D^2(x)$; in fact, as is easily verified, the map $D^2(x) \mapsto C$ is a bijection from $\hat{W}$ to $U(m)$, and therefore, bicontinuity being obvious, a homeomorphism. This completes the proof. □

We use this homeomorphism to construct an embedding $i'\colon U(m) \to \Pi_2$. That homeomorphism associates with each $g \in U(m)$ the 2-dimensional disc

$$D^2(I_m \oplus g) = \left\{ \begin{bmatrix} \alpha I_m & \beta g^{-1} \\ -\beta g & \bar{\alpha} I_m \end{bmatrix} \middle| \beta \geq 1, |\alpha|^2 + \beta^2 = 1 \right\}.$$

(Note incidentally that if $g_1 \neq g_2$, then

$$D^2(I_m \oplus g_1) \cap D^2(I_m \oplus g_2) = S_0^1.)$$

Denoting by $f_0\colon D^2 \to D_0^2$ the standard identification with $D_0^2 \subset SU(2m)$ of the standard disc $D^2 \subset \mathbb{R}^2$, we define, for each $g \in U(m)$, a map $f_g\colon D^2 \to SU(2m)$ in Π_2 by setting

$$f_g(y) = (I_m \oplus g)f_0(y)(I_m \oplus g^{-1}), \qquad y \in D^2.$$

The embedding $i'\colon U(m) \to \Pi_2$ we are seeking to define, is then given by $i'\colon g \mapsto f_g$.

It follows from the above lemma that the set of maps $i'(U(m)) \subset \Pi_2$ coincides with the set of maps of the form $\mathrm{Ad}_x \circ f_0$, where x ranges over the group $G = \{I \oplus C\} \subset U(2m)$, $G \simeq U(m)$, and Ad_x denotes conjugation by x; thus $i'(U(m))$ is the orbit containing the point f_0 of Π_2 under the conjugating action of the group G.

25.10. Lemma. *The homomorphism*

$$\beta_2 \circ i'_*: \pi_j(U(m)) \to \pi_{j+2}(SU(2m))$$

coincides, for $j \leq 2m$, with the isomorphism of unitary periodicity. It follows that the homomorphism

$$i'_*: \pi_j(U(m)) \to \pi_j(\Pi_2^*)$$

is an isomorphism for $j \leq 2m$.

Proof. Let $\varphi: S^j \to U(m)$ represent a typical element $[\varphi]$ of $\pi_j(U(m))$. Writing $\varphi(y) = g$, for $y \in S^j$, we have

$$(\beta_2 \circ i'_*)(\varphi)(y) = \beta_2(f_g), \qquad f_g: D^2 \to SU(2m),$$

by definition of i'. By analysing the map β_2 in detail (see (23) *et seqq.*) it can be shown that

$$\begin{aligned}\beta_2(f_g) &= D^2(I_m \oplus \varphi(y)) = D^2(I_m \oplus g) \\ &= \left\{\begin{bmatrix} \alpha I_m & \beta g^{-1} \\ -\beta g & \bar{\alpha} I_m \end{bmatrix} \middle| \beta \geq 1, |\alpha|^2 + \beta^2 = 1\right\},\end{aligned}$$

which coincides, in its essentials, with the explicit formula (13) (of Theorem 25.7) for the isomorphism of unitary periodicity (valid for $j \leq 2m$).

Since β_2 is an isomorphism (for all j), it follows that the homomorphism $i'_*: \pi_j(U(m)) \to \pi_j(\Pi_2^*)$ is an isomorphism for $j \leq 2m$. This completes the proof of the lemma. □

To complete the proof of the theorem it only remains to show that $i'(U(m)) = W$. To this end, consider the Euclidean space $\mathbb{R}^{8m^2}$ identified with the complex space $\mathbb{C}^{4m^2}$ of all $2m \times 2m$ complex matrices endowed with the bilinear form $\psi(A, B) = \mathrm{Re}(\mathrm{tr}\, A\bar{B}^T)$. The group $SU(2m)$ then embeds (isometrically) in the sphere $S^{8m^2-1} \subset \mathbb{R}^{8m^2}$ of radius $\sqrt{2m}$, as a smooth submanifold equipped with the special Riemannian metric (in fact, just the Euclidean metric on $\mathbb{R}^{8m^2}$) induced from the above bilinear form, invariant under both right and left translations by elements of $SU(2m)$ (the Killing form; cf. the beginning of §25.1).

Many of the metrical properties of the group $SU(2m)$ are more conveniently examined in the larger context of the ambient sphere S^{8m^2-1}, for instance, the property that there are no variations ("infinitesimal perturbations") of the embedded 2-dimensional disc $D_0^2 \subset SU(2m)$, leaving the boundary $S_0^1 = \partial D_0^2$ pointwise fixed, such that the perturbed disc $\tilde{D}_0^2$, while remaining minimal in

$SU(2m)$, is no longer totally geodesic (see above). To see this suppose that such a variation exists. Note that the circle $S_0^1 \subset SU(2m) \subset S^{8m^2-1}$ is a "great circle" in the sphere S^{8m^2-1}, in fact the equator of the "great 2-sphere" S_0^2, obtained as the cross-section of the sphere S^{8m^2-1} by a 3-dimensional subspace of $\mathbb{R}^{8m^2}$, i.e. by a 3-dimensional "plane" through the origin. Since by hypothesis the disc $\tilde{D}_0^2$, obtained from D_0^2 by means of a small perturbation, is not totally geodesic in $SU(2m)$, it will not be totally geodesic as a submanifold of the sphere S^{8m^2-1}. Hence the disc $\tilde{D}_0^2$ cannot form part of a "great 2-sphere" in S^{8m^2-1} (with equator S_0^1), in consequence of which its area must to a linear approximation exceed that of D_0^2, i.e. $\delta A > 0$. Thus every variation of a disc $D^2(x) \in W$ either preserves the property of total geodesicity (in which case the perturbed disc is again a hemisphere, with boundary S_0^1, of a great 2-sphere in S^{8m^2-1}, and the transformation effecting the perturbation is a "rotation about S_0^1" and can be achieved by mans of an inner automorphism of the group $SU(2m)$), or destroys the property of local minimality enjoyed by $D^2(x)$ (at least in a neighbourhood of some point).

25.11. Lemma. *The inclusion $i'(U(m)) \subset W$ holds.*

PROOF. Since each map $f \in i'(U(m))$ has the form $\mathrm{Ad}_x \circ f_0$, where Ad_x is the map effecting conjugation by $x \in G$ (see above), it suffices to verify that f_0, the standard map $D^2 \to D_0^2$, is an absolute minimum point for the Dirichlet functional D. Now, as already made clear in the preceding discussion, the disc D_0^2, being a hemisphere of the spherical cross-section of the sphere S^{8m^2-1} by a 3-dimensional plane through the origin of $\mathbb{R}^{8m^2}$, has least area among all (appropriate) discs in S^{8m^2-1} with boundary S_0^1, so that the area functional of such discs takes on its least value at f_0. Since $A[f_0] = D[f_0]$ provided $D[f_0]$ is evaluated with respect to conformal co-ordinates, and since we always have $A[f] \leq D[f]$, the fact that $A[f_0]$ is least for A implies that $D[f_0]$ is least for D. □

25.12. Lemma. *We have $i'(U(m)) = W$.*

PROOF. Let $f: D^2 \to SU(2m)$, $f|_{S^1} = j_0$, be an element of W, i.e. a map in Π_2 at which the Dirichlet functional D attains its least value (among all values taken on Π_2). It follows from the inequality $A[f] \leq D[f]$ together with the fact that equality holds precisely when the surface defined by f is parametrized by conformal co-ordinates (on D^2), that in fact the present absolute-minimum function f is so parametrized, whence

$$D[f] = D[f_0] = A[f] = A[f_0]$$

(cf. the proof of the preceding lemma). (Note incidentally that f must then be harmonic as a function of those co-ordinates.) Since the metric on $SU(2m)$ is induced from the standard metric on $S^{8m^2-1} \subset \mathbb{R}^{8m^2}$ (see above), it follows that, like D_0^2, the image $f(D^2)$ must be a "great hemisphere" in S^{8m^2-1}, i.e. a hemisphere of a great 2-sphere $\tilde{S}^2 \subset S^{8m^2-1}$. Thus we have in S^{8m^2-1} the two great 2-spheres $\bar{S}^2$ and S_0^2 (therefore totally geodesic in S^{8m^2-1}, and hence certainly in $SU(2m)$) whose intersection contains S_0^1 (which contains the point I_{2m}). Since each of these 2-spheres is the cross-section of S^{8m^2-1} by a 3-dimensional plane through the origin of $\mathbb{R}^{8m^2-1}$, they can be transformed into one another by means of a linear transformation of $\mathbb{R}^{8m^2}$ interchanging the two planes. Since the smallest subgroup of $SU(2m)$ containing S_0^2 is the isomorphic copy G_1 of $SU(2)$ obtained from (25) by letting β take on all complex values consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup G_2 containing $\bar{S}^2$ is likewise an isomorphic copy of $SU(2)$. In both of the groups G_1, G_2, the circle S_0^1 is the image of a maximal torus $T^1 \cong S^1$ in $SU(2)$ (i.e. a maximal connected commutative Lie subgroup), and $S_0^1 \subset T^{2m-1}$ for some maximal torus $T^{2m-1} \subset SU(2m)$. Letting $\alpha_0, \alpha\colon SU(2) \to SU(2m)$ denote the maps with images G_2, G_1, extending f_0, f respectively, it follows from Lie theory, in particular as it pertains to maximal tori (see e.g. [51]), that the representations α_0, α are equivalent, i.e. there is an element x of $SU(2m)$ such that $\alpha = \mathrm{Ad}_x \circ \alpha_0$. Restricting to $D^2 \subset SU(2)$ we infer that $f = \mathrm{Ad}_x \circ f_0$, whence it follows (via the proof of Lemma 25.9 and the ensuing definition of i') that $f \in i'U(m)$. This completes the proof of the lemma, and with it the proof of Theorem 25.8. □

In connexion with the above proof, observe that the points of the set W (the absolute-minimum points for both functionals A and D) turned out to be "totally geodesic" (i.e. the surfaces they defined in $SU(2m)$ turned out to be totally geodesic in $SU(2m)$). This is somewhat analogous to the one-dimensional situation since, as we saw in §21, a (piecewise-smooth) minimal path $[0, 1] \to SU(2m)$ is automatically a geodesic in $SU(2m)$. However, in the present 2-dimensional context, minimality (as opposed to absolute minimality) of a map $D^2 \to SU(2m)$ does not in general entail that the resulting surface will be totally geodesic in $SU(2m)$. In fact, it can be shown that the only totally geodesic discs in $SU(2m)$ with boundary S_0^1 are those in $\hat{W}$, or, more precisely, if $f \in \Pi_2$ is a critical point of the functional D and if $f(D^2)$ is totally geodesic in $SU(2m)$, then $f \in W$.

25.3. Orthogonal Periodicity via the Higher-Dimensional Calculus of Variations

There is a periodic isomorphism also for the orthogonal group. This result, the "theorem on orthogonal periodicity", is also due to Bott.

25.13. Theorem (On Orthogonal Periodicity). *For the stable orthogonal group $O = \lim_{n\to\infty} O(n)$, where $O(n)$ is embedded in the standard way in $O(n+1)$ for $n = 1, 2, \ldots$, there are isomorphisms*

$$\pi_i(O) \simeq \pi_{i+8}(O), \qquad i = 0, 1, 2, \ldots \tag{26}$$

It follows that each stable homotopy group $\pi_i = \pi_i(0)$ is isomorphic to the appropriate one of the following groups:

$$\pi_0 \simeq \mathbb{Z}_2,\ \pi_1 \simeq \mathbb{Z}_2,\ \pi_2 = 0,\ \pi_3 \simeq \mathbb{Z},\ \pi_4 = \pi_5 = \pi_6 = 0,\ \pi_7 \simeq \mathbb{Z},\ \pi_i \simeq \pi_{i+8}. \tag{27}$$

We shall in the present subsection sketch the proof of (26) only. (Some of the isomorphisms in (27) may however be gleaned from Part II, §24.4.) The standard proof of the isomorphism of orthogonal periodicity utilizes one-dimensional Morse theory and breaks up into eight steps (analogously to the way in which the standard proof of the isomorphism of unitary periodicity expounded above splits up naturally into two steps). However, as we shall indicate below, by using the 8-dimensional calculus of variations as applied to the Dirichlet functional on a certain space of 8-dimensional balls (rather than the 2-dimensional balls, i.e. discs, that it was natural to consider in the context of unitary periodicity) the isomorphism of orthogonal periodicity $\pi_i(0) \simeq \pi_{i+8}(0)$ may be achieved (with some sacrifice of rigour) in a single step.

Proceeding much as in the unitary case, we consider the Euclidean space $\mathbb{R}^{p^2}$ of real $p \times p$ matrices with the usual Euclidan scalar product, which in terms of matrices is given by the bilinear form $\varphi(A, B) = \mathrm{tr}(AB^{\mathrm{T}})$. The group $SO(p)$ is then a smooth Riemannian submanifold of the standard sphere $S^{p^2-1} \subset \mathbb{R}^{p^2}$ of radius $\sqrt{p}$, centre the origin, with the (two-sided invariant) Killing metric induced on it from the Euclidean metric on $\mathbb{R}^{p^2}$ (cf. Part II, Corollary 6.4.4). The Lie algebra $so(p)$ of the group $SO(p)$ is also embedded in $\mathbb{R}^{p^2}$, as the subspace of matrices X satisfying $X^{\mathrm{T}} = -X$. We denote the intersection $so(p) \cap SO(p)$ by $\Omega_1(p)$. (Note incidentally that if p is even this intersection can be identified with the compact symmetric space $O(p)/U(p/2)$ (see Part II, §6, for the definition of symmetric space).) Clearly $\Omega_1(p)$ consists precisely of those elements g of $SO(p)$ satisfying the equation $g^2 = -I$, i.e. is in one-to-one correspondence with the set of "complex structures" on $\mathbb{R}^p$.

We now take p in the form $p = 16r$; it can be shown (see [44]) that in $\Omega_1(16r)$ there exist eight anti-commuting complex structures, i.e. there are eight matrices $J_1, \ldots, J_8$ in $\Omega_1(16r)$ satisfying

$$J_i J_k = -J_k J_i, \quad i \neq k, \qquad J_j^2 = -I.$$

It follows that the J_i, as position vectors of points in $SO(16r)$, are pairwise orthogonal (with respect to the Euclidean metric on $\mathbb{R}^{(16r)^2}$), and moreover together with the identity matrix I form an orthonormal set. Hence the sphere

$$S_0^8 = \{x \in SO(16r) \mid x = a^0 I + a^1 J_1 + \cdots + a^8 J_8,\ (a^0)^2 + \cdots + (a^8)^2 = 1\}$$

is a cross-section of the sphere $S^q \subset \mathbb{R}^{(16r)^2}$ of dimension $q = (16r)^2 - 1$, by a 9-dimensional plane through the origin of $\mathbb{R}^{(16r)^2}$. Consequently, S_0^8 is a totally geodesic submanifold of S^q and therefore certainly of $SO(16r) \subset S^q$. Clearly

$$S_0^8 \cap so(16r) = S_0^8 \cap \Omega_1(16r) = \bar{S}_0^7,$$

where $\bar{S}_0^7$ is the 7-dimensional "great sphere" or "equator" of S_0^8, defined by

$a^0 = 0$, again totally geodesic (in S^q and therefore $SO(16r)$). Consider, on the other hand, the "equator"

$$S_0^7 = \{x \in S_0^8 \mid x = a^0 I + a^1 J_1 + \cdots + a^7 J_7\}$$

defined by $a^8 = 0$, the boundary of the totally geodesic 8-dimensional ball $D_0^8 \subset S_0^8$, the "upper hemisphere" of S_0^8, given by

$$D_0^8 = \{x \in S_0^8 \mid a^8 \geq 0\}.$$

Let D^8 denote the standard 8-dimensional ball (in standard position in the Euclidean space $\mathbb{R}^8$), let $S^7 = \partial D^8$, let $\hat{i}\colon D^8 \to D_1^8$ be the standard bijection from the ball D^8 to the upper hemisphere of the sphere S^8 (in standard position in $\mathbb{R}^9$), and let i'' be the standard isometric bijection identifying $\hat{i}(D^8) \subset S^8$ with $D_0^8 \subset S_0^8 \subset SO(16r)$ and coinciding on $\hat{i}(S^7)$ with a standard fixed isometry $j_0\colon \hat{i}(S^7) \to S_0^7$. Write

$$f_0 = i'' \circ \hat{i}\colon D^8 \to SO(16r).$$

Denote by Π_8^* (in notation reminiscent of that used in the unitary case) the space of all continuous maps $f\colon D^8 \to SO(16r)$ satisfying $f|_{S^7} = j_0 \circ \hat{i}$, and by $\Pi_8 \subset \Pi_8^*$ the subspace of all such maps $f\colon D^8 \to SO(16r)$ in the class $H_1^8(D^8)$ (see the preceding subsection). On the function space Π_8 we consider the (parameter-independent) 8-dimensional-volume functional $A[f]$ (the analogue of the area functional utilized in the unitary case) defined by

$$A[f] = \int_{D^8} \sqrt{\det(g_{ij})}\, dV,$$

(see Part I, §18.2) where (g_{ij}) is the metric on $SO(16r) \subset \mathbb{R}^{(16r)^2}$, and the Dirichlet functional (see (19))

$$D[f] = \int_{D^8} \left[\frac{1}{8}(y_k^i, y_l^j)\right]^4 dV = \int_{D^8} \left[\frac{1}{8}\sum_{k=1}^{8} g_{ij} y_k^i y_l^j\right]^4 dV,$$

which, as before, depends on the co-ordinatization of D^8. It is not difficult to show that for all $f \in \Pi_8$ we have $A[f] \leq D[f]$. Finally, let

$$\beta_8\colon \pi_j(\Pi_8^*) \simeq \pi_{j+8}(SO(16r)) \tag{28}$$

be the standard isomorphism constructed analogously to β_2 (see (23) *et seqq.*).

25.14. Theorem (Fomenko). *Let $SO(16r)$ be embedded in the Euclidean space $\mathbb{R}^{(16r)^2}$ as above, and let the spaces Π_8^* and Π_8 of maps of the 8-dimensional ball D^8 into $SO(16r)$ be defined also as above. Denote by W the subspace of Π_8 consisting of all those points (i.e. maps) $f \in \Pi_8$ at which the Dirichlet functional $D[f]$ attains its absolute minimum value on Π_8. The following statements hold:*

(i) *the subspace W is homeomorphic to the orthogonal group $O(r)$;*
(ii) *the inclusion $i\colon W \to \Pi_8 \to \Pi_8^*$ induces isomorphisms*

$$i'_*\colon \pi_j(O(r)) \simeq \pi_j(\Pi_8^*) \qquad \text{for } j \leq r-2,$$

(whence it follows that the $(r-2)$-dimensional skeletons of the spaces $O(r)$

and Π_8^*, *or rather of some realizations of these spaces as cell complexes, are homotopically equivalent*).

The composite of the isomorphism i'_* *with the isomorphism* β_8 *of* (28) *coincides with the* (*standard*) "*isomorphism of orthogonal periodicity*":

$$\beta_8 \circ i'_*: \pi_j(O(r)) \simeq \pi_{j+8}(SO(16r)), \qquad j \leq r-2.$$

We shall now indicate the proof of this theorem. Observe first that, just as the triviality of $\pi_2(U(2m))$ entails the connectedness of the space Π_2^* of continuous maps $D^2 \to U(2m)$ with prescribed restriction to ∂D^2, so the fact that $\pi_8(SO(16r)) \simeq \mathbb{Z}_2$ (a consequence of the theorem) implies that Π_8^* must have precisely two connected components. It will appear below that the space W is also made up of two connected components, one contained in each of the connected components of Π_8^*; the connected components of Π_8^* are in fact contractible (in the limit as $r \to \infty$) onto the respective components of W which they contain.

The present proof (like the more standard ones—see, e.g. [44]) involves the subset $\Omega_8 \subset SO(16r)$ of all complex structures (i.e. solutions of $J^2 = -I$) which anti-commute with the fixed complex structures $J_1, \ldots, J_7$ whose existence was noted above, and hence with every point of the 6-dimensional standard sphere $S_0^6 \subset S_0^7$ defined in S_0^7 by the equation $a^0 = 0$ (see above). It is clear that $J_8 \in \Omega_8$, that Ω_8 is contained in the vector subspace of $\mathbb{R}^{(16r)^2}$ orthogonal to the subspace spanned by the vectors $E, J_1, \ldots, J_8$, that $S_0^8 \cap \Omega_8 = \{\pm J_8\}$, and, consequently, that $D_0^8 \cap \Omega_8$ is the one-point space $\{J_8\}$. By means of a largely computational algebraic argument it can be shown that the space Ω_8 has two connected components, and in fact is diffeomorphic to $O(r)$ (see [44]).

With each point $x \in \Omega_8$ we associate the totally geodesic 8-sphere $S^8(x)$, having the standard 7-sphere S_0^7 as equator, given by

$$S^8(x) = \{a^0 I + a^1 J_1 + \cdots + a^7 J_7 + a^8 x \mid (a^0)^2 + \cdots + (a^8)^2 = 1\}.$$

Since the vectors $I, J_1, \ldots, J_7, x$ form an orthonormal subsystem in $\mathbb{R}^{(16r)^2}$, it follows that $S^8(x)$ is the cross-section of $S^q \subset \mathbb{R}^{(16r)^2}$ by a 9-dimensional plane through the origin, and is therefore certainly a totally geodesic submanifold of $SO(16r) \subset \mathbb{R}^{(16r)^2}$. For each $x \in \Omega_8$ we then denote by $D^8(x)$ the upper hemisphere of $S^8(x)$:

$$D^8(x) = \{a \in S^8(x) \mid a = a^0 I + \cdots + a^7 J_7 + a^8 x, a^8 \geq 0\}. \tag{29}$$

Having in this way associated with each point $x \in \Omega_8$ a totally geodesic 8-dimensional disc (i.e. ball) $D^8(x)$ such that $\partial D^8(x) = S_0^7$ and

$$D^8(x_1) \cap D^8(x_2) = S_0^7 \qquad \text{if} \quad x_1 \neq x_2,$$

we are now in a position to construct a canonical embedding $i': O(r) \to \Pi_8^*$ (the analogue of the embedding $i': U(m) \to \Pi_2$ of the preceding subsection). For each $x \in \Omega_8$ let $\omega(x): D^8 \to D^8(x)$ be the unique isometry whose restriction to the boundary is the standard map $j_0 \circ \hat{i}: S^7 \to S_0^7$ defined above $(= f_0|_{\partial D^8})$;

this then determines an embedding

$$\omega\colon \Omega_8 \to \Pi_8 \subset \Pi_8^*. \tag{30}$$

Denoting by $\varphi\colon O(r) \to \Omega_8$ the above-mentioned diffeomorphism, we now define i' by

$$i' = \omega \circ \varphi\colon O(r) \to \Pi_8 \subset \Pi_8^*. \tag{31}$$

25.15. Lemma. *The embedding $i'\colon O(r) \to \Pi_8^*$ induces isomorphisms*

$$\beta_8 \circ i'_*\colon \pi_j(O(r)) \simeq \pi_{j+8}SO(16r) \qquad \text{for} \quad j \leq r-2,$$

coinciding with the usual "isomorphisms of orthogonal periodicity".

Proof (sketch only). Let $f\colon S^j \to O(r)$ be a map representing an arbitrary element of the jth homotopy group of $O(r)$: $[f] \in \pi_j(O(r))$, and consider the family of maps $\omega(x)\colon D^8 \to D^8(x)$ (see (30)) indexed by the elements x of $\varphi(f(S^j)) \subset \Omega_8$ where $\varphi\colon O(r) \cong \Omega_8$ is the above-mentioned diffeomorphism. Since each map $\omega(x)$ restricts to the standard map $S^7 \to S^7_0$ on the boundary $\partial D^8 = S^7$, and the $D^8(x)$ intersect pairwise precisely in S^7_0, the union of these maps $\omega(x)$ yields a map

$$F\colon S^{j+8} \to \tilde{S} = \bigcup_{x \in \varphi(f(S^j))} D^8(x) \subset SO(16r), \tag{32}$$

with the property that $F|_{S^j} = \varphi \circ f$, where S^j is canonically situated (equatorially) in S^{j+8}. We claim that the correspondence $f \mapsto F$ induces an isomorphim

$$\pi_j(O(r)) \simeq \pi_{j+8}(SO(16r)), \qquad j \leq r-2,$$

coinciding with the isomorphism of orthogonal periodicity as usually constructed. We now summarize very briefly the usual construction (see, e.g. [44]).

For each $k = 0, 1, \ldots, 8$, denote by Ω_k the set of all complex structures $J \in SO(16r)$ which anti-commute with the fixed complex structures $J_1, \ldots, J_{k-1}$. (The particular set Ω_8 has already figured in our discussion.) Starting at $k = 7$ (with a view to working our way down to $k = 0$), we define a map

$$\gamma_7\colon \Omega_8 \to \Omega(SO(16r); J_7, -J_7)$$

by setting $\gamma_7(x) = D^1(x)$, the minimal geodesic in $SO(16r)$ with midpoint x, lying in Ω_7. (It can be shown that $D^1(x)$ is uniquely defined by these conditions.) By taking the union of those $D^1(x)$ for which $x \in \varphi(f(S^j))$, we obtain a map $F_7\colon S^{j+1} \to \Omega_7$ extending $\varphi \circ f$:

$$F_7\colon S^{j+1} \to \bigcup_{x \in \varphi(f(S^j))} D^1(x) \subset \Omega_7, \qquad F_7|_{S^j} = \varphi \circ f.$$

It follows via one-dimensional Morse theory (this and the similar steps succeeding it contain the burden of the usual proof), that the correspondence $f \mapsto F_7$ induces an isomorphism

$$\pi_j(O(r)) \simeq \pi_{j+1}(\Omega_7).$$

Continuing in the same vein, one next defines a map $\gamma_6: y \mapsto D^1(y)$, from Ω_7 to the space of minimal geodesics in $SO(16r)$ joining J_6 to $-J_6$ and lying in Ω_6, whence one obtains as before a map

$$F_6: S^{j+2} \to \bigcup_{x \in F_7(S^{j+1})} D^1(y) \subset \Omega_6, \qquad F_6|_{S^{j+1}} = F_7,$$

with the property (established using Morse theory) that the correspondence $F_7 \mapsto F_6$ induces an isomorphism

$$\pi_{j+1}(\Omega_7) \simeq \pi_{j+2}(\Omega_6).$$

Iterating this procedure, one defines further maps $\gamma_5, \ldots, \gamma_0$ (taking $J_0 = I$), and $F_5, \ldots, F_0$, where in particular F_0 maps S^{j+8} to $\Omega_0 = SO(16r)$. It can now be verified (we omit the details) that

$$\bar{S} = \bigcup_{x \in \varphi(f(S^j))} [\gamma_0 \circ \gamma_1 \circ \cdots \circ \gamma_7(x)]$$

(where $\bar{S}$ is as in (32)), and that F_0 essentially coincides with the map F of (32). Since the composite correspondence

$$f \mapsto F_7 \mapsto \cdots \mapsto F_0$$

induces an isomorphism

$$\pi_j O(r) \simeq \pi_{j+8} SO(16r), \qquad j \leq r-2,$$

(the isomorphism of orthogonal periodicity as usually constructed), it follows that the correspondence $f \mapsto F$ induces the same isomorphism. Since $i' = \omega \circ \varphi$ the lemma now follows from the definition of the map F in terms of ω. □

To complete the proof of Theorem 25.14 it remains to show that $i'(O(r)) = W$. As in the unitary case we verify this in two steps.

25.16. Lemma. *The map $i': O(r) \to \Pi_8^*$ given by* (31) *above, satisfies $i'(O(r)) \subset W$, where W consists of all those functionals in Π_8 at which the Dirichlet functional D has its least value on Π_8.*

This is established by an argument analogous to that used in proving Lemma 25.11, exploiting the inequality $A[f] \leq D[f]$ together with the fact that for each $g \in O(r)$ the disc $i'(g) = D^8(\varphi(g))$ is a hemisphere of the cross-section of the sphere $S^q \subset \mathbb{R}^{(16r)^2}$ by a plane through the origin.

25.17. Lemma. *We have $i'(O(r)) = W$.*

Proof. Initially the argument proceeds as in the proof of the analogous Lemma 25.12. Let $f \in W$, i.e. suppose $D[f]$ is least among all values taken by D on Π_8, and let $f_0: D^8 \to D_0^8 \subset SO(16r)$ be as before. In terms of standard co-ordinates on D^8 it is easy to see that $A[f_0] = D[f_0]$. Furthermore, in view of the boundary conditions on admissible maps $D^8 \to SO(16r)$ (namely that $\partial D^8 = S^7$ be mapped canonically onto $S_0^7 \subset SO(16r)$), the value $A[f_0]$ is the least taken by A on Π_2. Hence we can infer, via the general inequality $A[h] \le D[h]$, that

$$D[f] = A[f] = A[f_0].$$

It follows that $f(D^8)$, like $f_0(D^8)$, must be a hemisphere of a central-plane cross-section of $S^q \subset \mathbb{R}^{(16r)^2}$ (of course having S_0^7 as its boundary: $\partial(f_0(D^8)) = S_0^7$).

Now let x be a vector in $f(D^8)$ orthogonal to the eight vectors $I, J_1, \ldots, J_7$. It follows from the above description of $f(D^8)$ that

$$f(D^8) = \{a^0 I + a^1 J_1 + \cdots + a^7 J_7 + a^8 x \mid (a^0)^2 + \cdots + (a^8)^2 = 1, a^8 \ge 0\}. \tag{33}$$

Since $f(D^8)$ is a central-plane cross-section of S^q with the points $\pm I$ as antipodal points of its boundary S_0^7, there is a unique minimal geodesic arc $\gamma(t)$, $0 \le t \le 1$, of $SO(16r)$ (actually of S^q), such that $\gamma(0) = I$, $\gamma(\frac{1}{2}) = x$, $\gamma(1) = -I$ (namely half the circle obtained by sectioning S^q by the 2-dimensional plane spanned by the vectors I and x). Now any geodesic γ in $SO(16r)$ with $\gamma(0) = I$, has the form $\gamma(t) = \exp(\pi t A)$ for some $A \in so(16r)$. By means of a fairly direct matrix computation (see [44]) it can be shown that if $\gamma(1) = -I$ and γ is minimal, then A is a complex structure: $A^2 = -I$. It follows that

$$\gamma(t) = (\cos \pi t) I + (\sin \pi t) A,$$

whence $\gamma(\frac{1}{2})$ $(= A)$ is a complex structure. Since in our situation $\gamma(\frac{1}{2}) = x$, we infer that x is a complex structure, i.e. $x \in \Omega_1(16r)$. By (33)

$$\frac{1}{\sqrt{2}}(x + J_k) \in f(D^8) \subset SO(16r) \qquad \text{for} \quad k = 1, \ldots, 7,$$

whence

$$\frac{1}{\sqrt{2}}(x + J_k) \in SO(16r) \cap so(16r) = \Omega_1(16r), \qquad k = 1, \ldots, 7,$$

i.e. $\frac{1}{2}(x + J_k)^2 = -I$. Hence $xJ_k + J_k x = 0$, i.e. $x \in \Omega_8$. We conclude that $f(D^8) = D^8(x)$ (see (29)), so that $f \in i'(O(r))$, as claimed. This completes the proof of the lemma and with it our sketch of the proof of Theorem 25.14. □

From the proof of orthogonal periodicity given for instance in [44], it emerges that the stable orthogonal group O has the homotopy type of the 4-fold loop space $\Omega\Omega\Omega\Omega\, Sp$, and that the stable symplectic group Sp has the homotopy type of the 4-fold loop space $\Omega\Omega\Omega\Omega\, O$, whence one obtains an analogous "symplectic periodicity" theorem. We leave to the reader the detailed formulation of this theorem incorporating an approach to its validation via the higher-dimensional calculus of variations, analogous to that described above in the unitary and orthogonal cases.

EXERCISE

Derive the following homotopy equivalences:

(i) $BSp \sim \Omega\Omega\Omega SO$;

(ii) $BO \sim \Omega\Omega\Omega Sp$ (see §10.4).

Find using these the first eight homotopy groups $\pi_i O$, $i = 0, \ldots, 7$, namely, in order, $Z_2, Z_2, 0, Z, 0, 0, 0, Z$.

In the preceding subsection, in the course of establishing unitary periodicity we showed that the space $i'(U(m)) \subset \Pi_2$ is actually the orbit of the particular point $f_0 \in \Pi_2$ under the conjugating action of the group $G = \{I \oplus C \mid C \in U(m)\} \subset U(2m)$. In the present situation of orthogonal periodicity a version of this result holds also (not however needed in the above proof of orthogonal periodicity).

25.18. Proposition. *The two connected components of the space $W = i'(O(r)) \subset \Pi_8$ are orbits of the conjugating action on Π_8 of the group*

$$G = J_8\Omega_8 \subset SO(16r) \qquad (G \simeq O(r)). \tag{34}$$

PROOF. We need to show that given any one of the totally geodesic balls $D^8(x)$ defined in (29), where $x \in \Omega_8$ is in the same connected component of Ω_8 as J_8, there is an element $g \in G$ such that $gxg^{-1} = J_8$, since then

$$gD^8(x)g^{-1} = D^8(J_8) = D_0^8,$$

so that the corresponding map $\omega(x)\colon D^8 \to D^8(x)$ (see (30)) is obtained from the map $f_0\colon D^8 \to D_0^8 \subset SO(16r)$, by conjugating by g. Since each $g \in G$ has the form $g = J_8 y$, $y \in \Omega_8$, the equation $gxg^{-1} = J_8$ is equivalent to $yxy^{-1} = J_8$, or $yxy = -J_8$ (since $y^{-1} = -y$).

Now in [44] the aforementioned diffeomorphism $\Omega_8 \cong O(r)$ is constructed by finding two r-dimensional subspaces X_1, X_2 of $\mathbb{R}^{16r}$ with the property that the matrices of the form $J_7 y$, $y \in \Omega_8$, act as (distinct) orthogonal transformations $X_1 \to X_2$, and in fact account for all such transformations. It follows readily that G, as defined in (34), is indeed a group, since it is identifiable with the group of all orthogonal self-transformations of X_1 (or X_2). The equation $yxy = -J_8$ for which we seek a solution $y \in \Omega_8$, is equivalent to

$$(J_7 y)(J_7 x^{-1})^{-1} J_7 y = -J_7 J_8,$$

whence we see that the existence of such a solution is equivalent to the existence of a solution $z \in O(r)$ of the equation

$$zAz = B$$

for each (fixed) pair A, B of elements in the same connected component of $O(r)$. Since every such equation does indeed have a solution (we lay the onus of verification of this on the reader), the proposition follows. □

In summary, the idea underlying the above approach to unitary and orthogonal periodicity (and borne out by the above discussion), is that both these results should arise via the same mechanism (namely the higher-dimensional calculus of variations applied to the Dirichlet functional), the particular outcome depending only on the space on which that functional is considered—the space of maps of 2-dimensional discs in the unitary case, and of 8-dimensional discs in the orthogonal case.

The reader will have observed that, this uniform approach notwithstanding, our proofs of both unitary and orthogonal periodicity ultimately relied on the more standard arguments of one-dimensional Morse theory. It would be of considerable interest if a direct uniform proof (using the Dirichlet functional) could be found which avoided any appeal to the Morse theory of the one-dimensional functionals of action and length. Such a proof would in fact be feasible if one could establish directly the contractibility of the 2m-dimensional skeleton of Π_2 (or the $(r-2)$-skeleton of Π_8 in the orthogonal case) onto the space $i'(U(m))$ (resp. $i'(O(r))$) of points where the Dirichlet functional has its least value. It is precisely a contractibility result of this kind in the classical (i.e. one-dimensional) Morse theory, of the path space $\Pi_1 = \Omega(SU(2m); I_{2m}, -I_{2m})$ which allows one to obtain the crucial isomorphism $\pi_{j-1}(G^{\mathbb{C}}_{2m,m}) \simeq \pi_{j-1}(\Pi_1)$. However, the analogous result for the higher-dimensional calculus of variations applied to the Dirichlet functional is not available, a lack essentially due to the typical sort of difficulty which arises in higher-dimensional problems of the "Plateau" type, whereby the higher-dimensional functional in question may be degenerate on certain sets of positive measure contained in the extremal submanifolds.

§26. Morse Theory and Certain Motions in the Planar n-Body Problem

In this section we shall show how Morse theory may be applied to the analysis of certain motions in the many-body problem of celestial mechanics, i.e. the problem of describing the motions of n objects acting on each other by means of mutual (gravitational) forces. It is well known that to a first approximation the planets of our solar system move in a plane, the so-called "plane of the ecliptic". Furthermore, to a large degree of accuracy the centre of mass of the whole system may be identified with the sun's position, and the motion of the system may be assumed to be governed by the Newtonian gravitational potential of classical mechanics. The motion is then determined in the usual way by initial conditions, namely the positions and velocities of the gravitating masses (considered as point-particles) at some chosen initial instant of time. As is also well known, the general solution of the resulting system of differential equations is exceedingly complex: for instance, by the classical results of Bruns and Poincaré, there are no single-valued analytic integrals of the system (i.e. expressions in the position variables and their derivatives which are constant at all times) other than the "classical" ones (the integrals of energy, angular momentum, and of motion of the centre of mass). (However, in the case $n = 2$ a complete solution can be given; see Part I, §32.2.)

It has nevertheless proved possible to delineate among all solutions certain natural subclasses of solutions admitting a relatively simple description. One such subclass is that of all so-called "rigid-body" solutions, i.e. those particular solutions obtained under the additional assumption that all the masses of the system revolve together about the sun, at the same angular speed, in the plane of the ecliptic. Thus, in this special case, the relative positions of the masses remain unchanged and the whole system revolves like a (planar) rigid body about its centre of mass. In the literature such periodic solutions are sometimes referred to as "circular orbits" (with all particles having the same angular velolcity). It is a remarkable fact that the description of such planar "rigid-body" solutions of the many-body problem, reduces to the description of the critical points of a certain function, possibly Morse, on a smooth manifold, and that the resulting topological information about the manifold which we have seen (in for instance §15) to be obtainable from knowledge of the critical points, can then be used to draw important qualitative inferences about the geometric structure of the "circular" solutions under investigation.

For instance, the following question is of considerable interest: Given a system of point-masses, what planar configurations of the system are compatible with some "rigid-body" solution? (It is intuitively clear that far from every arrangement of the particles will quality for some planar "rigid-body" solution.) Clearly the admissible configurations will be determined by the masses of the n particles of the system. (In the particular case where all but one of the particles have the same mass, it turns out that these configurations are linked by the action of a certain discrete group of symmetries.) Such configurations are sometimes called "relative equilibria" of the system.

We now turn to the precise formulation of the problem. The planar n-body problem of celestial mechanics is determined essentially by a sequence of n real, positive numbers $m_1, \dots, m_n$, representing the masses of the n bodies, or rather point-particles, situated at n points of the (2-dimensional) Euclidean plane. We may clearly suppose that the origin O coincides at all times with the centre of mass of the system of particles. The position of the jth particle on the plane will then be denoted, in terms of the standard Euclidean co-ordinates, by (x_j, y_j), or by the complex co-ordinate $z_j = x_j + iy_j$. From the assumption that the origin is the centre of mass we infer immediately the relation $\sum_{j=1}^{n} m_j z_j = 0$. The "configuration space" of the system is a certain subspace (see below) contained in the linear subspace $M = M^{2n-2}$ (or complex hyperplane) of the Euclidean space $\mathbb{R}^{2n} = \mathbb{C}^n$, given by

$$M^{2n-2} = \left\{ (z_1, \dots, z_n) \in \mathbb{R}^{2n} \,\middle|\, \sum_{j=1}^{n} m_j z_j = 0 \right\}, \tag{1}$$

and the "phase space" of the system is (a certain submanifold of) the cotangent bundle $T^*M \cong M \times M$ (the direct square of M). (Note that since the co-ordinates are Euclidean, T and T^* may be identified; see Part I, §19.1.)