Graduate Texts in Mathematics

B.A. Dubrovin A.T. Fomenko S.P. Novikov

Modern Geometry-Methods and Applications

Part III. Introduction to Homology Theory



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Modern Geometry— Methods and Applications

Part III. Introduction to Homology Theory

Translated by Robert G. Burns

With 120 Illustrations



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> † Evidently Poincaré, a

> Riemann, a

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In expositions of the elements of topology it is customary for homology to be given a fundamental role. Since Poincaré, who laid the foundations of topology, homology theory has been regarded as the appropriate primary basis for an introduction to the methods of algebraic topology. From homotopy theory, on the other hand, only the fundamental group and covering-space theory have traditionally been included among the basic initial concepts. Essentially all elementary classical textbooks of topology (the best of which is, in the opinion of the present authors, Seifert and Threlfall's A Textbook of Topology) begin with the homology theory of one or another class of complexes. Only at a later stage (and then still from a homological point of view) do fibre-space theory and the general problem of classifying homotopy classes of maps (homotopy theory) come in for consideration. However, methods developed in investigating the topology of differentiable manifolds, and intensively elaborated from the 1930s onwards (by Whitney and others), now permit a wholesale reorganization of the standard exposition of the fundamentals of modern topology. In this new approach, which resembles more that of classical analysis, these fundamentals turn out to consist primarily of the elementary theory of smooth manifolds,† homotopy theory based on these, and smooth fibre spaces. Furthermore, over the decade of the 1970s it became clear that exactly this complex of topological ideas and methods were proving to be fundamentally applicable in various areas of modern physics. It was for these reasons that the present authors regarded as absolutely

Lividently the beginning ideas of topology, which can be traced back to Gauss, Riemann and Poincaré, actually arose, historically speaking, in this order. However, at the time of Gauss and Riemann, a correspondingly organized conceptual basis for a theory of topology was unrealizable. It was Poincaré who, in creating the homology theory of simplicial complexes, was able to provide

a quite different, precise foundation for algebraic topology.

essential material for a training in topology, in the first place precisely the theory of smooth manifolds, homotopy theory, and fibre spaces, and incoes porated this subject matter in Part II of their textbook Modern Geometry. It is assumed in the present text that the reader is acquainted with that material.

On the other hand, the solution of the more complex problems arising both within topology itself (the computation of homotopy groups, the classification of smooth manifolds, etc.) and in the numerous applications of the algebratopological machinery to algebraic geometry and complex analysis, requires a very extensive elaboration of the methods of homology theory. There is in the contemporary topological literature a complete lack of books from which one might assimilate the complex of methods of homology theory useful in applications within topology. It is part of the aim of the present book to remedy this deficiency.

In expounding homology theory we have, wherever possible, striven to avoid using the abstract terminology of homological algebra, in order that the reader continually remain cognizant of the fact that cycles and boundaries, and homologies between them, are after all concrete geometrical objects. In a few places, for instance in the section devoted to spectral sequences, this self-imposed restriction has inevitably led to certain defects of exposition. However, it is our experience that the usual expositions of the machinery of modern homological algebra lead to worse defects in the reader's understanding, essentially because the geometric significance of the material is lost from view. Certain fundamental methods of modern algebraic topology (notably those associated with spectral sequences and cohomology operations) are described without full justification, since this would have required a substantial increase in the volume of material. It must be remembered that those methods are based exclusively on the formal algebraic properties of the algebraic entities with which they are concerned, and in no way involve their explicit geometric prototypes whence they derive their raison d'être. In the final chapter of the book the methods of algebraic topology are applied to the investigation of deep properties of characteristic classes and smooth structures on manifilds. It is the intention of the authors that the present monograph provide a path for the reader giving access to the contemporary topological literature.

A large contribution to the final version of this book was made by the editor, Victor Matveevich Bukhshtaber. Under his guidance several sections were rewritten, and many of the proofs improved upon. We thank him for carrying out this very considerable task.

Translator's acknowledgements. Thanks are due to G. C. Burns and Abe Shenitzer for much encouragement, to several of my colleagues (especially Stan Kochman) for technical help, and to Eadie Henry for her advice, superb typing, and forbearance.

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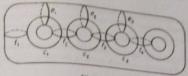


Figure 117

connected 3-manifolds (with repetitions) can in effect be obtained by enumerating, connected small state 0, all isotopy classes of self-diffeomorphisms of the surface of genus for each $r \ge 0$, all isotopy classes of self-diffeomorphisms of the surface of genus for each 2 of the surface of genus for any order) all possible finite products of the $\prod_i \, T_{s_i}^{\lambda_j}, \qquad s_j \in \{c_i, \, e_i, \, f_i\}.$

825. Unitary Bott Periodicity and Higher-Dimensional Variational Problems

In this section we shall prove the important topological result usually referred to as "Bott periodicity". For the sake of simplicity we shall, for the most part, concentrate our attention on the unitary groups: although "orthogonal Bott periodicity" is established along the same general lines as "unitary Bott periodicity", there are certain rather substantial technical difficulties to overcome in the former case. (We shall nonetheless outline a proof of "orthogonal periodicity" in the final subsection (§25.3).)

25.1. The Theorem on Unitary Periodicity

We shall prove this theorem in its "classical" form, namely as a result on the periodicity of the homotopy groups of the unitary groups, without considering its role as a "periodic" theorem for vector bundles.

25.1. Theorem (On Unitary Periodicity). The following isomorphisms hold between the homotopy groups of the special unitary groups:

$$\pi_{i-1}(SU(2m)) \simeq \pi_{i+1}(SU(2m))$$
 for $1 \le i \le 2m$.

It follows that for the "stable" unitary group $U = \lim_{m \to \infty} U(m)$, the direct limit of the unitary groups U(m) with respect to the standard embeddings $U(m) \subseteq U(m)$ U(m+1), we have

$$\pi_{i-1}(U) \simeq \pi_{i-1}(U) \quad \text{for} \quad i \ge 1,$$

whence

$$\pi_{2n}(U) = 0, \qquad \pi_{2n+1}(U) \simeq \mathbb{Z} \qquad \text{for} \quad n \geq 0.$$

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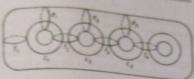


Figure 117

3. manifolds (with repetitions) can in effect be obtained by enumerating grach? 20, all isocopy classes of self-diffeomorphisms of the surface of genus grach? by enumerating (in any order) all nosels, as

$$\prod_j \, T_{s_j}^{s_j}, \qquad s_j \in \{c_i,\, e_i,\, f_i\}.$$

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$$\pi_{2n}(U) = 0, \qquad \pi_{2n+1}(U) \simeq \mathbb{Z} \qquad \text{for} \quad n \geq 0.$$

Consider the special unitary group SU(2m) of even degree, regarded as a Lie group, and denote up

$$\Omega = \Omega(SU(2m); I_{2m}, -I_{2m}),$$

where $I_{2m} \in SU(2m)$ is the identity linear transformation (identity matrix), the where $I_{2m} \in SU(2m)$ is the varieties space SU(2m) from the point function space of piecewise-smooth paths in the space SU(2m) from the point I_{2m} to $-I_{2m}$, and by

 $\Omega^* = \Omega^*(SU(2m); I_{2m}, -I_{2m})$

the full space of all continuous paths in SU(2m) from I_{2m} to $-I_{2m}$ (Recall the full space of an experimental than the inclusion $\Omega \to \Omega^*$ is a homotopy equivalence. We have concerned with the subspace shall be particularly concerned with the subspace

$$\bar{\Omega} = \bar{\Omega}(SU(2m); I_{2m}, -I_{2m})$$

of Ω consisting of all minimal geodesics γ (i.e. geodesics of least length among all piecewise-smooth paths) joining the points I_{2m} and $-I_{2m}$, with respect to the invariant metric on SU(2m) determined by the Killing form $\langle X, Y \rangle$ Re $\operatorname{tr}(X \bar{Y}^{\mathsf{T}})$, $X, Y \in \operatorname{su}(2m)$, on the Lie algebra $\operatorname{su}(2m)$ (see Part I, §24.4). (See [44] for general conditions under which such minimal geodesics exist.)

25.2. Lemma. The space Ω is homeomorphic to the complex Grassmannian manifold $G_{2m,m}^{\mathbb{C}}$, i.e. the manifold whose points are the m-dimensional complex planes through the origin in complex 2m-dimensional space C2m (see Part II §5.2).

PROOF. By Theorem 30.3.7 of Part I, in the Lie group SU(2m) the geodesics with respect to the above-mentioned Killing metric are precisely the oneparameter subgroups (and their translates (i.e. multiples) by elements of SU(2m)). Hence in order to characterize the geodesics in SU(2m) joining the points I_{2m} and $-I_{2m}$, it suffices to describe all one-parameter subgroups emanating from the identity I_{2m} of SU(2m) and reaching the point $-I_{2m}$. By Part I, Theorem 24.3.1 (and the definition of an invariant metric on a Lie group given in Part I, §24.4), the one-parameter subgroups $\gamma(t)$ passing through the identity I_{2m} have the form $\gamma(t) = \exp(tX)$ for some skew-Hermitian matrix X with zero trace (i.e. element of the Lie algebra su(2m); see Part I, §14.1). Since the parameter can always be chosen to vary from 0 to 1 along the geodesic arc from I_{2m} to $-I_{2m}$, we infer the conditions $\gamma(0) = I_{2m}$ (as required), and $\gamma(1) = \exp X = -I_{2m}$, from which we can ascertain X. To this end, recall the well-known result (a consequence of the classical process for bringing a matrix into its Jordan canonical form, or, if you like, of an orthogonalization process exploiting the operator Ad applied to the unitary case) to the effect that X is conjugate, by means of a matrix in SU(2m), to a diagonal matrix, i.e. there is an element $g_0 \in SU(2m)$ such that $g_0 X g_0^{-1} = X_0$ with X_0 of the form

$$X_0 = \begin{pmatrix} i\varphi_1 & 0 \\ & \ddots \\ 0 & i\varphi_{2m} \end{pmatrix}, \quad \text{where} \quad \varphi_1 + \cdots + \varphi_{2m} = 0.$$

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 $\omega^{\text{lexp}} X^{\text{lgo}^{-1}} = \exp(\text{go}^{-1})$

abence we have, for i aborting k1 + . . + k2m Having thus comple range remains to P oponential map sends solm) isometrically or latter arc is equal to th the Killing form on su

so that the length of th

Hence the least length $k_i = \pm 1$ for all i = 1corresponding matric diagonal. The upshot is SU(2m) joining th $\gamma(t) = \exp(tX)$ where

$$X_0 = \pi$$

 $LX = gX_0g^{-1}$ for We have thus es A of all minimal ge matrix X_0 (see (1) correspondence is SU(2m) is in turn (X_0) is the cen $S(U(m) \times U(m))$ (w

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positive specialists
$$g_0(\exp X)g_0^{-1} = \exp(g_0Xg_0^{-1}) = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_{2m}} \end{pmatrix} = g_0(-I_{2m})g_0^{-1} = -I_{2m}$$
whence we have, for $i = 1, \ldots, 2m$, that $\varphi_i = \pi k_i$ where i .

where we have, for $i=1,\ldots,2m$, that $\varphi_i=\pi k_i$ where the k_i are odd integers $k_i+\cdots+k_{2m}=0$.

 $\frac{k_1}{k_2}$ thus completely described the geodesics in SU(2m) from l_{2m} to pick out those of least length Since Having thus Having that to pick out those of least length. Since, in view of the I_{2m} of the metric on SU(2m) in terms of the Killing for l_{2m} . It refusals of the metric on SU(2m) in terms of the Killing form on su(2m), the definition of the metric on SU(2m) in terms of the Killing form on su(2m), the definition of the definition of the line segment $\{tX|0 \le t \le 1\}$ in the tangent space exponential map sends the line segment $\{tX|0 \le t \le 1\}$ in the tangent space exponential map exponential map is graph is graph onto the geodesic arc $\gamma(t) = \exp(tX)$, the length of this graph is graph to that of the line segment in $\sup(2\pi)$. Now, ss(2m) isometrated to that of the line segment in su(2m). Now, as noted above, the Killing form on su(2m) is given by

$$\langle A, B \rangle = \operatorname{Re} \operatorname{tr}(A\overline{B}^{\mathrm{T}}),$$

so that the length of the latter line segment is

$$\sqrt{\langle X, X \rangle} = \sqrt{\operatorname{tr} X \overline{X}^{\mathsf{T}}} = \pi \sqrt{\sum_{i=1}^{2m} (k_i)^2},$$

Hence the least length of a geodesic from I_{2m} to $-I_{2m}$ is $\pi \sqrt{2m}$, attained when $k_i = \pm 1$ for all $i = 1, \ldots, 2m$. Since $k_1 + \cdots + k_{2m} = 0$, it follows that the corresponding matrices X_0 must have the same number of i's as (-i)'s on the diagonal. The upshot of our argument is, therefore, that the minimal geodesics is SU(2m) joining the points I_{2m} and $-I_{2m}$ are precisely those of the form $y(t) = \exp(tX)$ where X is conjugate in SU(2m) to the matrix

i.e. $X = gX_0g^{-1}$ for some $g \in SU(2m)$.

We have thus established a one-to-one correspondence between the space of the Ω of all minimal geodesics from I_{2m} to $-I_{2m}$, and the conjugacy class of the Matrix X_0 (see (1)) in SU(2m), and it is intuitively clear that in fact this correspond correspondence is a homeomorphism. Since the conjugacy class of X_0 in SU(2m) is SU(2m) is in turn homeomorphic to the coset space $SU(2m)/C(X_0)$, where $C(X_0)$ is the $C(X_0)$ is in turn homeomorphic to the coset space SO(2m), $C(X_0) = SU(2m)$, and since clearly $C(X_0) = SU(2m)$. $S(U(m) \times U(m))$ (with the obvious interpretation), we finally conclude that Ω

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is homeomorphic to

$$SU(2m)/S((U(m) \times U(m)),$$

which (as indicated in Part II, §5.2) is identifiable with the Grassmannian manifold $G_{2m,m}^{C}$. This completes the proof of the lemma.

25.3. Lemma. Each minimal geodesic arc $\gamma(t)$, $0 \le t \le 1$, in SU(2m), from the 25.3. Lemma. Lemmin to I_{2m} is uniquely determined by its midpoint, i.e. by the identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point identity I_{2m} to I_{2m} is uniquely determined by its midpoint, i.e. by the point I_{2m} is uniquely determined by its midpoint, i.e. by the point I_{2m} is uniquely I_{2m} to I_{2m} . identity $I_{2m} = 10^{-1}$ $I_{2m} = 0$ of such minimal geodesics (shown in the preceding lemma $\gamma(\frac{1}{2})$, so that the space Ω of such minimal manifold G_{2m-1}^{C}) may be identified and the space Ω of such minimal geodesics (shown in the preceding lemma Ω). $\gamma(\frac{1}{2})$, so that the space to the Grassmannian manifold $G_{2m,m}^{C}$) may be identified with to be homeomorphism to be homeomorphism and the subspace $\{\gamma(\frac{1}{2})|\gamma\in\Omega\}$ of SU(2m). The latter subspace coincides with the subspace $\{\gamma(\frac{1}{2})|\gamma\in\Omega\}$ of SU(2m). The latter subspace coincides with the the subspace (122m) with its Lie algebra su(2m) (both considered as subspaces of the space ($\simeq \mathbb{R}^{8m^2}$) of all $2m \times 2m$ complex matrices).

PROOF. The first assertion follows easily from the form of the minimal geodesics y(t), established in the proof of Lemma 25.2 (see (1)):

$$\gamma(t) = \exp(tX) = (\cos \pi t)I_{2m} + (\sin \pi t)X;$$

putting t = 0, 1 we obtain $\gamma(0) = I_{2m}$, $\gamma(1) = -I_{2m}$, while at $t = \frac{1}{2}$, we have $\gamma(\frac{1}{2}) = X$, i.e. the midpoint of the geodesic arc is just X.

The remainder of the lemma then follows from the observation that those unitary $2m \times 2m$ complex matrices X which are at the same time skew. Hermitian, i.e. satisfy both $X\bar{X}^T=1$, and $X+\bar{X}^T=0$, are precisely the solutions of the matrix equation $X^2 = -I_{2m}$, and in SU(2m) these are just X_0 (as in (1)) and its conjugates. (Note incidentally the consequence of this, that the elements of Grassmannian manifold $G_{2m,m}^{\mathbb{C}}$ correspond one-to-one to the possible "complex structures" in SU(2m).)

25.4. Lemma. Every non-minimal geodesic γ in SU(2m) from I2m to -I2m has index at least 2m + 2.

PROOF. By the Index Theorem (21.7) the index of a geodesic arc γ in Ω is equal to the number of points in the interior of y (counted according to their multiplicities) conjugate to the initial point I_{2m} . Since by definition two points on a geodesic arc are conjugate along that arc if there is a non-zero Jacobi field along the arc vanishing at those points, we are led to consider Jacobi's differential equation (see §21 (8)).

By the proof of Lemma 25.2, we may assume that $\gamma(t) = \exp(tX)$, $0 \le t \le 1$, where

$$X = \begin{pmatrix} i\pi k_1 & 0 \\ & \ddots & \\ 0 & i\pi k_{2m} \end{pmatrix}, k_1 + \dots + k_{2m} = 0,$$

$$k_1 \ge k_2 \ge \dots \ge k_{2m}, \quad k_i \text{ odd.}$$

In particular $\dot{\gamma}(0) = X$. Consider the (real) linear transformation

$$K_X$$
: $su(2m) \rightarrow su(2m)$,

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pholors from the symme Spart I. Corollary 30. our transformation:

we may therefore choose

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and the Jacobi differen

$$\frac{d^2v_i}{dt^2} + \frac{1}{3}$$

Since there is an ison interchanges $\gamma(0)$ with as a whole (see §1 (19) point of y:

so that the system (4)

We are interested in $c_i(t) = c_i \sin \sqrt{\mu_i t}$ for integer multiples of $t_i(t)$ has the form c_i tero or vanishes on exp(tX) conjugate eigenvalues μ_i of the

occurring at the int With X as in (2)

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defined in terms of the curvature by

and in the $K_X(Y) = R(X, Y)X$ (= $\frac{1}{4}[[X, Y], X]$ by Part I, Corollary 30.3.61. It follows from the symmetry relation

$$\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$$

 p_{art} I, Corollary 30.3.6 or Theorem 30.2.1 (iv)) that K_x is a self-adjoint transformation: incar transformation:

$$\langle K_X(Y), W \rangle = \langle Y, K_X(W) \rangle$$

We may therefore choose an orthonormal basis e_1, \dots, e_k for $\mathfrak{su}(2m)$ such that $K_{\nu}(e_i) = \mu_i e_i$

where the μ_i are the eigenvalues of K_X . If we extend the e_i to vector fields $e_i(t)$ where the μ_i we were the μ_i by parallel transport, then an arbitrary vector field v(t) along γ can be along γ to singularly in the form expressed uniquely in the form

$$v(t) = \sum_{i=1}^{k} v_i(t)e_i(t),$$

and the Jacobi differential equation takes the form of the system (see §21 (9))

$$\frac{d^2v_i}{dt^2} + \sum_{j=1}^k \langle R(\dot{\gamma}, e_j)\dot{\gamma}, e_i \rangle v_j = 0, \qquad i = 1, \dots, k.$$
(4)

Since there is an isometry ("symmetry") of the Lie group SU(2m) which interchanges $\gamma(0)$ with any particular point $\gamma(t)$ and preserves the geodesic γ as a whole (see §1 (19) above, or Part II, §6.4), it follows that (3) holds at every point of y:

$$K_{\dot{\gamma}(t)}(e_i(t)) = \mu_i e_i(t),$$

so that the system (4) becomes

$$\frac{d^2v_i}{dt^2} + \lambda_i v_i = 0, \qquad i = 1, \dots, k.$$

We are interested in solutions of this system vanishing at t=0. If $\mu_i>0$, then $v_i(t) = c_i \sin \sqrt{\mu_i} t$ for some constant c_i , whence the zeros of $v_i(t)$ are at the integer multiples of $t = \pi/\sqrt{\mu_i}$. If $\mu_i = 0$, then $v_i(t) = c_i t$, and if $\mu_i < 0$, then $|u_i| = 1$ has the form $c_i \sinh \sqrt{|\mu_i|} t$; hence if $\mu_i \le 0$ then $v_i(t)$ is either identically 200 or vanishes only at t = 0. We conclude that the points on the geodesic crocks $\exp(iX)$ conjugate to the initial point $\gamma(0)$, are determined by the positive eigenvalues μ_i of the linear transformation K_x : $\mathfrak{su}(2m) \to \mathfrak{su}(2m)$, given by

$$K_X(Y) = \frac{1}{4}[[X, Y], X],$$

With X wi With X as in (2) it is easy to show that

$$[X, Y] = (i\pi(k_i - k_j)y_{ij}), \qquad Y = (y_{ij})$$

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$$\chi_{g}(Y) = \left(\frac{\pi^g}{4}(k_1 - k_2)^2 y_{ij}\right).$$

Now it is easy to check that for each pair it, with it < j the lan < li>m market Now it is easy to be in the (i.) Ith place and - [in the (i.) Ith place and place and any in the incidence P, with (i.) Ith and (i.) Ith place and any E, with entry death and also the matrices E, with (i,) th and (i,) the entries both vanc perox elsewhere), are eigenvectors of the linear transformation K and the perox elsewhere), are eigenvectors of the linear transformation K a comperiod electricities are eigenvalue $(\pi^2/4)(k_1-k_2)^2$. Furthermore, each diagnostic of k_1 matrix in su(2m) is clearly also an eigenvector of Ky (corresponding to the eigenvalue (1). Since a basis for su(2m) can be chosen from among the eigenvectors, it follows that the non-zero eigenvalues of K are just the numbers $(\pi^2/4)(k_1-k_2)^2$ for which $k_1>k_2$. By the first part of the proof each of these positive eigenvalues a gives rise to a sequence of conjugate points γ corresponding to the parameter values $s = n\pi/\sqrt{\mu}$, n = 1, 2, 3, ..., ne

$$i = \frac{2}{k_i - k_j}, \frac{4}{k_i - k_j}, \frac{6}{k_i - k_j}, \dots$$

Since the number of such values of r in the open interval (0, 1) is (k, -k, 12-1 and since each eigenvalue $(\pi^2/4)(k_r - k_j)^2 > 0$ has multiplicity 2, the lader Theorem tells us that the index it of y is given by the formula

$$\lambda = \sum_{k_i > k_j} (k_i - k_j - 2) \tag{6}$$

(Note that if y were minimal, then by Lemma 25.2 (see (1)), we should have $k_i = \pm 1$ for all i, so that the index would be zero (as might be expected): Now if, on the one hand, at least (m + 1) of the k, have the same sign (negative say), then at least one of the positive k_i must be ≥ 3 , whence

$$\lambda \geq \sum_{j=1}^{m+1} (3-(-1)-2) = 2m+2$$

On the other hand, if m of the k, are negative and m positive, then, since the cannot all be ± 1 (y being non-minimal), there must be one ≤ -3 and one 23, whence

hence
$$\lambda \ge \sum_{i=1}^{m+1} [(3-(-1)-2)+(1-(-3)-2)]+(3-(-3)-2)$$

$$= 4m > 2m + 2$$

This completes the proof of the lemma.

The theorem on unitary periodicity can now be deduced relatively easily. in two steps (the first of which involves the Fundamental Theorem of Mose Theory (22.5)), as follows.

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Lemma. The inclusion of the space Ω of minimal geodesics (homeomorphism between the constant space Ω = Ω = Ω | Ω | Lemma. The inclusion of the following process of minimal geodesics (homeomorphic the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$) in the path space $\Omega = \Omega(SU(2n))$, induces an isomorphism between the corresponding homotomy Ω . In all dimensions of the isomorphism Π (see Part II, Corollary 22.2.3) it then follows that $\pi_{\beta}(\Omega M) \simeq \pi_{i+1}(M)$ (see Π) $\simeq \pi_{i+1}(SU(2m))$

pg00F. By Theorem 21.3 the action functional E on the path space Ω has as pgOOF. By Theorem per arcs joining I_{2m} and I_{2m} . By the preceding the critical points of index 0 (the minimal geodesics) form retical points of index 0 (the minimal geodesics) form a subspace $G_{2m,m}^{C}$, while the index of all other critical points seminas the critical $G_{2m,m}^{C}$, while the index of all other critical points is at least homeomorphic to $G_{2m,m}^{C}$, while the index of all other critical points is at least homeomorphic to the property of the property of the property of a space obtained by attaching cells of $d_{2m}^{(N)} + 2$. It follows via Theorem 22.5 that $\Omega(SU(2m); I_{2m}^{(N)} - I_{2m})$ has the $\frac{2m}{2m} + 2$. It follows $\frac{2m}{2m} + 2$. It follows homotopy type G to the Grassmannian manifold $G^{\mathbb{C}}_{2m,m}$. Since attachment of cells of G at G to the homotopy G to G the homotopy G to G the homotopy G to G to G the homotopy G to G. 2m+2 to the 2m+2 has no effect on the homotopy groups of dimensions $\geq 2m+2$ has no effect on the homotopy groups of dimensions

$$\pi_i(\Omega) \simeq \pi_i(G_{2m,m}^{\mathbb{C}}) \quad \text{for } 0 \le i \le 2m,$$

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25.6. Lemma. There is an isomorphism

$$\pi_{i-1}(U(m)) \simeq \pi_i(G_{2m,m}^{\mathbb{C}}) \quad \text{for } 1 \le i \le 2m.$$

PROOF. As noted in Part II, §24.3(c), from the homotopy exact sequence of the standard fibration $U(m+1) \rightarrow S^{2m+1}$ with fibre U(m):

$$\cdots \to \pi_j(S^{2m+1}) \stackrel{\partial}{\to} \pi_{j-1} \, U(m) \stackrel{i_*}{\to} \pi_{j-1} \, U(m+1) \to \pi_{j-1}(S^{2m+1}) \to \cdots,$$

it follows readily that the inclusion i: $U(m) \rightarrow U(m+1)$ induces an isomorphism

$$i_*$$
: $\pi_{j-1} U(m) \simeq \pi_{j-1} U(m+1)$ for $j \le 2m$

(and also that the homomorphism i_* : $\pi_{2m}U(m) \to \pi_{2m}U(m+1)$ is onto). Since the maps

$$\pi_{j-1} U(m) \to \pi_{j-1} U(m+1) \to \pi_{j-1} U(m+2) \to \cdots$$
(7)

induced by the inclusions are all isomorphisms, these groups are all isomorphic to $\pi_{j-1}(U)$, the (j-1)st "stable homotopy group" of the unitary group.

From (7) and the homotopy exact sequence of the fibre bundle with total space U(2m), fibre $U(m) \subset U(2m)$, and base the coset space U(2m)/U(m) (the "complex Stiefel manifold"; cf. Part II, §5.2(d)):

$$\cdots \to \pi_{j}U(m) \stackrel{i_{\bullet}}{\to} \pi_{j}U(2m) \to \pi_{j}(U(2m)/U(m)) \stackrel{\partial}{\to} \pi_{j-1}U(m) \to \cdots,$$

it follows that

$$\pi_j(U(2m)/U(m)) = 0$$
 for $j \le 2m$.

Considering, finally, the homotopy exact sequene of the fibration

$$U(2m)/U(m) \rightarrow G_{2m,m}^{\mathbb{C}} = U(2m)/(U(m) \times U(m)),$$

with fillers Com, namely

and mosking (8), we obtain

$$g: \pi(G_{[m,m]} \cong \pi_{p-1}(U([m]))$$
 for $j \leq 2m$

as required

The theorem is now immediate from this lemma and the preceding the together with (7) and the isomorphism

$$\pi_j(SU(m)) \simeq \pi_j(U(m))$$
 for $j \neq 1$,

which follows easily by considering the homotopy exact sequence of the Sundle U(m) - S' with fibre SU(m) (see Part II, \$24.4)

We wish now to exhibit explicitly the isomorphism x (Em) $\pi_{\rm inv}(U(2m))$. $j \le 2m$, obtained above as the composite of the chain of isomore.

$$\begin{split} \pi_{j-1}(U(m)) &\stackrel{d}{\sim} \pi_{j}(G_{2m,m}^{*}) \rightarrow \pi_{j}(SU(SU(2m);I_{2m},-I_{2m})) \simeq \pi_{j-1}(SU(2m)) \\ &\simeq \pi_{j-1}(U(2m)). \end{split}$$

To this end, let $f_{i-1}: S^{-1} \to U(m)$ be a continuous map (representing the homotopy class $[f_{j-1}] \in \pi_{j-1}(U(m))$; starting with f_{j-1} , we wish to construct a corresponding map first SV"3 - SU(2m). We first distinguish in SU(2) considered in its matrix guise as the group of complex 2 × 2 matrices of tisform

$$x(\alpha, \beta) = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$
, where $|\alpha|^2 + |\beta|^2 = 1$, (1)

the subspace D2, homeomorphic to a 2-dimensional closed disc, consisting of those matrices (11) with β non-negative real. We embed SU(2) (and therefore D^2) into SU(2m) by means of the map

$$\varphi: \mathbf{x} \to \mathbf{x} \otimes I_{\mathbf{m}} = \begin{bmatrix} \alpha I_{\mathbf{m}} & \beta I_{\mathbf{m}} \\ -\bar{\beta} I_{\mathbf{m}} & \bar{\alpha} I_{\mathbf{m}} \end{bmatrix}$$
 (12)

We next consider the smooth curve 9 in D2, defined by

$$\hat{\gamma}(\beta, \tau) = \{x(\alpha, \beta) | \alpha = i\tau, \tau \text{ real}\},$$

and write $\gamma = \phi(\hat{\gamma})$. By Lemma 25.3 the complex Grassmannian manifold Gimm may be identified with the set of solutions in SU(2m) of the matrix equation $X^2 = -I_{2m}$, since the points of $\hat{\gamma}$ all satisfy $(\hat{\gamma}(\beta, \tau))^2 = -I_{2m}$ follows that, assuming this identification made, we have

$$\hat{\gamma}(\beta, \tau) \in G_{2m,m}^{\epsilon} \subset SU(2m)$$
 for $0 \le \beta \le 1$.

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We now distinguish in SU(2m) the subset consisting of all conjugates of the matrices in (12), of the form

$$g(y, \alpha, \beta) = \begin{bmatrix} I_m & 0 \\ 0 & f_{j-1}(y)^{-1} \end{bmatrix} \begin{bmatrix} \alpha I_m & \beta I_m \\ -\beta I_m & \overline{\alpha} I_m \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & f_{j-1}(y)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha I_m & \beta f_{j-1}(y) \\ -\beta f_{j-1}(y)^{-1} & \overline{\alpha} I_m \end{bmatrix}, \qquad (1)$$

where $\beta \ge 1$, $|\alpha|^2 + \beta^2 = 1$ and $y \in S^{j-1}$. This defines a map $g: S^{j-1} \times D^2 \to \mathbb{R}$ SU(2m), and since

 $g(y, \alpha, 0) = \begin{bmatrix} \alpha I_m & 0 \\ 0 & \overline{\alpha} I_m \end{bmatrix},$

for all y, it is not difficult to see that, in fact, the image in SU(2m) is a for all j, it is image in SU(2m); thus g determines a map

$$f_{j+1} \colon S^{j+1} \to SU(2m). \tag{14}$$

Restricting β to be real, $0 \le \beta \le 1$, and α to have the form ir, τ real, we obtain from (13) a map

 $S^{j-1} \times I \rightarrow G_{2m,m}^{\mathbb{C}} \subset SU(2m)$

whose image is a j-dimensional sphere

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$$S^{j} \subset G_{2m,m}^{\mathbb{C}} \subset SU(2m), \tag{15}$$

obtained essentially as the quotient space of $S^{j-1} \times \hat{\gamma}$ with, on the one hand. all points of the form $(y, \hat{y}(1, 0))$ (i.e. with $\alpha = 1, \beta = 0$, forming the lid of the cylinder) identified, and, on the other hand, all points $(y, \hat{y}(-1, 0))$ (constituting the base of the cylinder) identified. Putting $\beta = 1$ (whence $\alpha = 0$) in (13) we obtain the map

$$h: y \mapsto \begin{bmatrix} 0 & f_{j-1}(y) \\ -f_{j-1}(y)^{-1} & 0 \end{bmatrix}, \qquad S^{j-1} \to G_{2m,m}^{\mathbb{C}} \subset SU(2m);$$
 (16)

it can be shown that in fact $\partial[S^j] = [h]$, where $S^j \subset G_{2m,m}^{\mathbb{C}}$ is as in (15), and ∂ is (essentially) the boundary map determining the isomorphism (9). The upshot is that with the element $[f_{j-1}] \in \pi_{j-1}(U(m))$ (which may be identified with [h]), we have associated the element $[f_{j+1}] \in \pi_{j+1}(SU(2m))$ given by (14), i.e. by the formula

$$\{g(y, \alpha, \beta)\} = f_{j+1}(S^{j+1}) = \left\{ \begin{bmatrix} \alpha I_m & \beta f_{j-1}(y) \\ -\beta f_{j-1}(y)^{-1} & \overline{\alpha} I_m \end{bmatrix} \right\}.$$
with the iso

In fact, the map $[f_{j-1}] \mapsto [f_{j+1}]$ so defined coincides with the isomorphism of unitary f_{j-1} is associated of unitary periodicity given by (10): The first step, whereby $[f_{j-1}]$ is associated $\{b_{1j}, b_{2j}, b_{$ $(v_{13} \partial^{-1})$ with the S^j of (15), has already been noted. The next step is achieved by means of the by means of the embedding of $G_{2m,m}^{\mathbb{C}}$ in SU(2m) as the set of mid-points of the minimal geodesics from I_{2n} to $-I_{2n}$ (see Lemma 25.3); this embedding ares the others S' (or, more properly, "spheroid", i.e. map of the spheroid in $\Omega(SU(2nn))$ consisting of all the stranged whose mid-points are in S', which in turn, via the standard someoper S' and S' are S' and S' are S' associated with $S'''' \subset SU(2n)$. We have the

25.7. Theorem (Fornesko). Let $f_{j-1}\colon S^{j-1}\to U(m)$ represent an arbitrary dement of the homotopy group $x_{j-1}(U(m))$. The isomorphism of unitary periodics, (10) is given by the explicit formula $[f_{j-1}]\to [f_{j+1}]$, where $f_{j+1}\colon S^{j+1}\to S^{j+1}$ is defined by

$$\begin{split} f_{j+1} \colon S^{j+1} &\to \{g(y,\alpha,\beta) | y \in S^{j-1}, \, \beta \geq 1, \, |\alpha|^2 + \beta^2 = 1\} \subset SU(2n), \\ \text{with } g(y,\alpha,\beta) \text{ as in } (13). \end{split}$$

Thus, to repeat somewhat, the isomorphism of unitary periodicity (10) is, from a visual, geometric point of view, constructed in rather a simple way in the following two steps:

Step 1. Starting with an arbitrary spheroid f_{j-1} in U(m) one transforms the spheroid by means of the boundary homomorphism $\partial: \pi_i(G^0_{m,m}) \to \pi_{j-1}(U(m))$ (or rather its inverse; here it is in fact an isomorphism—see (9)), into a spheroid S^1 of one higher dimension in the Grassmannian manifold (see (16) for the explicit formula).

Step 2. By identifying the Grassmannian manifold $G_{2m,m}^C$ with the intersection of the group SU(2m) with its Lie algebra su(2m) (both regarded as subspaces of the space of all complex $2m \times 2m$ matrices), which intersection consists precisely of the mid-points of the minimal geodesics in SU(2m) from I_{2m} to $-I_{2m}$ (see Lemma 25.3), one can associate the spheroid S^j in $G_{2m,m}^C$ obtained in Step 1, with the spheroid $S^{j+1} \subset SU(2m)$ obtained as the union of those minimal geodesics whose mid-points are points of the spheroid $S^j \subset G_{2m,m}^C$. This spheroid, now in SU(2m) and of dimension j+1, is then the final image of the initial spheroid f_{j-1} under the isomorphism of unitary periodicity (the explicit formula for which is given in the above theorem).

In the case m = 2, j = 4, if the initial spheroid $f_{j-1} = f_3$: $S^3 \to SU(2) \subset U(2)$ is taken to be the map identifying the sphere S^3 with SU(2):

$$f_3(y) = \begin{bmatrix} \alpha & \beta \\ -\overline{\alpha} & \overline{\beta} \end{bmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1,$$

then the homotopy class $[f_3]$ is a generator of $\pi_3(SU(2)) \simeq \mathbb{Z}$. Applying repeatedly the above explicit formula for the isomorphism of unitary periodicity, starting with this $f_{j-1} = f_3$, one obtains in succession generators $[f_3]$. $[f_7]$, ..., in the cases $m = 2^2, 2^3, \ldots$, i.e. for each $k = 1, 2, \ldots$, a map

$$f_{2k+1}: S^{2k+1} \to SU(2^k),$$

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where $f^*(x)$ and $g^*(x) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $(x) \in \mathbb{R}^n$ pericular, the map |x| = 1, one then denote the

t is now not diffic defined above (see

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The proof of unita the one-dimension defined on (piecew the isomorphism of the isomorphism considering instead In the above, arried out in two domotopy groups

representing a generator of $\pi_{2k+1}(SU(2^k)) \simeq \mathbb{Z}$. For each k the map f_{2k+1} so the order of "Clifford algebra" of the (real) "duality" representing a generator $(z_k) = \mathbb{Z}$. For each k the map f_{2k+1} so that the complex analogue α_{2k+1}^c of the (real) duality map α_{2k+1} so plained is in fact the complex analogue α_{2k+1}^c of the (real) duality map α_{2k+1} so plained is in fact the complex analogue α_{2k+1}^c of the (real) duality map α_{2k+1} so plained in the theory of "Clifford algebras" and spinor representations. representations of "Clifford algebras" and spinor representations of accurring in the theory of "Clifford algebras" and spinor representations of accurring the structure of the "isomorphism of orthogonal group (cf. [5], where the "isomorphism of orthogonal group (cf. [5]) are the structure of obtaining in the theo, and an and spinor representations of accurring in the theo, accurring to the accurring to the theo, accurring to the theory accurrence to the theory accurrenc

period is related to the subsection by providing some of the details of this we conclude this subsection by providing some of the details of this we conclude the above pairs of values $m = 2^k$, j = 2k + 3. We conclude this we conclude this we conclude this we conclude this with the conclude this we can be concluded to the concluded this we can be concluded the concluded this we can be concluded this we analogy, since for the explicit formula for the isomorphism of unitary periods simplifying still further the geometric picture. Thus the second still further the geometric picture. furnishes another the geometric picture. Thus the map $\alpha_{2k+1}^{(i)}$ may

$$f: S^{n-1} \to GL(N, \mathbb{C}), \quad g: S^{m-1} \to GL(M, \mathbb{C}),$$

be any two continuous maps. Regarding S^{n-1} and S^{m-1} as embedded in the be any two controls of \mathbb{R}^n and \mathbb{R}^m respectively, we may clearly extend these maps to standard way in the standa

$$f * g = \omega \colon \mathbb{R}^{n+m} \setminus \{0\} \to GL(2MN, \mathbb{C}),$$

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$$(f*g)(x, y) = \begin{bmatrix} f(x) \otimes I_M & I_N \otimes g(y) \\ -I_N \otimes g^*(y) & f^*(x) \otimes I_M \end{bmatrix},$$

where $f^*(x)$ and $g^*(y)$ are the conjugate transposes of f(x) and g(y), and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m, (x,y) \neq (0,0)$. The restriction of this map to the standard unit sphere $S^{n+m-1} \subset \mathbb{R}^{n+m}$, then yields a map $S^{n+m-1} \to GL(2MN, \mathbb{C})$. Taking, in particular, the map $\alpha: S^1 \to GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$ to the unit circle: $\alpha(z) = z$, |z|=1, one then defines $\alpha_{2k+1}^{\mathbb{C}}$ to be the restriction to $S^{2k+1} \subset \mathbb{R}^{2k+2}$ of the map

$$\alpha * \cdots * \alpha : \mathbb{R}^{2k+2} \setminus \{0\} \to GL(2^k, \mathbb{C}).$$

It is now not difficult to verify directly that $\alpha_{2k+1}^{\mathbb{C}} \equiv f_{2k+1}$, where f_{2k+1} is as defined above (see (17) et seqq.).

25.2. Unitary Periodicity via the Two-Dimensional Calculus of Variations

The proof of unitary periodicity given in the preceding subsection is based on the one-dimensional calculus of variations applied to the action functional defined. defined on (piecewise-smooth) paths in the unitary group. It turns out that the isomorphism of unitary periodicity can be established more naturally by considering instead an appropriate 2-dimensional variational problem.

In the above, "classical", approach to unitary periodicity, the proof is Carried out in two quite distinct steps, in each of which the dimension of the homotopy groups under consideration increases by I. The fact that the proof breaks up in this way into two parts (the required increase of 2 in the dimension being achieved by means of two successive increases of 1 in the natural consequence of the method used, inasmuch as that method involves the one-dimensional calculus of variations of the action (and length) functionals on paths, i.e. maps of the one-dimensional disc D^1 (a line segment). We appropriate 2-dimensional analogue: Thus we take as the disc D^1 the interval appropriate 2-dimensional analogue: Thus we take as the disc D^1 the interval the points 0, 1), and now denote by Π_1^* the space $\Omega^*(SU(2m), I_{2m}, -I_{2m})$ the points 0, 1), and now denote by Π_1^* the space $\Omega^*(SU(2m), I_{2m}, -I_{2m})$ all continuous maps f of D^1 to SU(2m) satisfying $f|_{S^0} = i_0|_{S^0}$, where $i_0(S^0) = \{I_{2m}, -I_{2m}\}$, i.e. every f sends the end-points 0, 1 of D^1 to the same points $I_{2m}, -I_{2m}$ (respectively) of SU(2m). The action functional E on Π_1 , the subspace of Π_1^* consisting of all piecewise-smooth paths in Π_1^* (and homotopically equivalent to Π_1^*) is given by

$$E(\omega) = E_0^1(\omega) = \int_0^1 \left| \frac{d\omega}{dt} \right|^2 dt, \qquad \omega \in \Pi_1.$$

(The associated length functional

$$L_0^1(\omega) = \int_0^1 \left| \frac{d\omega}{dt} \right| dt,$$

was shown (essentially) in Part I, §31.2, to have the same extremal paths as the action functional E_0^1 in the class of all (piecewise-) smooth paths.) The set of points (i.e. paths) on which the action E_0^1 (and therefore also L_0^1) attains an absolute minimum, forms a subspace $W = \tilde{\Pi}_1$, shown in Lemma 25.2 to be homeomorphic to the Grassmannian manifold $G_{2m,m}^{\mathbb{C}}$, whence it was deduced (using Morse theory; see in particular Lemma 25.5) that the spaces Π_1 (and hence also Π_1^*) and $G_{2m,m}^{\mathbb{C}}$ have the homotopy type of cell complexes with identical (2m+1)-dimensional skeletons. Thus one might say that the "analytic part" of the isomorphism of unitary periodicity is given by the consequent isomorphism

$$\pi_i(G_{2m,m}^{\mathbb{C}}) \simeq \pi_i(\tilde{\Pi}_1) \simeq \pi_i(\Pi_1) \simeq \pi_{i+1}SU(2m), \qquad i \leq 2m,$$

since the subsequent step, $\pi_i(G_{2m,m}^{\mathbb{C}}) \simeq \pi_{i-1}(U(m))$, does not involve the action functional E_1^0 , being of a purely homotopy-theoretic character (see Lemma 25.6).

The above-described geometrical procedure for obtaining the isomorphism of unitary periodicity prompts the idea of trying to construct that isomorphism in a single step, rather than two distinct steps, by utilizing some 2-dimensional functional rather than the one-dimensional action functional on paths. It turns out that there is a feasible such method of obtaining the isomorphism of unitary periodicity, which moreover leads to a further simplification of the "geometrical picture" of that isomorphism. We shall in the present subsection expound this "one-step" procedure. (Note, however, that the construction we

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in $u \in L_p(X)$, i.e. $|u|^p$ in corresponding to with $0 \le |\alpha| \le m$ there should exist $r_s \in L_p(X)$ with the

where $\bar{D}^a g = \partial^{|\alpha|} g$

Thus, in particular $\mathbb{C}^{D^2} \to \mathbb{R}$ for which $\mathbb{D}(u)$, i.e. a square $\mathbb{C}^{D^2} \to \mathbb{R}$

where x_1 , x_2 are if $f_1 p_2 \rightarrow SU(2m)$ is in the g_2 co-ordinate full g_3 belong to haloes, in the g_4 g_5 g_6 g_6 g_6 g_7 g_8 g_8

295 altimately invokes the explicit formula (13), so that it does not constitute adependent proof of unitary periodicity.) give ultimated proof of unitary periodicity.)

independent production and propriate 2-dimensional problem; thus we we first need to describe the appropriate 2-dimensional problem; thus we we first need to with the specially chosen class of maps, $\frac{1}{2}$ define a suitable functional on a suitable specially chosen class of maps, $\frac{1}{2}$ define a suitable functional on a suitable specially chosen class of maps, $\frac{1}{2}$ define a suitable functional on a suitable specially chosen class of maps. sek to define a sulface. To begin with consider in SU(2m) the embedded circle parameter subgroup) a one-parameter subgroup)

$$S_0^1 = \left\{ \begin{bmatrix} \alpha I_m & 0 \\ 0 & \overline{\alpha} I_m \end{bmatrix} \middle| |\alpha| = 1 \right\}. \tag{18}$$

Let $D^2 \subset \mathbb{R}^2$ be the disc with centre (0,0) and radius 1, and let $j_0: S^1 \to SU(2m)$ Let $D^2 \subset \mathbb{R}^2$ be the (fixed) map identifying the boundary $S^1 = \partial D^2$ of the disc D^2 with the claim the obvious way. Denote by Π_2^* the topological \mathbb{R}^2 with the he the (fixed) find the obvious way. Denote by Π_2^* the topological space (topologized space) of an appropriate distance function, i.e. metrics circle S_0^{i} in the experimental distance function, i.e. metric) consisting of all py means of an appropriate distance function, i.e. metric) consisting of all by means of an error by means of $D^2 \to SU(2m)$ satisfying $f|_{S^1} = j_0$. (Thus $S^1_0 \subset SU(2m)$ is present 2-dimensional context, the role analogous continuous map $S_0 = SU(2m)$ continuous in the present 2-dimensional context, the role analogous to that played plays, in the present 2-dimensional sphere consisting of the periodic sphere in the present 2-dimensional sphere consisting of the present 2-dimensional sphere consisting and 2-d plays, in the property of the zero-dimensional sphere consisting of the two points I_{2m} , previously by the zero-dimensional sphere consisting of the two points I_{2m} , and the shown that Π_2^* has the homotopy type of a previously of $l_{m,m}$ it can be shown that Π_{2}^{*} has the homotopy type of a cell complex.

We next consider the subspace $\Pi_2 \subset \Pi_2^*$ consisting of all maps f in Π_2^* , belonging to a certain function space $H_1^2(D^2)$ which we shall now define: Given a region X of Euclidean space $\mathbb{R}^n(x^1,...,x^n)$, we shall say that a function $X \to \mathbb{R}$ belongs to the class of functions $H_m^p(X)$, if it satisfies the following two conditions:

(i) $u \in L_p(X)$, i.e. $|u|^p$ is integrable;

corresponding to each *n*-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of non-negative integers with $0 \le |\alpha| \le m$ if m > 1 and $|\alpha| = 1$ if m = 1, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, there should exist a "generalized derivative" D'u of u, i.e. a function $r_s \in L_p(X)$ with the property that for every function $g: X \to \mathbb{R}$ of class C^{∞}

$$\int_X g(x)r_a(x) dx = (-1)^{|a|} \int_X |\bar{D}^a g(x)| u(x) dx,$$

where $\bar{D}^{\alpha}g = \partial^{|\alpha|}g/(\partial x^1)^{\alpha_1}...(\partial x^n)^{\alpha_n}$

Thus, in particular, $H_1^2(D^2)$ consists of those square integrable functions $\mathbb{R}^{D^2} \to \mathbb{R}$ for which for each i = 1, 2 there exists a "generalized derivative" $D_i(u)$, i.e. a square integrable function $\overline{r_i}$ such that for every C^{∞} -function

$$\int_{D^2} g(x)\overline{r_i}(x) dx = \int_{D^2} \left| \frac{\partial g}{\partial x^i} \right| u(x) dx,$$

where x_1 , x_2 are the standard co-ordinates on \mathbb{R}^2 .) To say that a map $\int_{[0,T]}^{(1)} D^2 \to SU(2m)$ is in $H_1^2(D^2)$ is then understood to mean that each of the Sm^2 SU(2m) is in $H_1^2(D^2)$ is then understood to mean that $f:D^2 \to SU(2m)$ had co-ordinate functions is in $H_1^2(D^2)$. Thus the condition that $f:D^2 \to SU(2m)$ had somewhere G is a number of G. SU(2m) belong to $H_1^2(D^2)$ (as well as to Π_2^*) to qualify as a member of Π_2 , replaces, in the feplaces, in the 2-dimensional context, the previous condition that paths SU(2m). $D^{1 \to ces}$, in the 2-dimensional context, the previous condition $\Pi_1 (= \Omega)$, the SU(2m) be piecewise-smooth to qualify for membership in $\Pi_1 (= \Omega)$,

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appropriate condition for developing a "one-dimensional" Morse theory appropriate condition for developing a "one-dimensional" Morse the condition of the cond

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icable to path spaces.

Having specified Π_f , we now choose a suitable functional on Π_f . Having specified Π_f , which associates with each $f \in \Pi_f$, Ω_h , Ω_h , Ω_h . Having specified D[f], which associates with each $f \in \Pi_1$ the Dirichlet functional D[f], which associates with each $f \in \Pi_1$ the Dirichlet functional D[f], which associates with each $f \in \Pi_1$ the Dirichlet functional D[f]. Dirichlet functional by Dirichlet functional below), turns out to be appropriate for our integral of f (see the definition below), turns out to be appropriate for our integral of f (see the definition below), turns out to be appropriate for our integral of f (see the definition below). integral" of f (see the or and the second of the purpose. As was noted in Part I, §37.5, the 2-dimensional Direction functional in much as purpose. As was noted in the one-dimensional action functional in much the is the analogue of the one-dimensional action functional A is the 2-dimensional analogue of a same functional A. is the analogue of the same sense that the area functional A is the 2-dimensional analogue of the length sense that the area functional A is the functional D at L like is. sense that the area total area to the functional D at f. like the sense functional. In particular, the value of the functional D at f. like the sense function f. functional. In particular, i.e. on the co-ordinates chosen on the depends also on the parameters, i.e. on the co-ordinates chosen on the domain depends also on the parameter index. of f, whereas the values taken on by A and L are parameter-independent we of f, whereas the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional Dirichlet functional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the definition of the general n-dimensional (since now give the give t the 8-dimensional version will be needed in the following subsection). The the 8-different section of the sect let M and V With each map $f: V \to M$, $f \in H^p_1(V, M)$ (see above) we associate $g_{1,1}(v), v \in V$. With each map $f: V \to M$, $f \in H^p_1(V, M)$ (see above) we associate the "mixed" tensor $y_k^i = D^{s_k}(y^i)$, where the y^i are the local co-ordinates on M of the image under f of each $v \in V$: $(y^i) = y = f(v)$, and where $y_i = y_i$ $(0, \dots, 1, \dots, 0)$ is the *n*-tuple $(n = \dim V)$ with kth component 1 and all other components 0, and Dah is the corresponding generalized differential operator defined above. (Thus the upper index i of the tensor y's ranges from I to dim M and the lower, k, from 1 to dim V = n.) Given two such mixed tensors y_1^i , y_2^i arising from maps $f, \bar{f} \colon V \to M$, we define their (mixed) scalar product by

$$(y_k^i, \overline{y}/) = \hat{g}^{kl}g_{ij}y_k^i\overline{y}/,$$

where $(\hat{g}^{kl}) = (\hat{g}_{kl})^{-1}$. Then at each $f \in H_1^p(V, M)$ we define the Dirichlet functional D on HI(V, M) by

$$D[f] = \int_{V} \left[\frac{1}{n} (y_k^i, y_l^i) \right]^{n/2} dV, \tag{19}$$

where dV denotes the element of volume of the Riemannian manifold V. A map $f \in H_1^p(V, M)$ is then said to be harmonic if $\delta D[f; \eta] = 0$ for every vector field η on f(V) belonging to H_i^p , i.e. if the first-order variation of D[f] resulting from a variation of f by means of any such field η is zero (see, e.g. Part I, \mathfrak{P} 7. and cf. in particular Proposition 37.5.3 there). Thus the harmonic maps are the n-dimensional analogues of the geodesics. It can be shown by direct calculation that the resulting Euler-Lagrange equations for the harmonic maps are of the form $\nabla^k \nabla_k y^i = 0$ (see Part I, Theorem 37.1.2), where ∇_k denotes the operation of covariant differentiation with respect to the connexion on V compatible with its Riemannian metric.

In the particular case of present interest, the manifold V is the 2-dimensional Vunit disc in its standard position in Euclidean \mathbb{R}^2 , so that $g_{kl} = \delta_{kl}$, and the Dirichlet functional (19) takes the form

$$D[f] = \frac{1}{2} \int_{D^2} \left[(y_1^i, y_1^i) + (y_2^i, y_2^i) \right] dV = \frac{1}{2} \int_{D^2} g_{ij} (y_1^i y_1^j + y_2^i y_2^j) dV.$$
 (20)

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p(f) = \frac{1}{2}\int_{0}^{1} ad the first variations

The related functi (x,0) = (y'(u, 0)), th

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Thus the use of Agenes the pur hose maps f whi da continuous C d parameters ha aux the value of Before stating miliary facts:

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 $g_{ij}^{prop} \in \Pi_{ij}$, and g_{ij} is the metric on M = SU(2m) induced from the Euclidean on some \mathbb{R}^{N} with SU(2m) embedded in a standard way in second on the Euclidean co-ordinates on the where $f \in \Pi_1$, and g_{ij} with SU(2m) embedded in a standard way in \mathbb{R}^N with SU(2m) embedded in a standard way in \mathbb{R}^N in \mathbb{R}^N in \mathbb{R}^N by \mathbb{R}^N in \mathbb{R}^N i where on some \mathbb{R} some \mathbb{R} is excluded in a standard way in \mathbb{S}^n are u, v, then (20) are (cf. Part I, end of § 37.5)Nextmes (cf. Part I, end of §37.5)

$$p[f] = \frac{1}{2} \int_{D^2} \left[(y_a, y_a) + (y_b, y_c) \right] du dv, \qquad y = (y^b) \in SU(2m).$$
(21)

and the first variation δD of the functional D corresponding to the addition vector field $\eta \in H^2_1(D^2)$ on $f(D^2)$, is readily seen to be a and the first various $\eta \in H^2_1(D^2)$ on $f(D^2)$, is readily seen to be given by

$$\delta D[f;\eta] = \int_{D^2} \left[\left(\frac{\partial \eta}{\partial u}, y_v \right) + \left(\frac{\partial \eta}{\partial v}, y_v \right) \right] du \, dv. \tag{22}$$

The related functional A on the space Π_2 assigns to each map $f \in \Pi_2$, f(u,v) = (y'(u,v)), the integral (cf. Part I, §37.5)

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$$\int_{D^2} \left[\det \begin{pmatrix} (y_u, y_u) & (y_u, y_v) \\ (y_u, y_v) & (y_v, y_v) \end{pmatrix} \right]^{1/2} du \, dv;$$

thus A[f] is the 2-dimensional surface-area functional. As was noted in Part thus $A[f] \le D[f]$, with equality occurring precisely if the map f is (generalized) conformal, i.e. if u, v furnish conformal co-ordinates on the surface $f(D^2) \subset SU(2m)$ (so that in terms of these co-ordinates the induced Riemannian metric on $f(D^2)$ has diagonal form). It was also shown (see loc. oit) that under this condition on u, v as co-ordinates of $f(D^2)$, the Euler-Lagrange equations for a critical (in particular area-minimizing) surface f(D2) of the functional A, are equivalent to harmonicity (in the usual sense) of the radius vector $y = (y^i(u, v))$. This is analogous to the one-dimensional situation (of paths y in SU(2m) from I_{2m} to $-I_{2m}$) where the action and length functionals E and L satisfy $L^2(\gamma) \leq E(\gamma)$, with equality holding precisely when γ is parametrized by means of a natural parameter, in which case the critical paths for both functionals are the geodesic arcs from I_{2m} to $-I_{2m}$

Thus the use of the Dirichlet functional D (rather than the area functional Alserves the purpose, analogous to that served by E, of enabling us to ignore those maps f which, though not harmonic, can be made harmonic by means of a continuous change of co-ordinates on the disc D2: although such a change of parameters has of course no effect on the area functional, it will in general cause the value of the Dirichlet functional to change.

Before stating the theorem we are leading up to, we mention the following auxiliary facts:

$$\beta_2$$
: $\pi_j(\Pi_2^*) \simeq \Pi_{j+2}(SU(2m))$; (ii) the space Π_2^* is homotopically equivalent to the space $\tilde{\Pi}_2$ consisting of all continuous Ω_2^* (SU(2m) with the north pole (say) mapped always

continuous maps $S^2 \to SU(2m)$ with the north pole (say) mapped always to the same point of SU(2m).

(The isomorphism (23) is established by means of two applications of the isomorphism (23) is established by means of two applications of SU(2m), i.e. the (The isomorphism (2)) applications of Corollary 22.2.3 of Part II: if Ω denotes the loop space of SU(2m), i.e. the space Corollary 22.2.3 of Part II: if Ω denotes the loop space of SU(2m), i.e. the space corollary gives improve and ending at I_{2m} , then that corollary gives improve the space of SU(2m) in the space of SU(2m) in the space of SU(2m) in the space of SU(2m) is the space of SU(2m) in t Corollary 22.2.3 of Part II. The space of all loops beginning and ending at I_{2m} , then that corollary gives impediately of all loops beginning and ending at I_{2m} , then that corollary gives inspeciately of all loops beginning and ending at I_{2m} , then that corollary gives inspeciately of all loops beginning and ending at I_{2m} . of all loops beginning and the other hand, by applying it to the path $n_{j+1}(\Omega) \simeq n_{j+1}(\Omega) \simeq n_{j+1}(\Omega)$ and the other hand, by applying it to the path space $n_{j+1}(\Omega) \simeq n_{j+1}(\Omega) \simeq n_{j+1}(\Omega)$ $\pi_{j+1}(\Omega) \simeq \pi_{j+2}(SU(2m))$, with initial "point" the circle S_1^1 of Ω consisting of all trajectories in Ω with initial "point" the circle S_1^1 of Ω on Ω consisting prime given by maps $D^2 \to SU(2m)$, cf. Definition in Ω consisting of all trajectories being given by maps $D^2 \to SU(2m)$; cf. Definition 21.1 except that the fibre above S_0^1 itself is precisely Π^2 one 21.1 ex (such trajectories being g such trajectories being Π_2^* , one obtains sequ.), on noting that the fibre above S_0^1 itself is precisely Π_2^* , one obtains $\pi_i(\Pi_2^*) \simeq \pi_{i+1}(\Omega).$

25.8. Theorem (Fomenko). Let Π_2^* , Π_2 be the spaces of maps $D^2 \to SU(2m)$ 25.8. Incorem to the subspace of Π_2 consisting of all points (i.e. defined above, as which the Dirichlet functional D[f] attains its absolute minimum maps) $f \in \Pi_2$ at which the Dirichlet functional D[f] attains its absolute minimum value. The following assertions hold:

- (i) the subspace $W \subset \Pi_2$ is homeomorphic to the group U(m);
- (ii) the inclusion i: $W \to \Pi_2 \to \Pi_2^*$ induces an isomorphism

$$i'_*$$
: $\pi_j(U(m)) \simeq \pi_j(\Pi_2^*)$ for $j \leq 2m$,

(whence it follows that the 2m-dimensional skeletons of U(m) and Π_2^* , or rather of some realizations of these spaces as cell complexes, are homotopically equivalent).

The composite of the isomorphism i'_* with the isomorphism β_2 of (23) coincides with the isomorphism of unitary periodicity (see (10)):

$$\beta_2 \circ i'_* : \pi_j(U(m)) \simeq \pi_{j+2}(SU(2m)), \quad j \le 2m.$$
 (24)

(Thus by considering the set W of absolute minimum points of the 2-dimensional Dirichlet functional, the isomorphism of unitary periodicity can be achieved in a single step (involving an increase of 2 in the dimension of the homotopy groups), instead of the two steps required in the proof using the one-dimensional functionals of action and length.)

We give the proof in the form of a sequence of lemmas. First consider the 2-dimensional sphere

$$S_0^2 = \left\{ \begin{bmatrix} \alpha I_m & \beta I_m \\ -\beta I_m & \overline{\alpha} I_m \end{bmatrix} \middle| \beta \in \mathbb{R}, |\alpha|^2 + |\beta|^2 = 1 \right\} \subset SU(2m), \tag{25}$$

whose equator, defined by $\beta = 0$, coincides with the circle S_0^1 of (18). The upper hemisphere, $\beta \geq 0$, which we shall now denote by D_0^2 , played a significant role in the earlier explicit construction of the isomorphism of unitary periodicity (see (11) et seqq.). Using the fact that the inclusion $S_0^2 \to SU(2m)$ extends to a natural embedding of SU(2) into SU(2m) (see (12)), it can be shown that the sphere S_0^2 is a "totally geodesic" submanifold of SU(2m), and therefore a minimal submanifold (i.e. area-minimizing) (see the discussion following Lemma 25.10 below). (A submanifold of a manifold M is said to be totally geodesic if every geodesic in M tangent to the submanifold at any point of it

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St. Lemma. The subspi max Let x be any elen $\sum_{n=1}^{\infty} |3|_n \oplus \bar{\alpha} I_m ||\alpha| = 1$ Mix diagonal form:

 $x = (l_m \oplus DA^{-1})(A$ whence From the fact that (A G $p^{2}(x) = D^{2}(x_{1}) = \left\{ \left[\right].$

> Since $\beta \ge 0$, the matri fact, as is easily verifi and therefore, biconti the proof.

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Denoting by f_0 : D^2 gandard disc D^2 ¹⁰Π₂ by setting

The embedding i 1:9- 50

wholly in the submanifold or, equivalently, if every geodesic of the subsiold is a geodesic in M. That such submanifolds are locally wholly in the submanifold is a geodesic in M. That such submanifolds are locally volumehes winded is a geodesic of the submanifold are locally volume-manifold is a geodesic of the submanifold; in particular, in the control of the submanifold in the subma manifoldizing follows to the submanifold; in particular, in the Greenty minimizing follows the Riemannian curvature M when restricted to the submanifold; in particular, in the Greenty submand of the submanifold M is a Lie group, this restriction is a direct summand of the submanufold M is a Lie group on the ambient. minimor on M when the property of the group, this restriction is a direct summand of the full case where M is a Lie group, this restriction is a direct summand of the full case where M is a Lie group M.)

semannian cutvate it that the disc D_0^2 , the upper hemisphere of S_0^2 , is a totally thus we may take it that the disc D_0^2 , the upper hemisphere of S_0^2 , is a totally upper hemisphere. Thus we may take W_1 thus we may take W_2 thus we may take W_3 thus we may take W_4 thus we may take W_4 geodesic submatter SU(2m) of the form $D^2(x) = xD_0^2x^{-1}$, where $x \in SU(2m)$ and SU(2m) and $\sum_{x \le x^{-1}} = s \text{ for all } s \in S_0^1.$

25.9. Lemma. The subspace $\hat{W} \subset \Pi_2$ is homeomorphic to U(m).

p_{ROOF.} Let x be any element of SU(2m) satisfying $x_s = sx$ for all $s \in S_0^1$. Since PROOF. Let x be $\exists I_m | |\alpha| = 1$ } (see (18)), it follows easily that x must also have block diagonal form:

$$x = A \oplus D$$
, $A, D \in U(m)$,

whence

$$x = (I_m \oplus DA^{-1})(A \oplus A) = x_1(A \oplus A)$$
 where $x_1 = I_m \oplus DA^{-1}$.

From the fact that $(A \oplus A)d = d(A \oplus A)$ for all $d \in D_0^2$, $A \in U(m)$, we infer that

$$D^{2}(x) = D^{2}(x_{1}) = \left\{ \begin{bmatrix} \alpha I_{m} & \beta C^{-1} \\ -\beta C & \overline{\alpha} I_{m} \end{bmatrix} \middle| \beta \ge 1, |\alpha|^{2} + \beta^{2} = 1 \right\} \text{ where } C = DA^{-1}.$$

Since $\beta \ge 0$, the matrix $C \in U(m)$ is uniquely determined by the disc $D^2(x)$, in fact, as is easily verified, the map $D^2(x) \mapsto C$ is a bijection from \hat{W} to U(m). and therefore, bicontinuity being obvious, a homeomorphism. This completes the proof.

We use this homeomorphism to construct an embedding $i': U(m) \to \Pi_2$. That homeomorphism associates with each $g \in U(m)$ the 2-dimensional disc

$$D^{2}(I_{m} \oplus g) = \left\{ \begin{bmatrix} \alpha I_{m} & \beta g^{-1} \\ -\beta g & \overline{\alpha} I_{m} \end{bmatrix} \middle| \beta \geq 1, |\alpha|^{2} + \beta^{2} = 1 \right\}.$$

Note incidentally that if $g_1 \neq g_2$, then

$$D^2(I_m \oplus g_1) \cap D^2(I_m \oplus g_2) = S_0^1.$$

Denoting by $f_0: D^2 \to D_0^2$ the standard identification with $D_0^2 \subset SU(2m)$ of the standard disc $D^2
ightharpoonup \mathbb{R}^2$, we define, for each $g \in U(m)$, a map $f_g \colon D^2 \to SU(2m)$ in II, by setting

$$f_g(y) = (I_m \oplus g)f_0(y)(I_m \oplus g^{-1}), \quad y \in D^2.$$

The embedding $i': U(m) \to \Pi_2$ we are seeking to define, is then given by 1: grafa

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It follows from the above lemma that the set of maps $f'(U(m)) = \prod_2 \log g = \{I \oplus C\} \subset U(2m), G \cong U(m), \text{ and } Ad_2 \text{ of } G, \text{ where } x \text{ ranges over the group } G = \{I \oplus C\} \subset U(2m), G \cong U(m), \text{ and } Ad_2 \text{ denotes conjugation by } x \text{ that } f'(U(m)) \text{ is the orbit containing the point } f_0 \text{ of } \Pi_2 \text{ under the conjugating action of the group } G.$

25.10. Lemma. The homomorphism

$$\beta_2 \circ i'_* \colon \pi_j(U(m)) \to \pi_{j+2}(SU(2m))$$

coincides, for $j \le 2m$, with the isomorphism of unitary periodicity. It follows that the homomorphism $i'_*: \pi_i(U(m)) \to \pi_i(\Pi_2^*)$

 $i_*: \pi_j(U(m)) \to \pi_j(1)$

is an isomorphism for $j \leq 2m$.

PROOF. Let $\varphi: S^j \to U(m)$ represent a typical element $[\varphi]$ of $\pi_j(U(m))$. Writing $\varphi(y) = g$, for $y \in S^j$, we have

$$(\beta_2 \circ i'_*)(\varphi)(y) = \beta_2(f_g), \qquad f_g: D^2 \to SU(2m),$$

by definition of i'. By analysing the map β_2 in detail (see (23) et seqq.) it can be shown that

$$\begin{split} \beta_2(f_g) &= D^2(I_m \oplus \varphi(y)) = D^2(I_m \oplus g) \\ &= \left\{ \begin{bmatrix} \alpha I_m & \beta g^{-1} \\ -\beta g & \overline{\alpha} I_m \end{bmatrix} \middle| \beta \geq 1, |\alpha|^2 + \beta^2 = 1 \right\}, \end{split}$$

which coincides, in its essentials, with the explicit formula (13) (of Theorem 25.7) for the isomorphism of unitary periodicity (valid for $j \le 2m$).

Since β_2 is an isomorphism (for all j), it follows that the homomorphism $i_*: \pi_j(U(m)) \to \pi_j(\Pi_2^*)$ is an isomorphism for $j \le 2m$. This completes the proof of the lemma.

To complete the proof of the theorem it only remains to show that i'(U(m)) = W. To this end, consider the Euclidean space \mathbb{R}^{8m^2} identified with the complex space \mathbb{C}^{4m^2} of all $2m \times 2m$ complex matrices endowed with the bilinear form $\psi(A, B) = \text{Re}(\text{tr } A\overline{B}^T)$. The group SU(2m) then embeds (isometrically) in the sphere $S^{8m^2-1} \subset \mathbb{R}^{8m^2}$ of radius $\sqrt{2m}$, as a smooth submanifold equipped with the special Riemannian metric (in fact, just the Euclidean metric on \mathbb{R}^{8m^2}) induced from the above bilinear form, invariant under both right and left translations by elements of SU(2m) (the Killing form; cf. the beginning of §25.1).

Many of the metrical properties of the group SU(2m) are more conveniently examined in the larger context of the ambient sphere S^{8m^2-1} , for instance, the property that there are no variations ("infinitesimal perturbations") of the embedded 2-dimensional disc $D_0^2 \subset SU(2m)$, leaving the boundary $S_0^1 = \partial D_0^2$ pointwise fixed, such that the perturbed disc \widetilde{D}_0^2 , while remaining minimal in

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SII. Lemma. The inclusions of the conjugation of t

N.12. Lemma. We have PROOF. Let $f: D^2 \to SU$ which the Dirichlet fur

on II₂). It follows fro that equality holds proconformal co-ordinat function f is so param

id the proof of the prolatmonic as a function aduced from the state at like D²₀, the image missphere of a great

 $S^{(1)}(2n0)$ is no longer totally geodesic (see above). To see this suppose that $S^{(1)}(2n0)$ exists. Note that the circle $S^{(1)}_{ij} = SU(2n0) = S^{(1)}_{ij} = S^{(1)}_{ij}$ in fact the equator of the $S^{(1)}_{ij} = S^{(1)}_{ij} = S^{(1)}_{ij}$. $s^{(2m)}$ is no longer than the circle $S_0^* = SU(2m) = S^{(m)}$. To see this suppose that such sphere $S^{(2m)} = 1$, in fact the equator of the "great 2-sphere" $S^{(2m)} = 1$ is a $S^{(2m)} = 1$ to $S^{(2m)} = 1$ in fact the equator of the "great 2-sphere" $S^{(2m)} = 1$ is a $S^{(2m)} = 1$ to $S^{(2m)} = 1$ in $S^{(2m)} = 1$ in striation exists " in fact the equator of the "great 2-sphere \$2 at sphere \$2 at cacker by a 3-dimensional subthe sphere by the sphere space of the "great 2-sphere" by a 3-dimensional subspace of page 3. dimensional subspace of page 3. a by 3 obtained from D_0^2 by means of a small perturbation by solutions the file obtained from D_0^2 by means of a small perturbation. 3. dimension D_0^2 by means of a small perturbation, is not tosally as D_0^2 in SU(2m), it will not be totally geodesic as a submandation of the disc D_0^2 in SU(2m). D_{i}^{k} obtains, it will not be totally geodesic as a submandoid of the godesic as a Submandoid of the godesic as a Submandoid of the produces in the property of the disc \hat{B}_0^2 cannot form part of a "great 2-sphere" in Lighth equator \hat{S}_0^2), in consequence of which its area 2-sphere" in spect S_0^{second} (with equator S_0^{s}), in consequence of which its area must so a linear mation exceed that of D_0^2 , i.e. $\delta A > 0$. Thus every value (with expectation exceed that of D_0^2 , i.e. $\delta A>0$. Thus every variation of a linear W either preserves the property of total geodesicity to $d_{ij}^{\text{positivation}}(x) = 0$ which expressions the property of total geodesicity (in which case the property of total geodesicity (in which case the p'(x) & W which case the perturbed disc is again a hemisphere, with boundary S₃, of a great 2-sphere perturbed the distribution of the transformation effecting the perturbation is a "totation is a and can be achieved by mans of an inner automorphism of the about h_0 miner automorphism of the group SU(2m), or destroys the property of local minimality enjoyed by $D^2(x)$ as inhibourhood of some point). at least in a neighbourhood of some point).

35.11. Lemma. The inclusion i'(U(m)) = W holds

PROOF. Since each map $f \in I'(U(m))$ has the form $Ad_{\infty} \circ f_0$, where Ad_{∞} is the properties on the state of the sandard map $D^2 \to D_0^2$, is an absolute minimum point for the Dirichlet functional D. Now, as already made clear in the preceding discussion, the disc being a hemisphere of the spherical cross-section of the sphere S⁸⁻²⁻¹ by a 3-dimensional plane through the origin of R8m', has least area among all (appropriate) discs in S^{8m^2-1} with boundary S_0^1 , so that the area functional of such discs takes on its least value at f_0 . Since $A[f_0] = D[f_0]$ provided $D[f_0]$ sevaluated with respect to conformal co-ordinates, and since we always have $A[f] \leq D[f]$, the fact that $A[f_0]$ is least for A implies that $D[f_0]$ is least

25.12. Lemma, We have i'(U(m)) = W.

Prox. Let $f: D^2 \to SU(2m)$, $f|_{S^1} = j_0$, be an element of W, i.e. a map in Π_2 at which the Dirichlet functional D attains its least value (among all values taken os $\{I_2\}$. It follows from the inequality $A[f] \leq D[f]$ together with the fact that equality holds precisely when the surface defined by f is parametrized by conformal co-ordinates (on D^2), that in fact the present absolute-minimum function f is so parametrized, whence

$D[f] = D[f_0] = A[f] = A[f_0]$

of the proof of the preceding lemma). (Note incidentally that f must then be harmonic on SU(2m) is harmonic as a function of those co-ordinates.) Since the metric on SU(2m) is induced for nduced from the standard metric on $S^{nm^2-1} \subset \mathbb{R}^{nm^2}$ (see above), it follows that, like D^2 : hat, like D_0^2 , the image $f(D^2)$ must be a "great hemisphere" in S^{nm^2-1} , i.e. a have in S^{nm^2-1} the two being sphere of a great 2-sphere $\tilde{S}^2 \subset S^{8m^2-1}$. Thus we have in S^{8m^2-1} the two

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great 2-spheres S^2 and S_0^2 (therefore totally geodesic in S^{8m^2-1} and hence S_0^2 (which contains S_0) whose intersection contains S_0 . great 2-spheres S^2 and S_0 (theorem contains S_0) (which contains the point certainly in SU(2m)) whose intersection contains S_0 (which contains the point certainly in SU(2m)) of these 2-spheres is the cross-section of S^{2m-1} certainly in SU(2m)) whose 2-spheres is the cross-section of S^{2m-1} by a 1. Since each of these 2-spheres is the cross-section of S^{2m-1} , they can be transfer by a 1. I_{2m}). Since each of these 2-3 given in \mathbb{R}^{8m^2-1} , they can be transformed into dimensional plane through the origin of \mathbb{R}^{8m^2-1} , they can be transformed into dimensional plane through the dimensional plane through the smallest subgroup of SU(2m) containing S_{2}^{2} in the smallest subgroup of SU(2m) containing SU(one another by means of succession on another by means of succession of SU(2m) containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing S_0 is the isomorphism of SU(2m) obtained from (25) by letting B take on all containing SU(2m) obtained from (25) by letting SU(2m) of SU(2m) obtained from (25) by letting SU(2m) obtained from (25) by letting SU(2m) of SU(2m) of SU(2m) of SU(2m) of SU(2m) obtained from (25) by letting SU(2m) of SU(two planes. Since the shahes the shahes the phic copy G_1 of SU(2) obtained from (25) by letting β take on all complex values, phic copy G_1 of SU(2) obtained from (25) by letting β take on all complex values. phic copy G_1 of SU(2) obtained by the consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$, it can be deduced that the smallest subgroup consistent with $|\alpha|^2 + |\beta|^2 = 1$. consistent with $|\alpha| + |B| = 1$ is likewise an isomorphic copy of SU(2). In both of the groups, containing S^2 is likewise an isomorphic copy of SU(2). In both of the groups, S^2 is the image of a maximal torus $T^1 \simeq S^1$ in each of the groups. G_2 containing S^1 is the image of a maximal torus $T^1 \cong S^1$ in SU(2) (i.e. a symmetric G_1 , G_2 , the circle S_0^1 is the image of a maximal torus $T^1 \cong S^1$ in SU(2) (i.e. a symmetric G_1). G_1 , G_2 , the circle S_0 is maximal connected commutative Lie subgroup), and $S_0 \subset T^{2m-1}$ for some maximal connected $T^{2m-1} \subset SU(2m)$. Letting α_0 , $\alpha: SU(2) \to SU(2m)$ denote the maximal torus G_2 , G_1 , extending G_2 , G_3 respectively, it follows from Lie theory, in particular as it pertains to maximal tori (see e.g. [51]), that the representations α_0 , α are equivalent, i.e. there is an element x of SU(2m) such that $\alpha = \mathrm{Ad}_x \circ \alpha_0$. Restricting to $D^2 \subset SU(2)$ we infer that $f = \mathrm{Ad}_x \circ f_0$. whence it follows (via the proof of Lemma 25.9 and the ensuing definition of i') that $f \in i'U(m)$. This completes the proof of the lemma, and with it the proof of Theorem 25.8.

In connexion with the above proof, observe that the points of the set W (the absolute-minimum points for both functionals A and D) turned out to be "totally geodesic" (i.e. the surfaces they defined in SU(2m) turned out to be totally geodesic in SU(2m)). This is somewhat analogous to the one-dimensional situation since, as we saw in §21, a (piecewise-smooth) minimal path $[0,1] \to SU(2m)$ is automatically a geodesic in SU(2m). However, in the present 2-dimensional context, minimality (as opposed to absolute minimality) of a map $D^2 \to SU(2m)$ does not in general entail that the resulting surface will be totally geodesic in SU(2m). In fact, it can be shown that the only totally geodesic discs in SU(2m) with boundary S_0^1 are those in \hat{W} , or, more precisely, if $f \in \Pi_2$ is a critical point of the functional D and if $f(D^2)$ is totally geodesic in SU(2m), then $f \in W$.

25.3. Orthogonal Periodicity via the Higher-Dimensional Calculus of Variations

There is a periodic isomorphism also for the orthogonal group. This result, the "theorem on orthogonal periodicity", is also due to Bott.

25.13. Theorem (On Orthogonal Periodicity). For the stable orthogonal group $O = \lim_{n \to \infty} O(n)$, where O(n) is embedded in the standard way in O(n+1) for n = 1, 2, ..., there are isomorphisms

$$\pi_i(0) \simeq \pi_{i+8}(0), \qquad i = 0, 1, 2, \dots$$

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we shall in the presen We share present (27) Samuel Proof of the is Morse thee which the stand grounded above splits of which below, by using whe Dirichlet function the 2-dimensiona onest of unitary pe $\lesssim 0.5$ $\approx \pi_{i+8}(0)$ may be Proceeding much a gr of real p x p mat terms of matrices is gi so(p) is then a smo gride Res of radius Cling metric induce Corollary 6.4.4). The a Rr, as the subsp intersection $so(p) \cap$ ntersection can be i bee Part II, §6, for t precisely of those el in one-to-one corre We now take p

t follows that the onhogonal (with a somether with the i

4(16r) there exist that matrices J_1 ,

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303 that each stable homotopy group $\pi_i = \pi_i(0)$ is isomorphic to the polynomials one of the following groups: It following one of the following groups:

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We shall in the present subsection sketch the proof of (26) only. (Some of we shall in the proof of (26) only. (Some of the isomorphisms in (27) may however be gleaned from Part II, §24.4.) The the isomorphism of orthogonal periodicity utilizes one-standard proof of the isomorphism of orthogonal periodicity utilizes one-standard proof of the isomorphism of orthogonal periodicity utilizes onestandard proof of the isomorphism of united which the standard proof of the isomorphism of united the standard proof of the is dimensional Nto.

dimensional way in which the way in which the spounded above splits up naturally into two steps). However, as we shall expounded above by using the 8-dimensional calculus of various ways as we shall expounded are below, by using the 8-dimensional calculus of variations as we shall puriciple functional on a certain space of 8-dimensional calculus. indicate perodicate periodicate periodicat the Dirichlet 2-dimensional balls, i.e. discs, that it was natural to consider in the than the 2-dillect of unitary periodicity) the isomorphism of orthogonal periodicity context of unitary case, $\pi_{i+8}(0)$ may be achieved (with some sacrifice of rigour) in a single step.

 $\eta(0) \simeq \pi_{1+8}(0)$ much as in the unitary case, we consider the Euclidean space processing $p \times p$ matrices with the usual Euclidan scalar product, which in g^{μ} of tear product, which in terms of matrices is given by the bilinear form $\varphi(A, B) = \text{tr}(AB^{T})$. The group SO(p) is then a smooth Riemannian submanifold of the standard sphere $SO(p^{-1} \subset \mathbb{R}^{p^2})$ of radius \sqrt{p} , centre the origin, with the (two-sided invariant) Killing metric induced on it from the Euclidean metric on Rp2 (cf. Part II, Corollary 6.4.4). The Lie algebra so(p) of the group SO(p) is also embedded $\ln \mathbb{R}^{p^2}$, as the subspace of matrices X satisfying $X^T = -X$. We denote the intersection $so(p) \cap SO(p)$ by $\Omega_1(p)$. (Note incidentally that if p is even this intersection can be identified with the compact symmetric space O(p)/U(p/2)(see Part II, §6, for the definition of symmetric space).) Clearly $\Omega_1(p)$ consists precisely of those elements g of SO(p) satisfying the equation $g^2 = -1$, i.e. is in one-to-one correspondence with the set of "complex structures" on RP.

We now take p in the form p = 16r; it can be shown (see [44]) that in $\Omega_1(16r)$ there exist eight anti-commuting complex structures, i.e. there are eight matrices J_1, \ldots, J_8 in $\Omega_1(16r)$ satisfying

$$J_i J_k = -J_k J_i, \quad i \neq k, \qquad J_j^2 = -I.$$

It follows that the J_i , as position vectors of points in SO(16r), are pairwise orthogonal (with respect to the Euclidean metric on $\mathbb{R}^{(169)^2}$), and moreover logether with the identity matrix I form an orthonormal set. Hence the sphere

$$S_0^8 = \{x \in SO(16r) | x = a^0I + a^1J_1 + \dots + a^8J_8, (a^0)^2 + \dots + (a^8)^2 = 1\}$$

is a cross-section of the sphere $S^q \subset \mathbb{R}^{(16r)^2}$ of dimension $q = (16r)^2 - 1$, by a 9-dimension $q = (16r)^2 - 1$, by a ⁹-dimensional plane through the origin of $\mathbb{R}^{(16r)^2}$. Consequently, S_0^8 is a totally Brodesic submanifold of S^q and therefore certainly of $SO(16r) \subset S^q$. Clearly

$$S_0^8 \cap so(16r) = S_0^8 \cap \Omega_1(16r) = \overline{S_0^7},$$

Where $\overline{S_0}^7$ is the 7-dimensional "great sphere" or "equator" of S_0^8 , defined by

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ints of the set W turned out to be) turned out to ous to the one nooth) minimal However, in the absolute minat the resulting shown that the those in W. or.

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O(n + 1) for

 $a^0 = 0$, again totally geodesic (in S^q and therefore SO(16r)). Consider, on the other hand, the "equator"

$$S_0^7 = \{x \in S_0^8 | x = a^0 I + a^1 J_1 + \dots + a^7 J_7 \}$$

defined by $a^8 = 0$, the boundary of the totally geodesic 8-dimensional ball $D_0^8 \subset S_0^8$, the "upper hemisphere" of S_0^8 , given by

$$D_0^8 = \{ x \in S_0^8 | a^8 \ge 0 \}$$

Let D^8 denote the standard 8-dimensional ball (in standard position in the Let D^8 denote the standard D^8 , let $\hat{D}^8 \to D^8$, let $\hat{D}^8 \to D^8$ be the standard bijection Euclidean space \mathbb{R}^8), let $\hat{D}^8 \to D^8$ be the standard bijection Euclidean space 18 1, 18 to the upper hemisphere of the sphere S⁸ (in standard position from the ball D⁸ to the upper hemisphere of the sphere S⁸ (in standard position from the ball D⁸ to the upper hemisphere of the sphere S⁸ (in standard position from the ball D⁸). from the ball D to the standard isometric bijection identifying $\hat{i}(D^8) = g^8$ in \mathbb{R}^9), and let i'' be the standard isometric bijection identifying $\hat{i}(D^8) = g^8$ in \mathbb{R}^9), and let i be sometimes on $i(S^7)$ with a standard fixed isometry with $D_0^8 \subset S_0^8 \subset SO(16r)$ and coinciding on $i(S^7)$ with a standard fixed isometry $j_0: \hat{i}(S^7) \to S_0^7$. Write

 $f_0 = i'' \circ \hat{i} : D^8 \to SO(16r)$

Denote by Π₈ (in notation reminiscent of that used in the unitary case) the space of all continuous maps $f: D^8 \to SO(16r)$ satisfying $f|_{S^2} = j_0 \circ i$, and by $\Pi_8 \subset \Pi_8^*$ the subspace of all such maps $f: D^8 \to SO(16r)$ in the class $H_8^8(D^8)$ (see the preceding subsection). On the function space Π_8 we consider the (parameter-independent) 8-dimensional-volume functional A[f] (the analogue of the area functional utilized in the unitary case) defined by

$$A[f] = \int_{D^8} \sqrt{\det(g_{ij})} \, dV,$$

(see Part I, §18.2) where (g_{ij}) is the metric on $SO(16r) \subset \mathbb{R}^{(16r)^2}$, and the Dirichlet functional (see (19))

$$D[f] = \int_{D^8} \left[\frac{1}{8} (y_k^i, y_l^i) \right]^4 dV = \int_{D^8} \left[\frac{1}{8} \sum_{k=1}^8 g_{ij} y_k^i y_l^i \right]^4 dV,$$

which, as before, depends on the co-ordinatization of D8. It is not difficult to show that for all $f \in \Pi_8$ we have $A[f] \leq D[f]$. Finally, let

$$\beta_8: \pi_j(\Pi_8^*) \simeq \pi_{j+8}(SO(16r))$$
 (28)

be the standard isomorphism constructed analogously to β_2 (see (23) et seqq.)

25.14. Theorem (Fomenko). Let SO(16r) be embedded in the Euclidean space $\mathbb{R}^{(16r)^2}$ as above, and let the spaces Π_8^* and Π_8 of maps of the 8-dimensional ball D⁸ into SO(16r) be defined also as above. Denote by W the subspace of II₈ consisting of all those points (i.e. maps) $f \in \Pi_8$ at which the Dirichlet functional D[f] attains its absolute minimum value on Π_8 . The following statements hold:

- (i) the subspace W is homeomorphic to the orthogonal group O(r).
- (ii) the inclusion i: $W \to \Pi_8 \to \Pi_8^*$ induces isomorphisms

$$i'_*$$
: $\pi_j(O(r)) \simeq \pi_j(\Pi_8^*)$ for $j \le r - 2$,

(whence it follows that the (r-2)-dimensional skeletons of the spaces O(r)

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We shall now indie me white of #2(U) in maps D2 - C $\lim_{\delta i \setminus \mathcal{O}(16r)} = \mathbb{Z}_2 \left(a \right)$ specifiely two connec presented up of the 8 suscepted compone contractible (in the li they contain. The present proc the subset $\Omega_8 \subset SO$ which anti-commu aistence was noted sandard sphere So is clear that $J_8 \in \Omega$ orthogonal to the $S_0^{\dagger} \cap \Omega_8 = \{ \pm J_8 \},$ [J_s]. By means of a that the space $\Omega_{\rm B}$ h 10 O(r) (see [44]).

having the standa
$$S^{8}(x) = \{a^{0}$$

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Since the vectors follows that $S^8(x)$ through the origi of SO(16r) = P(1) hemisphere of S8

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we are now in a the analogue of For each $x \in \Omega_*$ to the boundary [1] or rather of some realizations of these spaces as cell complexes, are

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homoto $\beta_8 \circ i'_* \colon \pi_j(O(r)) \simeq \pi_{j+8}(SO(16r)),$

$$\beta_8 \circ l_*$$
: $\pi_j(O(r)) \simeq \pi_{j+8}(SO(16r)), \quad j \leq r-2$

We shall now indicate the proof of this theorem. Observe first that, just as We shall how $\pi_2(U(2m))$ entails the connectedness of the space Π_2^* of connectedness of Π_2^* of Π_2^* of connectedness $D^2 \to U(2m)$ with prescribed restriction to ∂D^2 triviality of $D^2 \to U(2m)$ with prescribed restriction to ∂D^2 , so the fact that $D^2 \to U(2m)$ with prescribed restriction to ∂D^2 , so the fact that imposs maps \mathbb{Z}_2 (a consequence of the theorem) implies that Π_*^* not the fact that π_* (or two connected components. It will appear below that susceptible will appear below that the space we are up of two connected components. It will appear below that the space we are up of two connected components, one contains the space we precisely two connected components, one contained in each of the is also made up on the sale of Π_8^* ; the connected components of Π_8^* are in fact confractible (in the limit as $r \to \infty$) onto the respective components of Π_8^* are in fact confractible (in the limit as $r \to \infty$) they contain.

The present proof (like the more standard ones—see, e.g. [44]) involves The present P_0 of all complex structures (i.e. solutions of $J^2 = -I$), the subset $\Omega_8 \subset SO(16r)$ of all complex structures (i.e. solutions of $J^2 = -I$) the subset J_1, \dots, J_n which anti-commute with the fixed complex structures J_1, \dots, J_n whose which and existence was noted above, and hence with every point of the 6-dimensional standard sphere $S_0^6 \subset S_0^7$ defined in S_0^7 by the equation $a^0 = 0$ (see above). It is clear that $J_8 \in \Omega_8$, that Ω_8 is contained in the vector subspace of $\mathbb{R}^{(16n)}$ orthogonal to the subspace spanned by the vectors E, J1, ..., J8, that $\mathfrak{L}_0^4 \cap \Omega_8 = \{\pm J_8\}$, and, consequently, that $D_0^8 \cap \Omega_8$ is the one-point space [Ja]. By means of a largely computational algebraic argument it can be shown that the space Ω_8 has two connected components, and in fact is diffeomorphic to O(r) (see [44]).

With each point $x \in \Omega_8$ we associate the totally geodesic 8-sphere $S^8(x)$, having the standard 7-sphere S_0^7 as equator, given by

$$S^{8}(x) = \{a^{0}I + a^{1}J_{1} + \dots + a^{7}J_{7} + a^{8}x|(a^{0})^{2} + \dots + (a^{8})^{2} = 1\}.$$

Since the vectors I, J_1, \ldots, J_7, x form an orthonormal subsystem in $\mathbb{R}^{(16r)^2}$, it follows that $S^8(x)$ is the cross-section of $S^q \subset \mathbb{R}^{(16r)^2}$ by a 9-dimensional plane through the origin, and is therefore certainly a totally geodesic submanifold of $SO(16r) \subset \mathbb{R}^{(16r)^2}$. For each $x \in \Omega_8$ we then denote by $D^8(x)$ the upper hemisphere of $S^8(x)$:

$$D^{8}(x) = \{ a \in S^{8}(x) | a = a^{0}I + \dots + a^{7}J_{7} + a^{8}x, a^{8} \ge 0 \}.$$
 (29)

Having in this way associated with each point $x \in \Omega_8$ a totally geodesic dimensional disc (i.e. ball) $D^8(x)$ such that $\partial D^8(x) = S_0^7$ and

$$D^8(x_1) \cap D^8(x_2) = S_0^7$$
 if $x_1 \neq x_2$,

the analogue of position to construct a canonical embedding $i: O(r) \to \Pi_s^*$ the analogue of the embedding i': $U(m) \to \Pi_2$ of the preceding subsection. For each $x \in \Omega_B$ let $\omega(x)$: $D^B \to D^B(x)$ be the unique isometry whose restriction. To the boundary is the standard map $j_0 \circ \hat{i}: S^7 \to S_0^7$ defined above $(=f_0)_{i \neq 0}$.

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this then determines an embedding

$$\omega\colon\Omega_n\to\Pi_n\subset\Pi_n^*$$

Denoting by $\varphi \colon O(r) \to \Omega_R$ the above-mentioned diffeomorphism, we note that define i' by $\Gamma = \omega \circ \varphi \colon O(r) \to \Pi_{\pi} \subset \Pi_{\pi}^{\pi}$

25.15. Lemma. The embedding i': O(r) → Π^{*} induces isomorphisms

 $\beta_n \circ i_n : \pi_j(O(r)) \simeq \pi_{j+n} SO(16r)$ for $j \le r - 2$

coinciding with the usual "isomorphisms of orthogonal periodicity"

PROOF (sketch only). Let $f: S^j \to O(r)$ be a map representing an arbitrary PROOF (sketch out), end of O(r): [f] $\in \pi_j(O(r))$, and consider the element of the jth homotopy group of O(r): [f] $\in \pi_j(O(r))$, and consider the family of maps $\omega(x)$: $D^{8} \to D^{8}(x)$ (see (30)) indexed by the elements $x \neq 0$ family of map $\phi(f(S')) \subset \Omega_8$ where $\phi: O(r) \cong \Omega_8$ is the above-mentioned diffeomorphism Since each map $\omega(x)$ restricts to the standard map $S^7 \to S_0$ on the boundary Since each map $\partial D^8 = S^7$, and the $D^8(x)$ intersect pairwise precisely in S_0^8 , the union of these maps ω(x) yields a map

$$F: S^{j+8} \to \overline{S} = \bigcup_{\mathbf{x} \in \phi(f(S^j))} D^8(\mathbf{x}) \subset SO(16r), \tag{3}$$

with the property that $F|_{S^J} = \varphi \circ f$, where S^J is canonically situated lemma torially) in S^{j+8} . We claim that the correspondence $f \mapsto F$ induces an isomorphism. phim

 $\pi_i(O(r)) \simeq \pi_{j+8}(SO(16r)), \quad j \le r-2,$

coinciding with the isomorphism of orthogonal periodicity as usually constructed. We now summarize very briefly the usual construction (see, e.g. [44])

For each k = 0, 1, ..., 8, denote by Ω_k the set of all complex startures $J \in SO(16r)$ which anti-commute with the fixed complex structures J_1,\ldots,J_{k-1} . (The particular set Ω_k has already figured in our discussion) staring at k = 7 (with a view to working our way down to k = 0), we define a map

$$\gamma_{\gamma} \colon \Omega_8 \to \Omega(SO(16r), J_{\gamma_1} - J_{\gamma})$$

by setting $y_1(x) = D^1(x)$, the minimal geodesic in SO(16r) with midpoint x lying in Ω_7 . (It can be shown that $D^1(x)$ is uniquely defined by these conditions By taking the union of those $D^1(x)$ for which $x \in \varphi(f(S^1))$, we obtain a map $F_{\gamma}: \mathbb{S}^{j+1} \to \Omega_{\gamma}$ extending $\varphi \circ f$:

$$F_{\gamma}\colon S^{j+1}\to \bigcup_{x+\alpha(f)\in S^{j}\eta}D^{1}(x)\subset \Omega_{\gamma}, \qquad F_{\gamma}|_{S^{j}}=\varphi\circ f.$$

It follows via one-dimensional Morse theory (this and the similar steps to ceeding it contain the burden of the usual proof), that the correspondence f - F, induces an isomorphism

$$\pi_j(O(r)) \simeq \pi_{j+1}(\Omega_\gamma)$$

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optimizing in the same vein, one next defines a map $76: y \mapsto D^1(y)$, from Ω_{γ} space of minimal geodesics in SO(16r) joining J_6 to J_7 and J_8 Continuing in the space of minimal geodesics in SO(16r) joining J_6 to J_6 and lying in the space one obtains as before a map the space one obtains as before a map

 $F_6: S^{j+2} \to \bigcup_{\mathbf{x} \in F_1(S^{j+1})} D^1(y) \subset \Omega_6, \qquad F_6|_{S^{j+1}} = F_{7},$

the property (established using Morse theory) that the correspondence induces an isomorphism E-F6 induces an isomorphism

 $\pi_{i+1}(\Omega_7) \simeq \pi_{i+2}(\Omega_6)$

this procedure, one defines further maps $\gamma_5, \dots, \gamma_0$ (taking $J_0 = I_0$), where in particular F_0 maps S^{j+8} to $\Omega_0 = SO(16)$. lierating this F_0 , where in particular F_0 maps S^{j+8} to $\Omega_0 = SO(16r)$. It can now be verified (we omit the details) that

 $\bar{S} = \bigcup_{x \in \varphi(f(S^j))} [\gamma_0 \circ \gamma_1 \circ \cdots \circ \gamma_7(x)]$

(where \bar{S} is as in (32)), and that F_0 essentially coincides with the map F of (32). Since the composite correspondence

 $f \mapsto F_7 \mapsto \cdots \mapsto F_0$

induces an isomorphism

 $\pi_i O(r) \simeq \pi_{j+8} SO(16r), \quad j \leq r-2.$

the isomorphism of orthogonal periodicity as usually constructed), it follows that the correspondence $f \mapsto F$ induces the same isomorphism. Since i' = fwo φ the lemma now follows from the definition of the map F in terms of ω.

To complete the proof of Theorem 25.14 it remains to show that i'(O(r)) =W. As in the unitary case we verify this in two steps.

3.16. Lemma. The map $i': O(r) \to \Pi_8^*$ given by (31) above, satisfies $i'(O(r)) \subset$ W, where W consists of all those functionals in Π_8 at which the Dirichlet functional D has its least value on Π_8 .

This is established by an argument analogous to that used in proving Lemma 25.11, exploiting the inequality $A[f] \le D[f]$ together with the fact that for each $g \in O(r)$ the disc $i'(g) = D^8(\varphi(g))$ is a hemisphere of the crosssection of the sphere $S^q \subset \mathbb{R}^{(16r)^2}$ by a plane through the origin.

25.17. Lemma. We have i'(O(r)) = W.

PROOF Initially the argument proceeds as in the proof of the analogous L_{emma} 25.12. Let $f \in W$, i.e. suppose D[f] is least among all values taken by $0 \text{ on } \Pi_8$, and let $f \in W$, i.e. suppose D[f] is least among at 0 ordinates of standard 0 ordinates or 0 ordinates ord O ordinates on D^8 it is easy to see that $A[f_0] = D[f_0]$. Furthermore, in view

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of the boundary conditions on admissible maps $D^a \to SO(16r)$ (namely that $\partial_0^2 = S^0 \to S^0$) be mapped canonically onto $S_0^2 = SO(16r)$), the value $A[f_a]$ is the least taken by A on Π_2 . Hence we can infer, via the general inequality $A[h] \le D[h]$, that $D[f] = A[f] = A[f_0].$

It follows that $f(D^n)$, like $f_0(D^n)$, must be a hemisphere of a central-plane cross-section of $S^q \subset \mathbb{R}^{(160)^2}$ (of course having S_0^7 as its boundary: $\partial_t f_0(D^n)$ as $(f_0(D^n))$ and $(f_0(D^n))$ and $(f_0(D^n))$ are the second to the second section of $(f_0(D^n))$ and $(f_0(D^n))$ are the second second section of $(f_0(D^n))$ and $(f_0(D^n))$ are the second second section of $(f_0(D^n))$ and $(f_0(D^n))$ are the second second

 $S_0^{(s)}$. Now let x be a vector in $f(D^n)$ orthogonal to the eight vectors I, J_1, \dots, J_n . It follows from the above description of $f(D^n)$ that

$$f(D^8) = \{a^0I + a^1J_1 + \dots + a^7J_7 + a^8x | (a^0)^2 + \dots + (a^8)^2 = 1, a^8 \ge 0\}$$

Since $f(D^8)$ is a central-plane cross-section of S^q with the points $\pm I$ as antipodal points of its boundary S_0^7 , there is a unique minimal geodesic arc $\gamma(t)$, $0 \le t \le 1$, of SO(16r) (actually of S^q), such that $\gamma(0) = I$, $\gamma(\frac{1}{2}) = x$, $\gamma(1) = -I$ (namely half the circle obtained by sectioning S^q by the 2-dimensional plane spanned by the vectors I and x). Now any geodesic γ in SO(16r) with $\gamma(0) = I$, has the form $\gamma(t) = \exp(\pi t A)$ for some $A \in SO(16r)$. By means of a fairly direct matrix computation (see [44]) it can be shown that if $\gamma(1) = -I$ and γ is minimal, then A is a complex structure: $A^2 = -I$. It follows that

$$\gamma(t) = (\cos \pi t)I + (\sin \pi t)A,$$

whence $\gamma(\frac{1}{2})$ (= A) is a complex structure. Since in our situation $\gamma(\frac{1}{2}) = x$, we infer that x is a complex structure, i.e. $x \in \Omega_1(16r)$. By (33)

$$\frac{1}{\sqrt{2}}(x+J_k)\in f(D^8)\subset SO(16r)\qquad \text{for}\quad k=1,\ldots,7,$$

whence

$$\frac{1}{\sqrt{2}}(x+J_k) \in SO(16r) \cap so(16r) = \Omega_1(16r), \qquad k = 1, ..., 7,$$

i.e. $\frac{1}{2}(x+J_k)^2=-1$. Hence $xJ_k+J_kx=0$, i.e. $x\in\Omega_8$. We conclude that $f(D^8)=D^8(x)$ (see (29)), so that $f\in i'(O(r))$, as claimed. This completes the proof of the lemma and with it our sketch of the proof of Theorem 25.14.

From the proof of orthogonal periodicity given for instance in [44], it emerges that the stable orthogonal group O has the homotopy type of the 4-fold loop space $\Omega\Omega\Omega\Omega$ Sp, and that the stable symplectic group Sp has the homotopy type of the 4-fold loop space $\Omega\Omega\Omega\Omega$ O, whence one obtains an analogous "symplectic periodicity" theorem. We leave to the reader the detailed formulation of this theorem incorporating an approach to its validation via the higher-dimensional calculus of variations, analogous to that described above in the unitary and orthogonal cases.

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PROOF. We need defined in (29), we there is an element

the map f_0 : D^8 form $g = J_8 y$, J_0 or $yxy = -J_8$ Now in [44]

by finding two the matrices of tions $X_1 \rightarrow X_2$ readily that G, the group of a $J^{(X)} = -J_8$ for

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5. Collet's Bott Periodicity and Higher-Dimensional Varietional Prote

point the following homotopy equivalences QQQ SO;

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80 - 0000 Sp (see §10.4). is $y_i = (KKL)$ by the first eight homotopy groups x_iO_i $i=0,\ldots,7$, namely, in order, O_i O_i 1, I, O, Z, O, O, O, Z

In the preceding subsection, in the course of establishing unitary periods that the space $i'(U(m)) \subset \Pi$, is account. In the precessing that the space $i'(U(m)) \subset \Pi_2$ is actually the orbit of spaces $f(U(m)) \subset \Pi_2$ is actually the orbit of spaces $f(U(m)) \subset \Pi_2$ in actually the orbit of solutive we have a point $f_0 \in \Pi_2$ under the conjugating action of the group $G = \Pi_2$ in the present situation of only. particular $C(C \in U(m)) \subset U(2m)$. In the present situation of orthogonal periodicity of this result holds also (not however needed in a | © CIC e this result holds also (not however needed in the above proof of schogonal periodicity).

Proposition. The two connected components of the space $W = \Gamma(O(r)) = \frac{1}{r} \left(\frac{1}{r} \right) \left(\frac{1}$ 11. are orbits of the conjugating action on 112 of the group

$$G = J_8\Omega_8 \subset SO(16r)$$
 $(G \simeq O(r)).$ (34)

PROOF. We need to show that given any one of the totally geodesic balls D⁸(x) defined in (29), where $x \in \Omega_8$ is in the same connected component of Ω_8 as I_8 . there is an element $g \in G$ such that $g \times g^{-1} = J_8$, since then

$$gD^8(x)g^{-1} = D^8(J_8) = D_0^8$$

so that the corresponding map $\omega(x)$: $D^8 \to D^8(x)$ (see (30)) is obtained from the map $f_0: D^8 \to D_0^8 \subset SO(16r)$, by conjugating by g. Since each $g \in G$ has the form $g = J_8 y$, $y \in \Omega_8$, the equation $g \times g^{-1} = J_8$ is equivalent to $y \times y^{-1} = J_8$. or $yxy = -J_x$ (since $y^{-1} = -y$).

Now in [44] the aforementioned diffeomorphism $\Omega_k \cong O(r)$ is constructed by finding two r-dimensional subspaces X_1, X_2 of \mathbb{R}^{16r} with the property that the matrices of the form $J_{\gamma}y,\,y\in\Omega_8$, act as (distinct) orthogonal transformasions $X_1 \rightarrow X_2$, and in fact account for all such transformations. It follows teadily that G, as defined in (34), is indeed a group, since it is identifiable with the group of all orthogonal self-transformations of X_1 (or X_2). The equation $pxy = -J_8$ for which we seek a solution $y \in \Omega_8$, is equivalent to

$$(J_{\gamma}y)(J_{\gamma}x^{-1})^{-1}J_{\gamma}y = -J_{\gamma}J_{8},$$

whence we see that the existence of such a solution is equivalent to the existence of a solution $z \in O(r)$ of the equation

$$zAz = B$$

for each (fixed) pair A, B of elements in the same connected component of only Survey of the lay the onus Oir. Since every such equation does indeed have a solution (we lay the onus of periferance). of verification of this on the reader), the proposition follows.

In summary, the idea underlying the above approach to unitary and onhogonal periodicity (and borne out by the above discussion), is that both

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these results should arise via the same mechanism (namely the higher these results of variations applied to the Dirichlet functions these results should arise the birichlet functional, the higher dimensional calculus of variations applied to the Dirichlet functional, the dimensional calculus of variation only on the space on which that functional, the particular outcome depending only on the space of maps of 2-dimensional discs in the unitary could particular outcome depending particular outco of 8-dimensional discs in the orthogonal case.

8-dimensional discs in the orthogonal periodicity ultimately reliable to the reader will have observed that, this uniform approach notwithstanding.

The reader will have and orthogonal periodicity ultimately reliable to the periodicity ultimately and the periodicity ultimately a The reader will have our more than a continuous periodicity ultimately relied on the our proofs of both unitary and orthogonal periodicity ultimately relied on the course of the continuous periodicity unitary and orthogonal periodicity ultimately relied on the continuous periodicity unitary and orthogonal periodicity ultimately relied to the continuous periodicity relie our proofs of both unitary on the more standard arguments of one-dimensional Morse theory. It would be of more standard arguments of one-dimensional Morse theory. It would be of more standard arguments if a direct uniform proof (using the Dirichlet functional) considerable interest if a direct uniform proof (using the Dirichlet functional) could be found which avoided any appeal to the Morse theory of the one. dimensional functionals of action and length. Such a proof would in fact be dimensional functional directly the contractibility of the 2m-dimensional feasible if one could establish directly the contractibility of the 2m-dimensional feasible if one could be skeleton of Π_8 in the orthogonal case) onto skeleton of H_2 (of the space i'(U(m)) (resp. i'(O(r))) of points where the Dirichlet functional has its least value. It is precisely a contractibility result of this kind in has its least tandent the classical (i.e. one-dimensional) Morse theory, of the path space Π_1 The classical I_{2m} , I_{2m} , which allows one to obtain the crucial isomorphism $\pi_{i-1}(G_{2m,m}^{\mathbb{C}}) \simeq \pi_{j-1}(\Pi_1)$. However, the analogous result for the higherdimensional calculus of variations applied to the Dirichlet functional is not available, a lack essentially due to the typical sort of difficulty which arises in higher-dimensional problems of the "Plateau" type, whereby the higherdimensional functional in question may be degenerate on certain sets of positive measure contained in the extremal submanifolds.

§26. Morse Theory and Certain Motions in the Planar n-Body Problem

In this section we shall show how Morse theory may be applied to the analysis of certain motions in the many-body problem of celestial mechanics, i.e. the problem of describing the motions of n objects acting on each other by means of mutual (gravitational) forces. It is well known that to a first approximation the planets of our solar system move in a plane, the so-called "plane of the ecliptic". Furthermore, to a large degree of accuracy the centre of mass of the whole system may be identified with the sun's position, and the motion of the system may be assumed to be governed by the Newtonian gravitational potential of classical mechanics. The motion is then determined in the usual way by initial conditions, namely the positions and velocities of the gravitating masses (considered as point-particles) at some chosen initial instant of time As is also well known, the general solution of the resulting system of differential equations is exceedingly complex: for instance, by the classical results of Bruns and Poincaré, there are no single-valued analytic integrals of the system (i.e. expressions in the position variables and their derivatives which are constant at all times) other than the "classical" ones (the integrals of energy, angular

and of motive and the solution can provide the solution of t It has not continued as the subclasses of sol address of sold success of sold success of sold success is that of a sold success of sold succ self successful and under additions revenue together Whe cliptic. Thus, in of the purchanged and pensantis centre of mass about to as "circula resoluty). It is a rema loody" solutions of the gilical points of a ce and that the resulting have seen (in for insta points, can then be U geometric structure For instance, the system of point-mas patible with some " every arrangement solution.) Clearly t masses of the n par one of the particles are linked by the configurations are We now turn to problem of celesti real, positive num rather point-parti plane. We may cl the centre of mass on the plane will ordinates, by $(x_j,$ assumption that relation $\sum_{j=1}^{n} m_j 2$ subspace (see beld hyperplane) of th

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and the "phase si bundle T*M ≅ ordinates are El and of motion of the centre of mass). (However, in the case $\eta = 2$

omplete solution can be described and the case n = 2 it has nevertheless proved possible to delineate among all solutions certain it has neverthelesses of solutions admitting a relatively simple described contains the case of the co-called private the case of the co-called private the case of thas nevertheress provided to define a mong all solutions certain the subclasses of solutions admitting a relatively simple description. One partial subclass is that of all so-called "rigid-body" solutions, i.e., those provided the additional account. patural subclasses of all so-called "rigid-body" solutions, i.e. those particular subclass is that of all so-called "rigid-body" solutions, i.e. those particular such subclass is that of all so-called "rigid-body" solutions, i.e. those particular such such as the subclasses of the path subclass is that of the additional assumption that all the masses of the solutions obtained under the additional assumption that all the masses of the solutions obtained about the sun, at the same angular speed, in the plane assets of the colliptic. Thus, in this special case, the relative positions of the plane system revolve to system revolves to the system revolves of the eliptic. Thus, in this special case, the relative positions of the masses of the anchanged and the whole system revolves like a (plane) of the ecliptic. In the whole system revolves like a (planar) rigid body main unchanged mass. In the literature such periodic solutions are sometimes about its centre of mass. In the literature such periodic solutions are sometimes about its as "circular orbits" (with all particles having the about its centre or circular orbits" (with all particles having the same angular referred to as "circular orbits" (with all particles having the same angular referred to as a remarkable fact that the description of such planar rigidvelolcity). It is a velocity of the many-body problem, reduces to the description of the body solutions of a certain function, possibly Morse, on body solutions of a certain function, possibly Morse, on a smooth manifold, entical points of a smooth manifold, and that the resulting topological information about the manifold which we and that the restance §15) to be obtainable from knowledge of the critical have seen the be used to draw important qualitative inferences about the geometric structure of the "circular" solutions under investigation.

For instance, the following question is of considerable interest: Given a system of point-masses, what planar configurations of the system are comnatible with some "rigid-body" solution? (It is intuitively clear that far from every arrangement of the particles will quality for some planar "rigid-body" solution.) Clearly the admissible configurations will be determined by the masses of the n particles of the system. (In the particular case where all but one of the particles have the same mass, it turns out that these configurations are linked by the action of a certain discrete group of symmetries.) Such configurations are sometimes called "relative equilibria" of the system.

We now turn to the precise formulation of the problem. The planar n-body problem of celestial mechanics is determined essentially by a sequence of n real, positive numbers m_1, \ldots, m_n , representing the masses of the n bodies, or rather point-particles, situated at n points of the (2-dimensional) Euclidean plane. We may clearly suppose that the origin O coincides at all times with the centre of mass of the system of particles. The position of the jth particle on the plane will then be denoted, in terms of the standard Euclidean coordinates, by (x_j, y_j) , or by the complex co-ordinate $z_j = x_j + iy_j$. From the assumption that the origin is the centre of mass we infer immediately the relation $\sum_{j=1}^{n} m_j z_j = 0$. The "configuration space" of the system is a certain subspace $(1 + m_j)^2 = 0$. Subspace (see below) contained in the linear subspace $M = M^{2n-2}$ (or complex linear). hyperplane) of the Euclidean space $\mathbb{R}^{2n} = \mathbb{C}^n$, given by

 $M^{2n-2} = \left\{ (z_1, \ldots, z_n) \in \mathbb{R}^{2n} \left| \sum_{j=1}^n m_j z_j = 0 \right\} \right\},$

and the "phase space" of the system is (a certain submanifold of) the cotangent bundle $T^*M \cong M \times M$ (the direct square of M). (Note that since the coordinates are Γ). ordinates are Euclidean, T and T^* may be identified; see Part 1, §19.1.)

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