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Steven Roman

An Introduction to the Language of Category Theory

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An Introduction to the Language of Category Theory

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To Donna

Preface

The purpose of this little book is to provide an introduction to the *basic concepts* of category theory. It is intended for the graduate student, advanced undergraduate student, nonspecialist mathematician or scientist working in a need-to-know area. The treatment is abstract in nature, with examples drawn mainly from abstract algebra. Although there are no formal prerequisites for this book, a basic knowledge of elementary abstract algebra would be of considerable help, especially in dealing with the exercises.

Category theory is a relatively young subject, founded in the mid 1940s, with the lofty goals of *unification*, *clarification* and *efficiency* in mathematics. Indeed, Saunders MacLane, one of the founding fathers of category theory (along with Samuel Eilenberg), says in the first sentence of his book *Categories for the Working Mathematician*: "Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows." Of course, unification and simplification are common themes throughout mathematics.

To illustrate these concepts, consider three sets with a binary operation:

1) the set $\mathbb{R}^*$ of nonzero real numbers under multiplication
2) the set $\mathcal{M}(n, k)$ of $n \times k$ matrices over the complex numbers under addition and
3) the set $\mathcal{B}$ of bijections of the integers under composition.

Now, very few mathematicians would take the time to prove that inverse elements are unique in each of these cases—They would simply note that each of these is an example of a *group* and prove in one quick line that the inverse of *any* element in *any* group is unique, to wit, if α and β are inverses for the group element a, then

$$\alpha = \alpha 1 = \alpha(\alpha\beta) = (\alpha a)\beta = 1\beta = \beta$$

This at once *clarifies* the role of uniqueness of inverses by showing that this property has *nothing whatever to do with real numbers, matrices or bijections*. It has to do only with associativity and the identity property. This also *unifies* the concept of uniqueness of inverses because it shows that uniqueness of inverses in each of these three cases is really a single concept. Finally, it makes life more *efficient* for mathematicians because they can prove uniqueness of inverses for *all* groups *in one fell swoop*, as it were.

Category theory attempts to do the same for *all* of mathematics (well, perhaps not *all*) as group theory does for the case described above.

But there is a problem. It has been my experience that most students of mathematics and the sciences (and even some mathematicians) find category theory to be very challenging indeed, primarily due to its extremely abstract nature. We must remember that the vast majority of students are *not* seeking to be category theorists—They are seeking a "modest" understanding of the *basic concepts* of category theory so that they can apply these ideas to their chosen area of specialty. This book attempts to supply this understanding in as gentle a manner as possible.

We envision this book as being used as independent reading or as a supplementary text for graduate courses in other areas. It could also be used as the textbook for either a short course or a leisurely one-quarter course in category theory.

The Five Basic Concepts of Category Theory

It can be said that there are five *basic* concepts in category theory, namely,

- *Categories*
- *Functors*
- *Natural transformations*
- *Universality*
- *Adjoints*

Some would argue that each of these concepts was "invented" or "discovered" in order to produce the next concept in this list. For example, Saunders MacLane himself is reported to have said: "I did not invent category theory to talk about functors. I invented it to talk about natural transformations."

Whether this be true or not, many students of mathematics are finding that the language of category theory is popping up in many of their classes in abstract algebra, lattice theory, number theory, differential geometry, algebraic topology and more. Also, category theory has become an important topic of study for many computer scientists and even for some mathematical physicists. Hopefully, this book will fill a need for those who require an understanding of the basic concepts of the subject. If the need or desire should arise, one can then turn to more lengthy and advanced treatments of the subject.

A Word About Definitions

To my mind, there are two types of definitions. *Standard definitions* are, well, standard. They are intended to be in common usage and last through the ages. However, after about 40 years of teaching and the writing of about 40 books, I have come to believe in the virtue of *nonstandard, temporary, ad hoc definitions* that are primarily intended for pedagogical purposes, although one can hope that some ad hoc definitions turn out to be so useful that they eventually become a standard part of the subject matter.

Let me illustrate a nonstandard definition. One of the most important (some would say *the* most important) concepts in category theory is that of an *adjoint*. There are left adjoints and right adjoints, but the two concepts come together in something called an *adjunction*.

Now, there are many approaches to discussing adjoints and adjunctions. In my experience, adjunctions are usually just defined without much preliminary leg work. However, one of the goals of this book is to make the more difficult concepts, such as that of an adjunction a bit more palatable by "sneaking up" on them, as it were. In order to do this with adjunctions, we gently transition through the following concepts,

$$\text{initial objects in comma categories} \to \text{universality} \to \text{naturalness} \to \text{binaturalness (adjunctions)}$$

During this transition process, we will find it extremely useful to use certain maps that, to my knowledge, do not have a specific name. So this is the perfect place to introduce a nonstandard definition, which in this case is the *mediating morphism map*.

The only downside to making nonstandard definitions is that they are not going to be recognized outside the context of this book and therefore must be used very circumspectly. But I think that is a small price to pay if they help the learning process.

That said, I will use nonstandard definitions only as often as I feel absolutely necessary and will try to identify them as such upon first use, either by the term "nonstandard" or by a phrase such as "we will refer to ...".

Thanks

I would like to thank my students Phong Le, Sunil Chetty, Timothy Choi, Josh Chan, Tim Tran and Zachary Faubion, who attended my lectures on a much expanded version of this book and offered many helpful suggestions.

Steven Roman

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Categories

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Foundations

Before giving the definition of a category, we must briefly (and somewhat informally) discuss a notion from the foundations of mathematics. In category theory, one often wishes to speak of "the category of (all) sets" or "the category of (all) groups." However, it is well known that these descriptions cannot be made precise within the context of sets alone.

In particular, not all "collections" that one can define informally through the use of the English language, or even formally through the use of the language of set theory, can be considered sets without producing some well-known logical paradoxes, such as the Russell paradox of 1901 (discovered by Zermelo a year earlier). More specifically, if $\phi(x)$ is a well-formed formula of set theory, then the collection

$$X = \left\{\text{sets } \ x \,\middle|\, \phi(x) \text{ is } \ \text{true}\right\}$$

cannot always be viewed as a set. For example, the family of all sets, or of all groups, cannot be considered a set. Nonetheless, it is desirable to be able to apply some of the operations of sets, such as union and cartesian product, to such families. One way to achieve this goal is through the notion of a **class**. Every set is a class and the classes that are not sets are called **proper classes**. Now we can safely speak of the *class* of all sets, or the *class* of all groups. Classes have many of the properties of sets. However, while every set is an element of another set, no class can be an element of another class. We can now state that the family X defined above is a class without apparent contradiction.

Another way to avoid the problems posed by the logical paradoxes is to use the concept of a set $\mathcal{U}$ called a **universe**. The elements of $\mathcal{U}$ are called **small sets**. Some authors refer to the *subsets* of $\mathcal{U}$ as *sets* and some use the term *classes*. In order to carry out "ordinary mathematics" within the universe $\mathcal{U}$, it is assumed to be closed under the basic operations of set theory, such as the taking of ordered pairs, power sets and unions.

These two approaches to the problem of avoiding the logical paradoxes result in essentially the same theory and so we will generally use the language of sets and classes, rather than universes.

The Definition

We can now give the definition of a category.

Definition

A **category** $\mathcal{C}$ *consists of the following:*

1) (**Objects**) *A class* **Obj**$(\mathcal{C})$ *whose elements are called the* **objects**. *It is customary to write* $A \in \mathcal{C}$ *in place of* $A \in$ **Obj**$(\mathcal{C})$.
2) (**Morphisms**) *For each (not necessarily distinct) pair of objects* $A, B \in \mathcal{C}$, *a set* $\hom_{\mathcal{C}}(A, B)$, *called the* **hom-set** *for the pair* (A, B). *The elements of* $\hom_{\mathcal{C}}(A, B)$ *are called* **morphisms**, **maps** *or* **arrows** *from* A *to* B. *If* $f \in \hom_{\mathcal{C}}(A, B)$, *we also write*

$$f\colon A \to B \quad \text{or} \quad f_{AB}$$

The object $A = \operatorname{dom}(f)$ *is called the* **domain** *of* f *and the object* $B = \operatorname{codom}(f)$ *is called the* **codomain** *of* f.
3) *Distinct hom-sets are disjoint, that is,* $\hom_{\mathcal{C}}(A, B)$ *and* $\hom_{\mathcal{C}}(C, D)$ *are disjoint unless* $A = C$ *and* $B = D$.
4) (**Composition**) *For* $f \in \hom_{\mathcal{C}}(A, B)$ *and* $g \in \hom_{\mathcal{C}}(B, C)$ *there is a morphism* $g \circ f \in \hom_{\mathcal{C}}(A, C)$, *called the* **composition** *of* g *with* f. *Moreover, composition is associative:*

$$f \circ (g \circ h) = (f \circ g) \circ h$$

whenever the compositions are defined.
5) (**Identity morphisms**) *For each object* $A \in \mathcal{C}$ *there is a morphism* $1_A \in \hom_{\mathcal{C}}(A, A)$, *called the* **identity morphism** *for* A, *with the property that if* $f_{AB} \in \hom_{\mathcal{C}}(A, B)$ *then*

$$1_B \circ f_{AB} = f_{AB} \quad \text{and} \quad f_{AB} \circ 1_A = f_{AB}$$

The class of all morphisms of $\mathcal{C}$ *is denoted by* **Mor**$(\mathcal{C})$. □

A variety of notations are used in the literature for hom-sets, including

$$(A, B), \quad [A, B], \quad \mathcal{C}(A, B) \quad \text{and} \quad \operatorname{Mor}(A, B)$$

(We will drop the subscript $\mathcal{C}$ in $\hom_{\mathcal{C}}$ when no confusion will arise.)

We should mention that not all authors require property 3) in the definition of a category. Also, some authors permit the hom-sets to be classes. In this case, the categories for which the hom-classes are sets is called a **locally small category**. Thus, all of our categories are locally small. A category $\mathcal{C}$ for which both the class **Obj**$(\mathcal{C})$ and the class **Mor**$(\mathcal{C})$ are sets is called a **small category**. Otherwise, $\mathcal{C}$ is called a **large category**.

Two arrows belonging to the same hom-set $\hom(A, B)$ are said to be **parallel**. We use the phrase "f is a morphism **leaving** A" to mean that the domain of f is A and "f is a morphism **entering** B" to mean that the codomain of f is B.

When we speak of a composition $g \circ f$, it is with the tacit understanding that the morphisms are **compatible**, that is, $\operatorname{dom}(g) = \operatorname{codom}(f)$.

The concept of a category is *very general*. Here are some examples of categories. In most cases, composition is the "obvious" one. We suggest that you just skim this list of examples at this point. If you are not familiar with some of the concepts in these example (such as smooth manifolds), not to worry. The purpose of this list is to give you a general idea of the wide range of categories in mathematics.

Example 1

The Category **Set** of Sets

Obj is the class of all sets.

hom(A, B) is the set of all functions from A to B.

The Category **Mon** of Monoids

Obj is the class of all monoids.

hom(A, B) is the set of all monoid homomorphisms from A to B.

The Category **Grp** of Groups

Obj is the class of all groups.

hom(A, B) is the set of all group homomorphisms from A to B.

The Category **AbGrp** of Abelian Groups

Obj is the class of all abelian groups.

hom(A, B) is the set of all group homomorphisms from A to B.

The Category $\mathbf{Mod}_R$ of R-modules, where R is a ring

Obj is the class of all R-modules.

hom(A, B) is the set of all R-maps from A to B.

The Category $\mathbf{Vect}_F$ of Vector Spaces over a Field F

Obj is the class of all vector spaces over F.

hom(A, B) is the set of all linear transformations from A to B.

The Category **Rng** of Rings

Obj is the class of all rings (with unit).

hom(A, B) is the set of all ring homomorphisms from A to B.

The Category **CRng** of Commutative Rings with identity

Obj is the class of all commutative rings with identity.

hom(A, B) is the set of all ring homomorphisms from A to B.

The Category **Field** of Fields

Obj is the class of all fields.

hom(A, B) is the set of all ring embeddings from A to B.

The Category **Poset** of Partially Ordered Sets

Obj is the class of all partially ordered sets.

hom(A, B) is the set of all **monotone** functions from A to B, that is, functions $f : P \to Q$ satisfying

$$p \le q \quad \Rightarrow \quad f(p) \le f(q)$$

The Category **Rel** of relations
Obj is the class of all sets.
hom(A, B) is the set of all binary relations from A to B, that is, subsets of the cartesian product $A \times B$.

The Category **Top** of Topological Spaces
Obj is the class of all topological spaces.
hom(A, B) is the set of all continuous functions from A to B.

The Category **SmoothMan** of Manifolds with Smooth Maps
Obj is the class of all manifolds.
hom(A, B) is the set of all smooth maps from A to B. □

■ Example 2

The category of *all* categories does not exist, on foundational grounds. The well-known Russell paradox shows that the set of all sets does not exist and an analogous argument has been constructed to show that the category of all categories does not exist. However, the argument is a bit involved and is not really in the spirit of this introductory book, so we will not go into the details. On the other hand, the class $\mathcal{S}$ of all *small* categories does form the objects of another category, whose morphisms are called *functors*, to be defined a bit later in the chapter. □

Here are some slightly more unusual categories.

■ Example 3

Let F be a field. The category $\mathbf{Matr}_F$ of matrices over F has objects equal to the set $\mathbb{Z}^+$ of positive integers. For $m, n \in \mathbb{Z}^+$, the hom-set hom(m, n) is the set of all $n \times m$ matrices over F, composition being matrix multiplication. Why do we reverse the roles of m and n? Well, if $M \in$ hom(m, n) and $N \in$ hom(n, k), then M has size $n \times m$ and N has size $k \times n$ and so the product NM makes sense and has size $k \times m$, that is, it belongs to hom(m, k), as required. Incidentally, this is a case in which the category is named after its morphisms, rather than its objects. □

■ Example 4

A single monoid M defines a category with a single object M, where each element is a morphism. We define the composition $a \circ b$ to be the product ab. This example applies to other algebraic structures, such as groups. All that is required is that there be an identity element and that the operation be associative. □

■ Example 5

Let $(P, \leq)$ be a partially ordered set. The objects of the category $\mathbf{Poset}(P, \leq)$ are the elements of P. Also, hom(a, b) is empty unless $a \leq b$, in which case hom(a, b) contains a single element, denoted by ab. Note that the hom-sets specify the relation $\leq$ on P. As to composition, there is really only one choice: If $ab\colon a \to b$ and $bc\colon b \to c$ then it follows that $a \leq b \leq c$ and so $a \leq c$, which implies that hom$(a, c) \neq \emptyset$. Thus, we set $bc \circ ab = ac$. The hom-set hom(a, a) contains only the identity morphism for the object a.

As a specific example, you may recall that each positive natural number $n \in \mathbb{N}$ is defined to be the set of all natural numbers that preceed it:

$$n = \{0, 1, \ldots, n - 1\}$$

and the natural number 0 is defined to be the empty set. Thus, natural numbers are ordered by membership, that is, $m < n$ if and only if $m \in n$ and so n is the set of all natural numbers *less than* n. Each natural number n defines a category whose objects are its elements and whose morphisms describe this order relation. The category n is sometimes denoted by bold face $\boldsymbol{n}$.□

Example 6

A category for which there is *at most one* morphism between every pair of (not necessarily distinct) objects is called a **preordered category** (some authors use the term **thin category**). If $\mathcal{C}$ is a thin category, then we can use the *existence* of a morphism to define a binary relation on the objects of $\mathcal{C}$, namely, $A \preceq B$ if there exists a morphism from A to B. It is clear that this relation is reflexive and transitive. Such relations are called **preorders**. (The term *preorder* is used in a different sense in combinatorics.)

Conversely, any preordered class $(P, \preceq)$ is a category, where the objects are the elements of P and there is a morphism f_{AB} from A to B if and only if $A \preceq B$ (and there are no other morphisms). Reflexivity provides the identity morphisms and transitivity provides the composition.

More generally, if $\mathcal{C}$ is any category, then we can use the *existence* of a morphism to define a preorder on the objects of $\mathcal{C}$, namely, $A \preceq B$ if there is at least one morphism from A to B.□

Example 7

Consider a deductive logic system, such as the propositional calculus. We can define two different categories as follows. In both cases, the well-formed formulas (wffs) of the system are the objects of the category. In one case, there is one morphism from the wff α to the wff β if and only if we can deduce β given α. In the other case, we define a morphism from α to β to be a *specific deduction* of β from α, that is, a specific ordered list of wffs starting with α and ending with β for which each wff in the list is either an axiom of the system or is deducible from the previous wffs in the list using the rules of deduction of the system. □

The Categorical Perspective

The notion of a category is extremely general. However, the definition is *precisely* what is needed to set the correct stage for the following two key tenets of mathematics:

1) Morphisms (e.g. linear transformations, group homomorphisms, monotone maps) play an essentially equal role alongside the mathematical structures that they morph (e.g. vector spaces, groups, partially ordered sets).
2) Many mathematical notions are best described in terms of morphisms between structures rather than in terms of the individual elements of these structures.

In order to implement the second tenet, one must grow accustomed to the idea of focusing on the appropriate *maps* between mathematical structures and not on the *elements* of these structures. For example, as we will see in due course, such important notions as a basis for a vector space, a direct product of vector spaces, the field of fractions of an integral domain and the quotient of a group by a normal subgroup can be described using maps rather than elements. In fact, many of the most important properties of these notions follow from their morphism-based descriptions.

Note also that one of the consequences of the second tenet is that important mathematical notions tend to be defined *only up to isomorphism*, rather than uniquely.

An immediate example seems in order, even though it may take some time (and further reading) to place in proper perspective.

Example 8

Let V and W be vector spaces over a field F. The external direct product of V and W is usually defined in elementary linear algebra books as the set of ordered pairs

$$V \times W = \{(v, w) \,|\, v \in V, w \in W\}$$

with componentwise operations:

$$(v, w) + (v', w') = (v + v', w + w')$$

and

$$r(v, w) = (rv, rw)$$

for $r \in F$. One then defines the **projection maps**

$$\rho_1 \colon V \times W \to V \quad \text{and} \quad \rho_2 \colon V \times W \to W$$

by

$$\rho_1(v, w) = v \quad \text{and} \quad \rho_2(v, w) = w$$

However, the importance of these projection maps is not always made clear, so let us do this now.

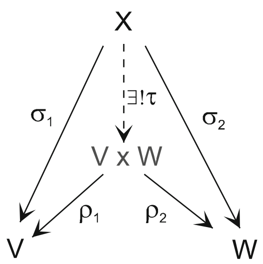

Figure 1

As shown in Figure 1, the ordered triple $(V \times W, \rho_V, \rho_W)$ has the following **universal property**: Given any vector space X over F and any "projection-like" pair of linear transformations

$$\sigma_1 \colon X \to V \quad \text{and} \quad \sigma_2 \colon X \to W$$

from X to V and W, respectively, there is a *unique* linear transformation $\tau\colon X \to V \times W$ for which

$$\rho_1 \circ \tau = \sigma_1 \quad \text{and} \quad \rho_2 \circ \tau = \sigma_2$$

Indeed, these two equations uniquely determine $\tau(x)$ for any $x \in X$ because

$$\tau(x) = (\rho_1(\tau(x)), \rho_2(\tau(x))) = (\sigma_1(x), \sigma_2(x))$$

It remains only to show that τ is linear, which follows easily from the fact that σ_1 and σ_2 are linear. Now, the categorical perspective is that this universal property is the essence of the direct product, at least up to isomorphism. In fact, it is not hard to show that if an ordered triple

$$(U, \lambda_1\colon U \to V, \lambda_2\colon U \to W)$$

has the universal property described above, that is, if for any vector space X over F and any pair of linear transformations

$$\sigma_1\colon X \to V \quad \text{and} \quad \sigma_2\colon X \to W$$

there is a *unique* linear transformation $\tau\colon X \to U$ for which

$$\lambda_1 \circ \tau = \sigma_1 \quad \text{and} \quad \lambda_2 \circ \tau = \sigma_2$$

then U and $V \times W$ are isomorphic as vector spaces. Indeed, in some more advanced treatments of linear algebra, the direct product of vector spaces is *defined* as *any* triple that satisfies this universal property. Note that, using this definition, *the direct product is defined only up to isomorphism*.

If this example seems to be a bit overwhelming now, don't be discouraged. It can take a while to get accustomed to the categorical way of thinking. It might help to redraw Figure 1 a few times without looking at the book. □

Functors

If we are going to live by the two main tenets of category theory described above, we should discuss morphisms between categories! Structure-preserving maps between categories are called *functors*. At this time, however, there is much to say about categories as individual entities, so we will briefly describe functors now and return to them in detail in a later chapter.

The unabridged dictionary defines the term *functor*, from the New Latin *functus* (past participle of *fungi*: to perform) as "something that performs a function or operation." The term *functor* was apparently first used by the German philosopher Rudolf Carnap (1891–1970) to represent a special type of function sign. In category theory, the term *functor* was introduced by Samuel Eilenberg and Saunders Mac Lane in their paper *Natural Isomorphisms in Group Theory* [8].

Since the structure of a category consists of *both* its objects and its morphisms, a functor should map objects to objects and morphisms to morphisms. This requires two different maps. Also, there are two versions of functors: *covariant* and *contravariant*.

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ *is a pair of functions (as is customary, we use the same symbol F for both functions):*

1) *The* **object part** *of the functor*

$$F\colon \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{D})$$

maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$

2) *The* **arrow part**

$$F\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{D})$$

maps morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ as follows:

a) *For a* **covariant functor**,

$$F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$$

for all $A, B \in \mathcal{C}$, that is, F maps a morphism $f\colon A \to B$ in $\mathcal{C}$ to a morphism $Ff\colon FA \to FB$ in $\mathcal{D}$.

b) *For a* **contravariant functor**,

$$F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FB, FA)$$

for all $A, B \in \mathcal{C}$, that is, F maps a morphism $f\colon A \to B$ in $\mathcal{C}$ to a morphism $Ff\colon FB \to FA$ in $\mathcal{D}$. (Note the reversal of direction).

We will refer to the restriction of F to $\hom_{\mathcal{C}}(A, B)$ *as a* **local arrow part** *of F.*

3) *Identity and composition are preserved, that is,*

$$F1_A = 1_{FA}$$

and for a covariant functor,

$$F(g \circ f) = Fg \circ Ff$$

and for a contravariant functor,

$$F(g \circ f) = Ff \circ Fg$$

whenever all compositions are defined. □

As is customary, we use the same symbol F for both the object part and the arrow part of a functor. We will also use a double arrow notation for functors. Thus, the expression $F\colon \mathcal{C} \Rightarrow \mathcal{D}$

implies that $\mathcal{C}$ and $\mathcal{D}$ are categories and is read "F is a functor from $\mathcal{C}$ to $\mathcal{D}$." (For readability's sake in figures, we use a thick arrow to denote functors.)

A functor $F\colon \mathcal{C} \Rightarrow \mathcal{C}$ from $\mathcal{C}$ to itself is referred to as a **functor on** $\mathcal{C}$. A functor $F\colon \mathcal{C} \Rightarrow$ **Set** is called a **set valued functor**. We say that functors $F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$ with the same domain and the same codomain are **parallel** and functors of the form $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ are **antiparallel**.

The term *covariant* appears to have been first used in 1853 by James Joseph Sylvester (who was quite fond of coining new terms) as follows: "Covariant, a function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms do to a similarly derived transform of its primitive." In plainer terms, an operation is covariant if it varies in a way that preserves some related structure or operation. In the present context, a covariant functor preserves the direction of arrows and a *contravariant* functor reverses the direction of arrows.

One way to view the concept of a functor is to think of a (covariant) functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ as a mapping of one-arrow diagrams in $\mathcal{C}$,

$$A \xrightarrow{f} B$$

to one-arrow diagrams in $\mathcal{D}$,

$$FA \xrightarrow{Ff} FB$$

with the property that "identity loops" and "triangles" are preserved, as shown in Figure 2.

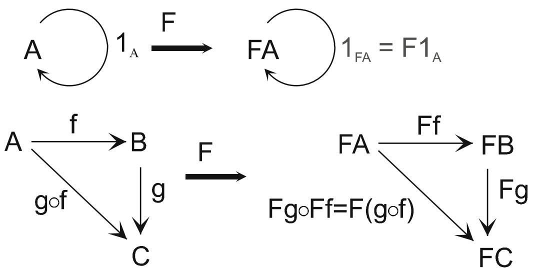

Figure 2

A similar statement holds for contravariant functors.

Composition of Functors

Functors can be composed in the "obvious" way. Specifically, if $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{E}$ are functors, then $G \circ F\colon \mathcal{C} \Rightarrow \mathcal{E}$ is defined by

$$(G \circ F)(A) = G(FA)$$

for $A \in \mathcal{C}$ and

$$(G \circ F)(f) = G(Ff)$$

for $f \in \hom_{\mathcal{C}}(A, B)$. We will often write the composition $G \circ F$ as GF.

Special Types of Functors

▪ **Definition**

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a functor.

1) *F is* **full** *if all of its local arrow parts are surjective.*
2) *F is* **faithful** *if all of its local arrow parts are injective.*
3) *F is* **fully faithful** *(i.e., full and faithful) if all of its local arrow parts are bijective.*
4) *F is an* **embedding** *of $\mathcal{C}$ in $\mathcal{D}$ if it is fully faithful and the object part of F is injective.* □

We should note that the term *embedding*, as applied to functors, is defined differently by different authors. Some authors define an embedding simply as a full and faithful functor. Other authors define an embedding to be a faithful functor whose object part is injective. We have adopted the strongest definition, since it applies directly to the important Yoneda lemma (coming later in the book).

Note that a faithful functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ need not be an embedding, for it can send two morphisms from *different* hom sets to the same morphism in $\mathcal{D}$. For instance, if $FA = FA'$ and $FB = FB'$ then it may happen that

$$Ff_{AB} = Fg_{A'B'}$$

which does not violate the condition of faithfulness. Also, a full functor need not be surjective on $\mathbf{Mor}(\mathcal{C})$.

A Couple of Examples

Here are a couple of examples of functors. We will give more examples in the next chapter.

▪ **Example 9**

The **power set functor** $\wp$: **Set** $\Rightarrow$ **Set** sends a set A to its power set $\wp(A)$ and sends each set function $f\colon A \to B$ to the induced function $f\colon \wp(A) \to \wp(B)$ that sends X to fX. (It is customary to use the same notation for the function and its induced version.) It is easy to see that this defines a faithful but not full covariant functor.

Similarly, the **contravariant power set functor** F: **Set** $\Rightarrow$ **Set** sends a set A to its power set $\wp(A)$ and a set function $f\colon A \to B$ to the induced *inverse* function $f^{-1}\colon \wp(B) \to \wp(A)$ that sends $X \subseteq B$ to $f^{-1}X \subseteq A$. The fact that F is contravariant follows from the well known fact that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$ □

Example 10

The following situation is quite common. Let $\mathcal{C}$ be a category. Suppose that $\mathcal{D}$ is another category with the property that every object in $\mathcal{C}$ is an object in $\mathcal{D}$ and every morphism $f: A \to B$ in $\mathcal{C}$ is a morphism $f: A \to B$ in $\mathcal{D}$.

For instance, every object in **Grp** is also an object in **Set**: we simply ignore the group operation. Also, every group homomorphism is a set function. Similarly, every ring can be thought of as an abelian group by ignoring the ring multiplication and every ring map can be thought of as a group homomorphism.

We can then define a functor $F: \mathcal{C} \Rightarrow \mathcal{D}$ by sending an object $A \in \mathcal{C}$ to itself, thought of as an object in $\mathcal{D}$ and a morphism $f: A \to B$ in $\mathcal{C}$ to itself, thought of as a morphism in $\mathcal{D}$.

Functors such as these that "forget" some structure are called **forgetful functors**. In general, these functors are faithful but not full. For example, distinct group homomorphisms $f, g: A \to B$ are also distinct as functions, but not every set function between groups is a group homomorphism.

For any category $\mathcal{C}$ whose objects are sets, perhaps with additional structure and whose morphisms are set functions, also perhaps with additional structure, the "most forgetful" functor is the one that forgets all structure and thinks of an object simply as a set and a morphism simply as a set function. This functor is called the **underlying-set functor** $U: \mathcal{C} \Rightarrow$ **Set** on $\mathcal{C}$. □

The Category of All Small Categories

As mentioned earlier, it is tempting to define the category of all categories, but this does not exist on foundational grounds. On the other hand, the category **SmCat** of all *small* categories does exist. Its objects are the small categories and its morphisms are the covariant functors between categories. Of course, **SmCat** is a *large* category.

Concrete Categories

Despite the two main tenets of category theory described earlier, most common categories do have the property that their objects are sets whose elements are "important" and whose morphisms are ordinary set functions on these elements, usually with some additional structure (such as being group homomorphisms or linear transformations). This leads to the following definition.

Definition

A category $\mathcal{C}$ is **concrete** *if there is a faithful functor $F: \mathcal{C} \Rightarrow$* **Set**. *Put more colloquially, $\mathcal{C}$ is concrete if the following hold:*

1) *Each object A of $\mathcal{C}$ can be thought of as a set FA (which is often A itself). Note that distinct objects may be thought of as the same set.*
2) *Each distinct morphism $f: A \to B$ in $\mathcal{C}$ can be thought of as a distinct set function $Ff: FA \to FB$ (which is often f itself).*
3) *The identity 1_A morphism can be thought of as the identity set function $F1: FA \to FA$ and the composition $f \circ g$ in $\mathcal{C}$ can be thought of as the composition $Ff \circ Fg$ of the corresponding set functions.* □

Categories that are not concrete are called **abstract categories**. Many concrete categories have the property that FA is A and Ff is f. This applies, for example, to most of the previously defined categories, such as **Grp**, **Rng**, **Vect** and **Poset**. The category **Rel** is an example of a category that is not concrete.

In fact, the subject of which categories are concrete and which are abstract can be rather involved and we will not go into it in this introductory book, except to remark that all small categories are concrete, a fact which follows from Yoneda's lemma, to be proved later in the book.

Subcategories

Subcategories are defined as follows.

■ Definition

Let $\mathcal{C}$ be a category. A **subcategory** *$\mathcal{D}$ of $\mathcal{C}$ is a category for which consists of a nonempty subclass* **Obj**$(\mathcal{D})$ *of* **Obj**$(\mathcal{C})$ *and a nonempty subclass* **Mor**$(\mathcal{D})$ *of* **Mor**$(\mathcal{C})$ *with the following properties:*

1) **Obj**$(\mathcal{D}) \subseteq$ **Obj**$(\mathcal{C})$, *as classes.*
2) *For every $A, B \in \mathcal{D}$,*

$$\hom_{\mathcal{D}}(A, B) \subseteq \hom_{\mathcal{C}}(A, B)$$

and the identity map 1_A in $\mathcal{D}$ is the identity map 1_A in $\mathcal{C}$, that is,

$$(1_A)_{\mathcal{D}} = (1A)_{\mathcal{C}}$$

3) *Composition in $\mathcal{D}$ is the composition from $\mathcal{C}$, that is, if*

$$f\colon A \to B \quad \textit{and} \quad g\colon B \to C$$

are morphisms in $\mathcal{D}$, then the $\mathcal{C}$-composite $g \circ f$ is the $\mathcal{D}$-composite $g \circ f$.

If equality holds in part 2) for all $A, B \in \mathcal{D}$, then the subcategory $\mathcal{D}$ is **full**. □

■ Example 11

The category **AbGrp** of abelian groups is a full subcategory of the category **Grp**, since the definition of group morphism is independent of whether or not the groups involved are abelian. Put another way, a group homomorphism between abelian groups is just a group homomorphism.

However, the category **AbGrp** of abelian groups is a *nonfull* subcategory of the category **Rng** of rings, since not all additive group homomorphisms $f\colon R \to S$ between rings are ring maps. Similarly, the category of differential manifolds with smooth maps is a nonfull subcategory of the category **Top**, since not all continuous maps are smooth. □

The Image of a Functor

Note that if $F\colon \mathcal{C} \Rightarrow \mathcal{D}$, then the image $F\mathcal{C}$ of $\mathcal{C}$ under the functor F, that is, the set

$$\{FA \mid A \in \mathcal{C}\}$$

of objects and the set

$$\{Ff \mid f \in \hom_{\mathcal{C}}(A,B)\}$$

of morphisms need *not* form a subcategory of $\mathcal{D}$. The problem is illustrated in Figure 3.

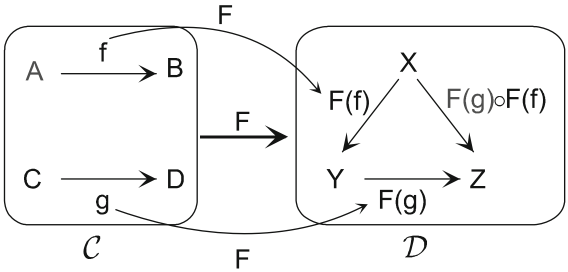

Figure 3

In this case, the composition $F(g) \circ F(f)$ is not in the image $F\mathcal{C}$. The only way that this can happen is if the composition $g \circ f$ does not exist because f and g are not compatible for composition. For if $g \circ f$ exists, then

$$F(g) \circ F(f) = F(g \circ f) \in F\mathcal{C}$$

Note that in this example, the object part of F is not injective, since $F(A) = F(C) = X$. This is no coincidence.

■ Theorem 12

If the object part of a functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is injective, then $F\mathcal{C}$ is a subcategory of $\mathcal{D}$, under the composition inherited from $\mathcal{D}$.

■ Proof

The only real issue is whether the $\mathcal{D}$-composite $Fg \circ Ff$ of two morphisms in $F\mathcal{C}$, when it exists, is also in $F\mathcal{C}$. But this composite exists if and only if

$$Ff\colon FA \to FB \quad \text{and} \quad Fg\colon FB \to FC$$

and so the injectivity of F on objects implies that

$$f\colon A \to B \quad \text{and} \quad g\colon B \to C$$

Hence, $g \circ f$ exists in $\mathcal{C}$ and so

$$F(g) \circ F(f) = F(g \circ f) \in F\mathcal{C}$$

Diagrams

The purpose of a *diagram* is to describe a portion of a category $\mathcal{C}$. By "portion" we mean one or more objects of $\mathcal{C}$ along with *some* of the arrows connecting these objects.

Informally, we can say that a diagram in $\mathcal{C}$ consists of a class of points (or nodes) in the plane, each labeled with an object of $\mathcal{C}$ and for each pair (A, B) of nodes a collection of arcs from the node labeled A to the node labeled B, each of which is labeled with a morphism from A to B.

The simplest way to form a diagram is with a functor—any functor.

Definition

*Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A $\mathcal{J}$-**diagram** (or just **diagram**) in $\mathcal{C}$ with **index category** $\mathcal{J}$ is a functor $J\colon \mathcal{J} \Rightarrow \mathcal{C}$.* □

Since the image $J(\mathcal{J})$ is indexed by the objects and morphisms of the index category $\mathcal{J}$, the objects in $\mathcal{J}$ are often denoted by lower case letters such as m, n, p, q. Figure 4 illustrates this definition.

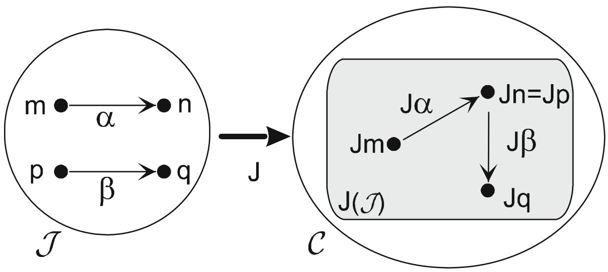

Figure 4

Observe that, as in this example, the image $J(\mathcal{J})$ need not be a subcategory of $\mathcal{C}$. In this example, J sends n and p to the same object in $\mathcal{C}$ but since α and β are not compatible for composition, the image of J need not contain the composition $J\beta \circ J\alpha$. Thus, the image of a functor simply contains *some* objects of $\mathcal{C}$ as well as *some* morphisms between these objects.

It is worth emphasizing that *any* functor $F\colon \mathcal{J} \Rightarrow \mathcal{C}$ is a diagram and so we have introduced nothing new other than a point of view and some concomitant terminology.

The Digraph-Based Version of a Diagram

Of course, another way to view the diagram (functor) in Figure 4 is simply to label the nodes and arrows of the index category $\mathcal{J}$ with the images of the objects and morphisms of $\mathcal{J}$ under the functor J, as shown in Figure 5. This description of a diagram has the disadvantage that it does not show as clearly as in Figure 4 the confluence of Jn and Jp, but it does have some advantages, as we will see later in the book.

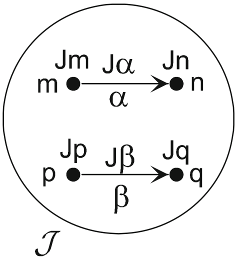

Figure 5

Because generally speaking, the *sole* purpose of the objects and morphisms of the index category is to uniquely identify the nodes and arcs of the diagram, Figure 5 is really little more than a *digraph* whose nodes and arcs are labeled with objects and morphisms from $\mathcal{C}$, respectively. Here is the formal definition of a labeled digraph, along with some terminology that we will need later in the book.

Definition

1) *A* **directed graph** (*or* **digraph**) $\mathcal{D}$ *consists of a nonempty class* $\mathcal{V}(\mathcal{D})$ *of* **vertices** *or* **nodes** *and for every ordered pair* (v, w) *of nodes, a (possibly empty) set* $\mathcal{A}(v, w)$ *of* **arcs** *from* v *to* w. *We say that an arc in* $\mathcal{A}(v, w)$ **leaves** v *and* **enters** w. *Two arcs from* v *to* w *are said to be* **parallel**. *The arcs from* v *to* v *are called* **loops**.
2) *The cardinal number of arcs entering a node is called the* **in-degree** *of the node and the cardinal number of arcs leaving a node is called the* **out-degree** *of the node. The sum of the in-degree and the out-degree is called the* **degree** *of the node.*
3) *A* **labeled digraph** $\mathcal{D}$ *is a digraph for which each node is labeled by elements of a labeling class and each arc is labeled by elements of a labeling class. We require that parallel arcs have distinct labels. A labeled digraph is* **uniquely labeled** *if no two distinct nodes have the same label and no two distinct arcs have the same label.* □

A **directed path** (or just **path**) in a labeled digraph $\mathcal{D}$ is a sequence of arcs of the form

$$e_1 \in \mathcal{A}(v_1, v_2), e_2 \in \mathcal{A}(v_2, v_3), \ldots, e_{n-1} \in \mathcal{A}(v_{n-1}, v_n)$$

where the ending node of one arc is the starting node of the next arc. The **length** of a path is the number of arcs in the path.

Thus, to create what we will call the **digraph version** of a diagram, we first draw a digraph whose nodes are labeled with the distinct objects of the index category $\mathcal{J}$ and whose arcs are labeled with the distinct morphisms of $\mathcal{J}$, subject to the obvious condition that the morphism $f: A \to B$ labels an arc from the node labeled A to the node labeled B. This is referred to as the **underlying digraph** for the diagram.

Then, as shown on the right in Figure 6, we further label the nodes and arcs of the digraph with the image of the functor J. Note that the labels from the index category $\mathcal{J}$ are distinct, but the labels from $\mathcal{C}$ are not necessarily distinct (in the previous example, $Jn = Jp$). This view of a diagram will be useful when we define morphisms between diagrams.

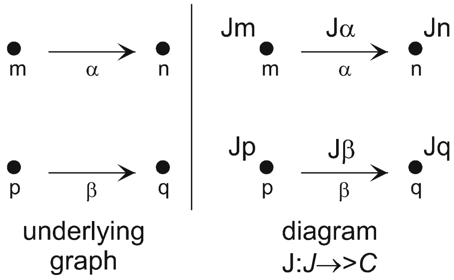

Figure 6

Note that if the object part of the functor J is not injective, then two distinct nodes of the underlying graph will be labeled with the same object in $\mathcal{C}$. Although this is useful on occasion (we will use it precisely once), for most applications of diagrams (at least in this book) the object part and the local arrow parts of J are injective and so the nodes and arcs are *uniquely* labeled from $\mathcal{C}$.

Since as we have remarked, the purpose of the objects and morphisms of the index category is to uniquely identify the nodes and arcs of the underlying digraph, once the graph is drawn on paper, the nodes and arcs are uniquely identified by their location and so the labels from $\mathcal{J}$ are not needed and are typically omitted. This is why diagrams are often drawn simply as in Figure 7, for example.

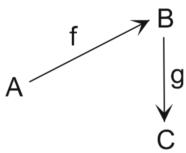

Figure 7

We will use blackboard letters $\mathbb{D}, \mathbb{E}, \mathbb{F}, \ldots$ to denote diagrams and if we need to emphasize the functor, we will write

$$\mathbb{D}(J: \mathcal{J} \Rightarrow \mathcal{C})$$

Commutative Diagrams

We consider that any directed path in a diagram is labeled by the *composition* of the morphisms that label the arcs of the path, taken in the reverse order of appearance in the path. For example, the label of the path

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in Figure 7 is $g \circ f$.

A diagram $\mathbb{D}$ in a category $\mathcal{C}$ is said to **commute** if for every pair (A, B) of objects in $\mathbb{D}$ and any pair of directed paths from A to B, *one of which has length at least two*, the corresponding path labels are equal. A diagram that commutes is called a **commuting diagram** or **commutative diagram**.

For example, the diagram in Figure 1 commutes since

$$\rho_1 \circ \tau = \sigma_1 \quad \text{and} \quad \rho_2 \circ \tau = \sigma_2$$

Note that we exempt the case of two parallel paths of length one so that a diagram such as the one in Figure 8 can be commutative without forcing f and g to be the same morphism. The commutativity condition for this diagram is thus $f \circ e = g \circ e$.

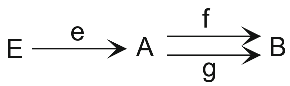

Figure 8

Special Types of Morphisms

For functions, the familiar concepts of *invertibility* (both one-sided and two-sided) and *cancellability* (both one-sided and two-sided) are both categorical concepts. However, the familiar concepts of injectivity and surjectivity are *not* categorical because they involve the *elements* of a set.

In the category **Set**, morphisms are just set functions. For this particular category, the concepts of right-invertibility, right-cancellability and surjectivity are equivalent, as are the concepts of left-invertibility, left-cancellability and injectivity. However, things fall apart totally in arbitrary categories. As mentioned, the concepts of injectivity and surjectivity are not even categorical concepts and so must go away. Moreover, the concepts of invertibility and cancellability are not equivalent in arbitrary categories!

Let us explore the relationship between invertibility and cancellability for morphisms in an arbitrary category. In the exercises, we will ask you to explore the relationship between these concepts and the noncategorical concepts of injectivity and surjectivity, when they exist in the context of a particular category.

We begin with the formal definitions.

■ Definition

Let $\mathcal{C}$ be a category.

1) *A morphism $f\colon A \to B$ is* **right-invertible** *if there is a morphism $f_R\colon B \to A$, called a* **right inverse** *of f, for which*

$$f \circ f_R = 1_B$$

2) *A morphism $f\colon A \to B$ is* **left-invertible** *if there is a morphism $f_L\colon A \to B$, called a* **left inverse** *of f, for which*

$$f_L \circ f = 1_A$$

3) *A morphism $f\colon A \to B$ is* **invertible** *or an* **isomorphism** *if there is a morphism $f^{-1}\colon B \to A$, called the* (**two-sided**) **inverse** *of f, for which*

$$f^{-1} \circ f = 1_A \quad \textit{and} \quad f \circ f^{-1} = 1_B$$

In this case, the objects A and B are **isomorphic** *and we write $A \approx B$.* □

Note that the *categorical* term *isomorphism* says nothing about injectivity or surjectivity, for it must be defined in terms of morphisms only!

In fact, this leads to an interesting observation. For categories whose objects are sets and whose morphisms are set functions, we can define an isomorphism in two ways:

1) (Categorical definition) An isomorphism is a morphism with a two-sided inverse.
2) (Non categorical definition) An isomorphism is a bijective morphism.

In most cases of algebraic structures, such as groups, rings or vector spaces, these definitions are equivalent. However, there are cases where only the categorical definition is correct.

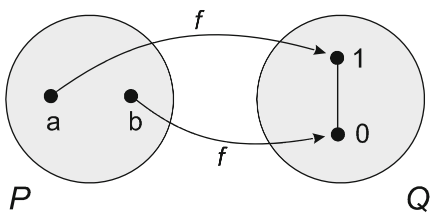

Figure 9

For example, as shown in Figure 9, let $P = \{a, b\}$ be a poset in which a and b are incomparable and let $Q = \{0, 1\}$ be the poset with $0 < 1$. Let $f\colon P \to Q$ be defined by $fa = 0$ and $fb = 1$. Then f is a bijective morphism of posets, that is, a bijective monotone map. However, it is not an isomorphism of posets!

Proof of the following familiar facts about inverses is left to the reader.

Theorem 13

1) *Two-sided inverses, when they exist, are unique.*
2) *If a morphism is both left and right-invertible, then the left and right inverses are equal and are a (two-sided) inverse.*
3) *If the composition $f \circ g$ of two isomorphisms is defined, then it is an isomorphism as well and*

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

□

Definition

Let $\mathcal{C}$ be a category.

1) *A morphism $f\colon A \to B$ is* **right-cancellable** *if*

$$g \circ f = h \circ f \quad \Rightarrow \quad g = h$$

for any parallel morphisms $g, h\colon B \to C$. A right-cancellable morphism is called an **epic** *(or* **epi***).*

2) *A morphism $f\colon A \to B$ is* **left-cancellable**, *if*

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h$$

for any parallel morphisms $g, h\colon C \to A$. A left-cancellable morphism is called a **monic** *(or a* **mono***).*

□

In general, invertibility is stronger than cancellability. We also leave proof of the following to the reader.

Theorem 14

Let f, g be morphisms in a category $\mathcal{C}$.

1) *f left-invertible $\Rightarrow$ f left-cancellable (monic)*
2) *f right-invertible $\Rightarrow$ f right-cancellable (epic)*
3) *f invertible $\Rightarrow$ f monic and epic.*

Moreover, the converse implications fail in general. □

It is also true that a morphism can be both monic and epic (both right and left cancellable) but fail to be an isomorphism. (*Hint*: Check the examples of categories given earlier). On the other hand, one-sided cancellability together with one-sided invertibility (on the other side, of course) do imply an isomorphism.

Theorem 15

Let $f\colon A \to B$ be a morphism in a category $\mathcal{C}$.

1) *If f is monic (left-cancellable) and right-invertible, then it is an isomorphism.*
2) *If f is epic (right-cancellable) and left-invertible, then it is an isomorphism.* □

1

Initial, Terminal and Zero Objects

Anyone who has studied abstract algebra knows that the trivial object (the trivial vector space $\{0\}$, the trivial group $\{1\}$, etc.) often plays a key role in the theory, if only to the point of constantly needing to be excluded from consideration. In general categories, there are actually two concepts related to these trivial or "zero" objects.

Definition

Let $\mathcal{C}$ be a category.

1) *An object $I \in \mathcal{C}$ is* **initial** *if for every $A \in \mathcal{C}$, there is exactly one morphism from I to A.*
2) *An object T is* **terminal** *if for every $A \in \mathcal{C}$, there is exactly one morphism from A to T.*
3) *An object that is both initial and terminal is called a* **zero object**. □

Note that if C is either initial or terminal then $\hom(C, C) = \{1_C\}$. The following simple result is key.

Theorem 16

Let $\mathcal{C}$ be a category. Any two initial objects in $\mathcal{C}$ are isomorphic and any two terminal objects in $\mathcal{C}$ are isomorphic.

Proof

If A and B are initial, then there are unique morphisms $f: A \to B$ and $g: B \to A$ and so $g \circ f \in \hom(A, A) = \{1_A\}$. Similarly, $f \circ g = 1_B$ and so $A \approx B$. A similar proof holds for terminal objects. □

Example 17

In the category **Set**, the empty set is the only initial object and each singleton-set is terminal. Hence, **Set** has no zero object. In **Grp**, the trivial group $\{1\}$ is a zero object. □

Zero Morphisms

In the study of algebraic structures, one also encounters "zero" functions, such as the zero linear transformation and the map that sends each element of a group G to the identity element of another group H. Here is the subsuming categorical concept.

Definition

Let $\mathcal{C}$ be a category with a zero object 0. *Any morphism $f: A \to B$ that can be factored through the zero object, that is, for which*

$$f = h_{0B} \circ g_{A0}$$

for morphisms $h: 0 \to B$ and $g: A \to 0$ is called a **zero morphism**. □

To explain this rather strange-looking concept, let us take the case of linear algebra, where the zero linear transformation $z: V \to W$ between vector spaces is usually defined to be the map that sends any vector in V to the zero vector in W. This definition is not categorical because it involves the zero *element* in W. To make it categorical, we interpose the *zero vector space* $\{0\}$. Indeed, the zero transformation z can be written as the composition $z = h \circ g$, where

$$g\colon V \to \{0\} \quad \text{and} \quad h\colon \{0\} \to W$$

Here, both g and h are uniquely defined by their domains and ranges, without mention of any elements. The point is that g has no choice but to send every vector in V to the zero vector in $\{0\}$ and h must send the zero vector in $\{0\}$ to the zero vector in W. Using g and h, we can avoid having to explicitly mention any individual vectors!

In the category of groups, the zero morphisms are precisely the group homomorphisms that map every element of the domain to the identity element of the range. Similar maps exist in **CRng** and **Mod**.

It is clear that any morphism entering or leaving 0 is a zero morphism.

Theorem 18

Let $\mathcal{C}$ be a category with a zero object 0.

1) *There is exactly one zero morphism between any two objects in $\mathcal{C}$.*
2) *Zero morphisms "absorb" other morphisms, that is, if $z\colon A \to B$ is a zero morphism, then so are $f \circ z$ and $z \circ g$, whenever the compositions make sense.* □

Duality

The concept of duality is prevalent in category theory.

Dual or Opposite Categories

For every category $\mathcal{C}$, we may form a new category $\mathcal{C}^{\mathrm{op}}$, called the **opposite category** or the **dual category** whose objects are the same as those of $\mathcal{C}$, but whose morphisms are "reversed", that is,

$$\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) = \hom_{\mathcal{C}}(B, A)$$

For example, in the category $\mathbf{Set}^{\mathrm{op}}$ the morphisms from A to B are the set functions from B to A. This may seem a bit strange at first, but one must bear in mind that morphisms are not necessarily functions in the traditional sense: By definition, they are simply elements of the hom-sets of the category. Therefore, there is no reason why a morphism from A to B cannot be a function from B to A.

The rule of composition in $\mathcal{C}^{\mathrm{op}}$, which we denote by $\circ_{\mathrm{op}}$, is defined as follows: If $f \in \hom_{\mathcal{C}^{\mathrm{op}}}(A, B)$ and $g \in \hom_{\mathcal{C}^{\mathrm{op}}}(B, C)$, then

$$g \circ_{\mathrm{op}} f \in \hom_{\mathcal{C}^{\mathrm{op}}}(A, C)$$

is the morphism $f \circ g \in \hom_{\mathcal{C}}(C, A)$. In short,

$$g \circ_{\mathrm{op}} f = f \circ g$$

Note that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and so every category is a dual category.

It might occur to you that we have not really introduced anything *new*, and this is true. Indeed, every category is a dual category (and conversely), since it is dual to its own dual. But we have introduced a new way to look at old things and this will prove fruitful. Stay tuned.

The Duality Principle

Let p be a property that a category $\mathcal{C}$ may possess, for example, p might be the property that $\mathcal{C}$ has an initial object. We say that a property p^{op} is a **dual property** to p if for all categories $\mathcal{C}$,

$$\mathcal{C} \text{ has } p^{\mathrm{op}} \quad \text{iff} \quad \mathcal{C}^{\mathrm{op}} \text{ has } p$$

Note that this is a symmetric definition and so we can say that two properties are dual (or not dual) to one another. For instance, since the initial objects in $\mathcal{C}^{\mathrm{op}}$ are precisely the terminal objects in $\mathcal{C}$, the properties of having an initial object and having a terminal object are dual. The property of being isomorphic is *self-dual*, that is, $A \approx B$ in $\mathcal{C}$ if and only if $A \approx B$ in $\mathcal{C}^{\mathrm{op}}$.

In general, if s is a statement about a category $\mathcal{C}$, then the **dual statement** is the same statement stated for the dual category $\mathcal{C}^{\mathrm{op}}$, but expressed in terms of the original category. For example, consider the statement

the category $\mathcal{C}$ has an initial object

Stated for the dual category $\mathcal{C}^{\mathrm{op}}$, this is

the category C^{op} has an initial object

Since the initial objects in $\mathcal{C}^{\mathrm{op}}$ are precisely the terminal objects in $\mathcal{C}$, this is equivalent to the statement

the category C has a terminal object

which is therefore the dual of the original statement.

A statement and its dual are not, in general, logically equivalent. For instance, there are categories that have initial objects but not terminal objects. However, for a special and very common type of conditional statement, things are different.

Let $\Pi = \{q_i \mid i \in I\}$ be a set of properties and let $\Pi^{\mathrm{op}} = \left\{q_i^{\mathrm{op}} \mid i \in I\right\}$ be the set of dual properties. Let p be a single property. Consider the statement

1) If a category $\mathcal{C}$ has Π, then it also has p (abbreviated $\Pi \Rightarrow p$).

   Since all categories have the form $\mathcal{C}^{\mathrm{op}}$ for some category $\mathcal{C}$, this statement is logically equivalent to the statement

2) If a category $\mathcal{C}^{\mathrm{op}}$ has Π, then it also has p.

   and this is logically equivalent to

3) If a category $\mathcal{C}$ has Π^{op}, then it also has p^{op} (abbreviated $\Pi^{\mathrm{op}} \Rightarrow p^{\mathrm{op}}$).

The fact that

$$\Pi \Rightarrow p \quad \text{iff} \quad \Pi^{\mathrm{op}} \Rightarrow p^{\mathrm{op}}$$

is called the **principle of duality** for categories. Note that if Π is **self-dual**, that is, if $\Pi = \Pi^{\mathrm{op}}$, then the principle of duality becomes

$$\Pi \Rightarrow p \quad \text{iff} \quad \Pi \Rightarrow p^{\mathrm{op}}$$

Of course, the empty set of properties is self-dual. Moreover, the condition $\emptyset \Rightarrow p$ means that all categories possess property p. Hence, we deduce that

> if all categories possess a property p, then all categories also possess any dual property p^{op}

For example, all categories possess the property that initial objects (when they exist) are isomorphic. Hence, the principle of duality implies that all terminal objects (when they exist) are isomorphic.

New Categories From Old Categories

There are many ways to define new categories from old categories. One of the simplest ways is to take the Cartesian product of the objects in two categories. There are also several important ways to turn the morphisms of one category into the objects of another category.

The Product of Categories

If $\mathcal{B}$ and $\mathcal{C}$ are categories, we may form the **product category** $\mathcal{B} \times \mathcal{C}$, in the expected way. Namely, the objects of $\mathcal{B} \times \mathcal{C}$ are the ordered pairs (B, C), where B is an object of $\mathcal{B}$ and C is an object of $\mathcal{C}$. A morphism from $B \times C$ to $B' \times C'$ is a pair (f, g) of morphisms, where $f\colon B \to B'$ and $g\colon C \to C'$. Composition is done componentwise:

$$(f, g) \circ (h, k) = (f \circ h, g \circ k)$$

A functor $F\colon \mathcal{A} \times \mathcal{B} \Rightarrow \mathcal{C}$ from a product category $\mathcal{A} \times \mathcal{B}$ to another category is called a **bifunctor**.

The Category of Arrows

Given a category $\mathcal{C}$, we can form the **category of arrows** $\mathcal{C}^{\to}$ of $\mathcal{C}$ by taking the objects of $\mathcal{C}^{\to}$ to be the morphisms of $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

Figure 10

A morphism in $\mathcal{C}^{\rightarrow}$, that is, a *morphism between arrows* is defined as follows. A morphism from $f\colon A \to B$ to $g\colon A' \to B'$ is a *pair* of arrows

$$(\alpha\colon A \to A', \beta\colon B \to B')$$

in $\mathcal{C}$ for which the diagram in Figure 10 commutes, that is, for which

$$g \circ \alpha = \beta \circ f$$

We leave it to the reader to verify that $\mathcal{C}^{\rightarrow}$ is a category, with compositon defined pairwise:

$$(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta)$$

and with identity morphisms $(1_A, 1_B)$.

Comma Categories

Comma categories form one of the most important classes of categories and they should be studied carefully since we will encounter them many times in the sequel. To help absorb the concept, we will define the simplest form of comma category first and then generalize twice.

Arrows Entering (or Leaving) an Object

The simplest form of comma category is defined as follows. Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$. In this context, the object A is referred to as a **source object**. The category of **arrows leaving** the source object A, denoted by $(A \to \mathcal{C})$ has for its objects the set of all pairs,

$$\{(B, f\colon A \to B) \mid B \in \mathcal{C}\} \tag{19}$$

The objects B are referred to as **target objects**. Note that there is in general only one source object but many target objects. For pedogogical reasons, we will refer to the pairs (19) as **comma objects**.

Note that since a morphism uniquely determines its codomain, we could have defined the objects of $(A \to \mathcal{C})$ to be just the morphisms $f\colon A \to B$ themselves but it is customary to include the codomains explicitly.

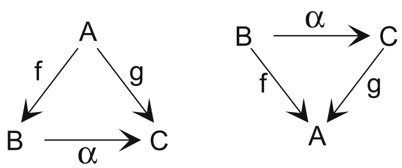

Figure 11

The morphisms

$$\alpha\colon (B, f\colon A \to B) \to (C, g\colon A \to C)$$

in the comma category $(A \to \mathcal{C})$ are defined by taking the morphisms $\alpha\colon B \to C$ in $\mathcal{C}$ between the target objects for which the triangle shown on the left in Figure 11 commutes, that is, for which

$$\alpha \circ f = g \tag{20}$$

and changing the domain and codomain to $(B, f\colon A \to B)$ and $(C, g\colon A \to C)$, respectively. Note that although some authors say that a morphism from $(B, f\colon A \to B)$ to $(C, g\colon A \to C)$ *is* a morphism $\alpha\colon B \to C$ that satisfies (20), this is not quite correct, since the two morphisms have different domains and codomains. To temporarily help clarify this distinction, we will write $\overline{\alpha}$ for the morphism in the comma category, but will drop this notation quickly, since other authors do not use it at all.

Now we can define composition in the comma category by

$$\overline{\alpha} \circ \overline{\beta} = \overline{\alpha \circ \beta}$$

whenever $\alpha \circ \beta$ is defined. As to the identity on an object $(B, f\colon A \to B)$, we have

$$\overline{1_B} \circ \overline{a} = \overline{1_B \circ \alpha} = \overline{\alpha} \quad \text{and} \quad \overline{\alpha} \circ \overline{1_B} = \overline{\alpha \circ 1_B} = \overline{\alpha}$$

and so $\overline{1_B}$ is the identity morphism for the object $(B, f\colon A \to B)$. We leave a check on associativity to you. The category of arrows leaving A is also called a **coslice category**.

Dually, the category $(\mathcal{C} \to A)$ of **arrows entering** a target object A has for its objects the pairs

$$\{(B, f\colon B \to A) \mid B \in \mathcal{C}\}$$

and as shown on the right in Figure 11, a morphism

$$\overline{\alpha}\colon (B, f\colon B \to A) \to (C, g\colon C \to A)$$

in $(\mathcal{C} \to A)$ comes from a morphism $\alpha\colon B \to C$ in $\mathcal{C}$ for which

$$g \circ \alpha = f$$

by changing domain and codomain. The category of arrows entering A is also called a **slice category**. In this case, there is only one target object A and many source objects B.

The First Generalization

There are many occassions (in fact, most occassions) when we would like to exercise control over what types of objects can be the target objects in a comma category. This is accomplished using a functor. Specifically, as shown in Figure 12, let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a functor and let $A \in \mathcal{D}$ be a source object.

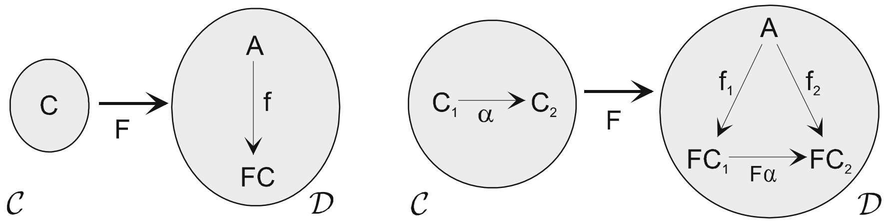

Figure 12

As shown on the left in the figure, the objects of the comma category $(A \to F)$ are the pairs

$$\big\{(C, f\colon A \to FC \mid C \in \mathcal{C}\big\} \tag{21}$$

whose target objects FC come from the image of the category $\mathcal{C}$ under the functor F. Again for pedological reasons, we will refer to the pairs (21) as **comma objects**.

As a quick example, suppose we wish to consider group homomorphims $f\colon A \to R$ from a fixed abelian group A (the source) into the additive portion of various rings R (the targets). To accomplish this, we use the forgetful functor $F\colon \mathbf{Rng} \Rightarrow \mathbf{AbGrp}$ to forget the multiplicative structure of a ring and so our morphisms take the form $f\colon A \to FR$ for $R \in \mathbf{Rng}$.

As to morphisms in the comma category $(A \to F)$, as shown on the right in Figure 12, a morphism

$$\alpha\colon (C_1, f_1\colon A \to FC_1) \to (C_2, f_2\colon A \to FC_2)$$

comes from a morphism $\alpha\colon C_1 \to C_2$ in $\mathcal{C}$ between "pre-target" objects with the property that

$$F\alpha \circ f_1 = f_2$$

with the appropriate change in domain and codomain. Note that the comma category $(A \to \mathcal{C})$ defined earlier is just $(A \to I_{\mathcal{C}})$, where $I_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$.

Dually, we can define the **comma category** $(F \to A)$ by reversing the arrows. Thus, a comma object in $(F \to A)$ is a pair

$$(C, f\colon FC \to A)$$

consisting of a source object C and an arrow from FC to the target object A. A morphism

$$\alpha\colon (C_1, f_1\colon FC_1 \to A) \to (C_2,\ f_2\colon FC_2 \to A)$$

effects a change in source objects, which is accomplished by a morphism $\alpha\colon C_1 \to C_2$ between "pre-source" objects for which

$$f_2 \circ F\alpha = f_1$$

The Final Generalization

As a final generalization, we can allow both the source and the target objects to vary over the image of separate functors. Specifically, let $F\colon \mathcal{B} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{C} \Rightarrow \mathcal{D}$ be functors with the same codomain. As shown in Figure 13, an object of the **comma category** $(F \to G)$ is a triple

$$(B, C, f\colon FB \to GC)$$

where $B \in \mathcal{B}$, $C \in \mathcal{C}$ and f is a morphism in $\mathcal{D}$.

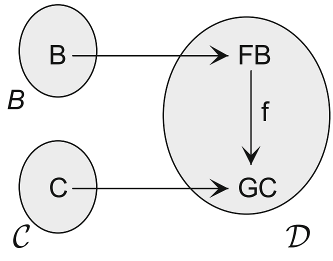

Figure 13

As to morphisms, as shown in Figure 14,

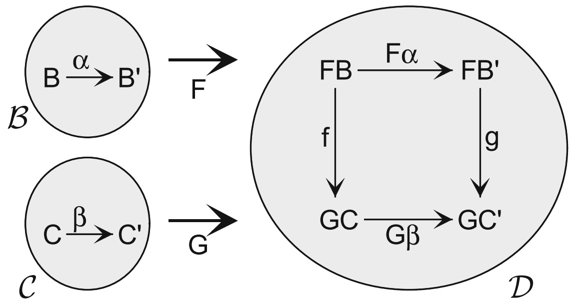

Figure 14

a morphism from $(B, C, f\colon FB \to GC)$ to $(B', C', f'\colon FB' \to GC')$ is a pair of morphisms

$$(\alpha\colon B \to B', \beta\colon C \to C')$$

for which the square commutes, that is,

$$G\beta \circ f = f' \circ F\alpha$$

The composition of pairs is done componentwise.

Example 22

Let $\mathcal{C}$ be a category and let $F\colon \mathcal{C} \Rightarrow$ **Set** be a set-valued functor. The objects of the **category of elements Elts**(F) are ordered pairs (C, a), where $C \in \mathcal{C}$ and $a \in FC$. A morphism $f\colon (C, a) \to (D, b)$ is a morphism $f\colon C \to D$ for which $Ff(a) = b$. We leave it to the reader to show that this is a special type of comma category. □

Hom-Set Categories

Rather than treating individual arrows as the objects of a new category, we can treat entire hom-sets

$$\{\hom_{\mathcal{C}}(A, X) \mid X \in \mathcal{C}\}$$

as the objects of a category $\mathcal{C}(A, -)$. As to the morphisms, referring to the left half of Figure 15, let $\hom_{\mathcal{C}}(A, X)$ and $\hom_{\mathcal{C}}(A, Y)$ be hom-sets. Then for each morphism $f\colon X \to Y$ in $\mathcal{C}$, there is a morphism

$$f^{\leftarrow}\colon \hom_{\mathcal{C}}(A, X) \to \hom_{\mathcal{C}}(A, Y)$$

defined in words as "follow by f," that is,

$$f^{\leftarrow}(\alpha) = f \circ \alpha$$

for all $\alpha \in \hom_{\mathcal{C}}(A, X)$.

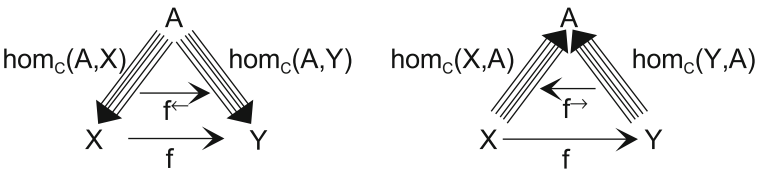

Figure 15

We can also define a category $\mathcal{C}(-, A)$ whose objects are

$$\{\hom_{\mathcal{C}}(X, A) \mid X \in \mathcal{C}\}$$

As shown on the right half of Figure 15, for each morphism $f\colon X \to Y$ in $\mathcal{C}$, there is a morphism in $\mathcal{C}(-, A)$ from $\hom_{\mathcal{C}}(Y, A)$ and $\hom_{\mathcal{C}}(X, A)$:

$$f^{\rightarrow}\colon \hom_{\mathcal{C}}(Y, A) \to \hom_{\mathcal{C}}(X, A)$$

defined by "preceed by f," that is,

$$f^{\rightarrow}(\alpha) = \alpha \circ f$$

Note that any category $\mathcal{C}$ can be viewed as a hom-set category by adjoining a new initial "object" $*$ not in $\mathcal{C}$ and defining a new morphism $f_A\colon * \to A$ from $*$ to each object $A \in \mathcal{C}$. Then each object $A \in \mathcal{C}$ can be identified with its hom-set hom$(*, A)$. Also, the morphisms $f\colon A \to B$ in $\mathcal{C}$ are identified with the morphisms

$$f^{\leftarrow}\colon \hom(*, A) \to \hom(*, B)$$

of hom-sets.

The Categorical Product

Recall that in Example 8, we tried to motivate the categorical perspective by describing how the external direct product of vector spaces can be defined using morphisms (linear transformations) rather than elements. The key to this description is the projection maps.

At this point, we want to generalize this example, so that we can use it in further examples and exercises. We will revisit this again in more detail in the chapter on cones and limits, so we will be brief here.

Here is the formal definition of the product for general categories.

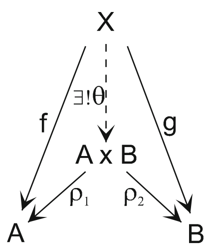

Figure 16

■ Definition

Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$, as shown in Figure 16. A **product** *of A and B is a triple*

$$(A \times B, \rho_1\colon A \times B \to A, \rho_2\colon A \times B \to B)$$

where $A \times B$ is an object in $\mathcal{C}$ and ρ_1 and ρ_2 are morphisms in $\mathcal{C}$ with the property that for any triple

$$(X, f\colon X \to A, g\colon X \to B)$$

where $X \in \mathcal{C}$, there exists a unique map $\theta\colon X \to A \times B$, called the **mediating morphism** *for which the diagram in Figure 16 commutes, that is, for which*

$$\rho_1 \circ \theta = f \quad \textit{and} \quad \rho_2 \circ \theta = g$$

The maps ρ_1 and ρ_2 are called the **projection maps**. □

Of course, by now you realize that the projection maps are critical to the concept of a categorical product. However, although it is quite misleading, it is common practice to denote a product simply as $A \times B$, without explicit mention of the projection maps. It can be shown, as we will do later, that all products of A and B are isomorphic.

It follows from the definition that two morphisms $\alpha, \beta\colon X \to A \times B$ into a product are equal if and only if

$$\rho_1 \circ \alpha = \rho_1 \circ \beta \quad \text{and} \quad \rho_2 \circ \alpha = \rho_2 \circ \beta$$

This is a common application of the uniqueness of the mediating morphism (and worth remembering!).

Here are some simple examples of the categorical product. It is interesting to note that the familiar product of groups, rings and vector spaces, for example, is an example of the same categorical concept as the upper bound in a poset!

■ **Example 23**

1) In **Set**, the product is the cartesian product, with the usual projections.
2) In **Grp**, **Mod**, **Vect** and **Rng**, the product is the usual direct product of groups, modules, vector spaces and rings, defined coordinatewise.
3) In **Poset**(P), the product is the least upper bound. □

If a category $\mathcal{C}$ has the property that every pair of objects in $\mathcal{C}$ has a product, which is the case for **Grp**, **Mod**, **Vect** and **Rng** but not for **Poset**(P), we say that $\mathcal{C}$ **has binary products**. On the other hand, the category **Field** does not have products.

As mentioned, we will go into more details about the product in a later chapter. For now, this is all you need to know to handle any subsequent discussions.

The Product of Morphisms

We can use the categorical product to define the product of morphisms in a category. Let $\mathcal{C}$ be a category with binary products and let $f_1\colon A_1 \to B_1$ and $f_2\colon A_2 \to B_2$ be morphisms in $\mathcal{C}$. We wish to define the **product morphism**

$$f_1 \times f_2 : (A_1 \times A_2, \alpha_1, \alpha_2) \to (B_1 \times B_2, \beta_1, \beta_2) \qquad \textbf{(24)}$$

where the maps α_i and β_i are the corresponding projection maps for the two products.

Let us "discover" the definition using some typical categorical reasoning, which we will call the **mediating morphism trick** so we can refer to it again later in the book. The mediating morphism trick is this: To get a map from any object X into a product such as $B_1 \times B_2$, we simply define two maps $\delta_1 : X \to B_1$ and $\delta_2 : X \to B_2$ from X into the *components* of the product and then invoke the definition of product! This definition says that there is a *unique* (mediating morphism) map $\tau : X \to B_1 \times B_2$ for which

$$\beta_1 \circ \tau = \delta_1 \quad \text{and} \quad \beta_2 \circ \tau = \delta_2$$

So, to use the mediating morphism trick to get a map from $A_1 \times A_2$ to $B_1 \times B_2$, all we need is a pair of maps: one from $A_1 \times A_2$ to B_1 and one from $A_1 \times A_2$ to B_2.

The whole story is shown in Figure 17.

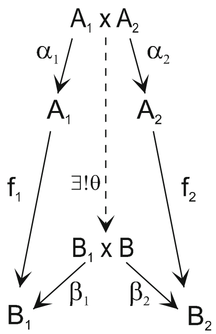

Figure 17

We have the projections $\alpha_i : A_1 \times A_2 \to A_i$ and the maps $f_i : A_i \to B_i$, whose compositions gives us the desired maps from $A_1 \times A_2$ to the components B_1 and B_2. Hence, there is a *unique* map (24) for which

$$\beta_1 \circ (f_1 \times f_2) = f_1 \circ \alpha_1 \quad \text{and} \quad \beta_2 \circ (f_1 \times f_2) = f_2 \circ \alpha_2 \qquad \textbf{(25)}$$

Equations (25) *define* the product $f_1 \times f_2$.

Note that in categories where the product is a cartesian product of *sets* and the projections are ordinary projection *set functions*, these equations give the coordinates of the ordered pair $(f_1 \times f_2)(x_1, x_2)$ and so

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

as we would hope. However, in more unusual categories, we must rely on Equations (25).

1

Exercises

1. Prove that identity morphisms are unique.
2. Does the following description form a cateogry $\mathcal{C}$? Explain. Let the objects of $\mathcal{C}$ be A and B and let the hom sets be

$$\begin{aligned} \hom(A,A)&=\{1_A\}, \quad \hom(B,B)=\{1_B\} \\ \hom(A,B)&=\{f\}, \quad \hom(B,A)=\{g,h\} \end{aligned}$$

3. If $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is fully faithful, prove that

$$FC \approx FC' \quad \Rightarrow \quad C \approx C'$$

4. Indicate how one might define a category without mentioning objects.
5. A category with only one object is essentially just a monoid. How?
6. Let V be a real vector space. Define a category $\mathcal{C}$ as follows. The objects of $\mathcal{C}$ are the vectors in V. For $u, v \in V$, let

$$\hom(u,v) = \left\{a \in \mathbb{R} \,\middle|\, \text{there is } r \geq 1 \text{ such that } rau = v\right\}$$

   Let composition be ordinary multiplication. Show that $\mathcal{C}$ is a category.
7. a) Prove that the composition of monics is monic.
   b) Prove that if $f \circ g$ is monic, then so is g.
   c) Prove that if $f \circ g$ is epic, then so is f.
8. Find a category with nonidentity morphisms in which every morphism is monic and epic, but no nonidentity morphism is an isomorphism.
9. Prove that any two initial objects are isomorphic and any two terminal objects are isomorphic.
10. Find the initial, terminal and zero objects in $\mathbf{Mod}_R$ and $\mathbf{CRng}$.
11. Find the initial, terminal and zero objects in the following categories:
    a) $\mathbf{Set} \times \mathbf{Set}$
    b) $\mathbf{Set}^{\rightarrow}$
12. In each case, find an example of a category with the given property.
    a) No initial or terminal objects.
    b) An initial object but no terminal objects.
    c) No initial object but a terminal object.
    d) An initial and a terminal object that are not isomorphic.
13. Let $\mathbb{D}$ be a diagram in a category $\mathcal{C}$. Show that there is a smallest subcategory $\mathcal{D}$ of $\mathcal{C}$ for which $\mathbb{D}$ is a diagram in $\mathcal{D}$.
14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Prove that the product category $\mathcal{C} \times \mathcal{D}$ is indeed a category.
15. A Boolean homomorphism $g\colon \wp(B) \to \wp(A)$ is a map that preserves union, intersection and complement, that is,

$$g\left(\bigcup B_i\right) = \bigcup\nolimits_g (B_i)$$
$$g\left(\bigcap B_i\right) = \bigcap\nolimits_g (B_i)$$
$$g(B^c) = (gB)^c$$

For the contravariant power set functor F: **Set** $\Rightarrow$ **Set**, show that the image **PS** $= F$(**Set**) is the subcategory of **Set** whose objects are the power sets $\wp(A)$ and whose morphisms are the Boolean homomorphisms g: $\wp(B) \to \wp(A)$ satisfying $g(B) = A$.

16. Let F: $\mathcal{B} \Rightarrow \mathcal{D}$ and G: $\mathcal{C} \Rightarrow \mathcal{D}$ be functors with the same codomain.
    a) Let R be a commutative ring with unit. Show that the category ($R \to$ **CRng**) is the category of R-algebras.
    b) Let t be a terminal element of a category $\mathcal{C}$. Describe $(\mathcal{C} \to t)$.
17. Show by example that the following do *not* hold in general.
    a) monic $\Rightarrow$ injective
       Hint: Let $\mathcal{C}$ be the category whose objects are the subsets of the integers $\mathbb{Z}$ and for which $\hom_{\mathcal{C}}(A, B)$ is the set of all *nonnegative* set functions from A to B, along with the identity function when $A = B$. Consider the absolute value function α: $\mathbb{Z} \to \mathbb{N}$.
    b) injective $\Rightarrow$ left-invertible
       Hint: Consider the inclusion map κ: $\mathbb{Z} \to \mathbb{Q}$ between rings.
    c) epic $\Rightarrow$ surjective
       Hint: Consider the inclusion map κ: $\mathbb{N} \to \mathbb{Z}$ between monoids.
    d) surjective $\Rightarrow$ right-invertible
       Hint: Let $C = \langle a \rangle$ be a cyclic group and let $H = \langle a^2 \rangle$. Consider the canonical projection map π: $C \to C/H = \{H, aH\}$.
18. Prove the following:
    a) For morphisms between sets, monoids, groups, rings or modules, any monic is injective. *Hint*: Let f: $A \to X$ be monic. Extend the relevant algebraic structure on A coordinatewise to the cartesian product $A \times A$ and let

$$S = \left\{(a, b) \in A \times A \mid f(a) = f(b)\right\}$$

       Let ρ_1: $S \to A$ be projection onto the first coordinate and let ρ_2: $S \to A$ be projection onto the second coordinate. Apply $f \circ \rho_i$ to $(a, b) \in S$.
    b) For morphisms between sets, groups or modules, epic implies surjective. *Hint*: suppose that f: $A \to X$ is not surjective and let $I = \mathrm{im}(f)$. Find two distinct morphisms p, q: $X \to Y$ that agree on I, then $p \circ f = q \circ f$ but $p \neq q$, in contradiction to epicness. (For groups, this is a bit hard.)
    c) However, for morphisms between monoids or rings, epic does not imply surjective. *Hint*: Consider the inclusion map κ: $\mathbb{N} \to \mathbb{Z}$ between monoids and the inclusion map κ: $\mathbb{Z} \to \mathbb{Q}$ between rings.

19) (For those familiar with the tensor product) We want to characterize the epimorphisms in **CRng**, the category of commutative rings with identity. Let $A, B \in$ **CRng** and f: $A \to B$. Then B is an A-module with scalar multiplication defined by

$$ab = f(a)b$$

for $a \in A$ and $b \in B$. Consider the tensor product $B \otimes B$ of the A-module B with itself. Show that f is an epic if and only if $1 \otimes b = b \otimes 1$ for all $b \in B$. *Hint*: any ring map $\lambda: A \to R$ defines an A-module structure on R.

20) Let $\mathcal{C}$ be a category with a zero object. Show that the following are equivalent:
  1) C is an initial object.
  2) C is a terminal object.
  3) $\iota_C = 0_{CC}$
  4) $\hom_{\mathcal{C}}(C, C) = \{0_{CC}\}$

Image Factorization Systems

An **image factorization system** for a category $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ where

a) $\mathcal{E}$ is a nonempty class of epics of $\mathcal{C}$, closed under composition.
b) $\mathcal{M}$ is a nonempty class of monics of $\mathcal{C}$, closed under composition.
c) Any isomorphism of $\mathcal{C}$ belongs to $\mathcal{E}$ and $\mathcal{M}$.
d) Every morphism $f: A \to B$ can be factored as $f = m \circ e$ where $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Moreover, this factorization is unique in the following sense: If $f = m' \circ e'$ with $m' \in \mathcal{M}$ and $e' \in E$, then there is an isomorphism $\theta: I \to J$ for which the following diagram commutes:
   that is, $\theta \circ e = e'$ and $m' \circ \theta = m$.

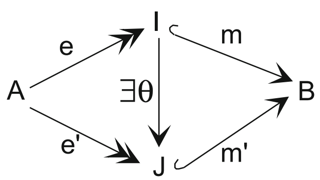

Figure 18

21. Find an image factorization system for **Set**.
22. Find an image factorization system for **Grp**.
23. Prove the *diagonal fill-in theorem*: Let $(\mathcal{E}, \mathcal{M})$ be an image factorization system. Let $f: A \to C$ and $g: B \to D$ be morphisms in $\mathcal{C}$ and let $e \in \mathcal{E}$ and $m \in \mathcal{M}$, with the square in Figure 19 commutes.

$$
\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists h"'] \\
C \arrow[r, hook, "m"'] & D
\end{tikzcd}
$$

Figure 19

Then there exists a unique morphism $h \colon B \to C$ for which the diagram in Figure 19 commutes.

Functors and Natural Transformations

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Let us now take a closer look at functors, beginning with some additional examples.

Examples of Functors

We have already discussed the power set functor and the forgetful functor. Let us consider some other examples of functors.

Example 26

1) For a given positive integer n, we can define a *matrix functor* F_n: **CRng** $\Rightarrow$ **Grp** sending a commutative ring R to the general linear group $GL_n(R)$ of nonsingular $n \times n$ matrices over R. Each ring homomorphism $f\colon R \to S$ is sent to the map that works elementwise on the entries of a matrix.
2) Another functor G: **CRng** $\Rightarrow$ **Grp** is defined by setting $GR = R^*$, the group of units of R and $Gf = f\big|_{R^*}$ for any ring homomorphism $f\colon R \to S$. This makes sense since a ring homomorphism maps units to units. □

Example 27

If P is a poset, then a nonempty subset D of P is a **down-set** if $d \in D$ and $x \leq d$ imply that $x \in D$. Let **Poset** be the category of all posets. Define the **down-set functor** $\mathcal{O}$: **Poset** $\Rightarrow$ **Poset** as follows. A poset P is sent to the family $\mathcal{O}(P)$ of all down-sets in P, ordered by set inclusion. If $f\colon P \to Q$ is a monotone map, then the inverse image of a down-set in Q is a down-set in P and so we may take $\mathcal{O}(f)\colon \mathcal{O}(Q) \to \mathcal{O}(P)$ to be the induced inverse map f^{-1}. Since

$$1_P^{-1} = 1_{\mathcal{O}(P)}$$

and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

it follows that $\mathcal{O}$ is a contravariant functor on **Poset**. □

Example 28

Let $A \in \mathcal{C}$ and consider the comma category $(\mathcal{C} \to A)$ of arrows entering A. Each object of $(\mathcal{C} \to A)$ is an ordered pair $(X, f\colon X \to A)$, as X ranges over the objects of $\mathcal{C}$. The **domain functor** $F\colon (\mathcal{C} \to A) \Rightarrow \mathcal{C}$ sends an object $(X, f\colon X \to A)$ to its domain X and a morphism

$$\overline{\alpha}: (X, f: X \to A) \to (Y, g: Y \to A)$$

which is a map $\alpha: X \to Y$ satisfying

$$g \circ \alpha = f$$

to the underlying morphism α. Thus $F\overline{\alpha} = \alpha$. We leave it to you to show that F is indeed a functor. □

Example 29

Here is a functor tongue-twister. Let $\mathcal{C}$ be a category. We can define a functor $F: \mathcal{C} \Rightarrow$ **SmCat** that takes an object $A \in \mathcal{C}$ to the comma category $(\mathcal{C} \to A) \in$ **SmCat**, with target object A. For this reason, we might call the functor F an **target functor** (a nonstandard term). A morphism $f: A \to A'$ between target objects in $\mathcal{C}$ must map under F to a *functor*, that is,

$$f: A \to A' \quad \underset{F}{\Rightarrow} \quad Ff: (\mathcal{C} \to A) \Rightarrow (\mathcal{C} \to A')$$

between the relevant comma categories. As shown on the left in Figure 20, the object portion of Ff must take an object $(C, \alpha: C \to A)$ in $(\mathcal{C} \to A)$ to an object in $(\mathcal{C} \to A')$. We take

$$Ff[(C, \alpha: C \to A)] = (C, f \circ \alpha: C \to A')$$

and so Ff is essentially the "follow by f" map $f^{\leftarrow}$ on objects.

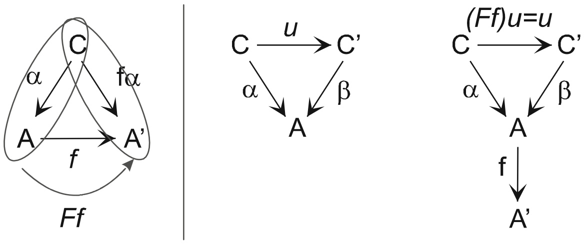

Figure 20

As to the arrow part, as shown on the right in Figure 20, recall that a morphism

$$\overline{u}: (C, \alpha: C \to A) \to (C', \beta: C' \to A)$$

in $(\mathcal{C} \to A)$ comes from a qualifying morphism $u: C \to C'$, that is, a morphism for which

$$\beta \circ u = \alpha$$

Now, Ff must take $\overline{u}$ to a morphism

$$(Ff)(\overline{u})\colon (C, f \circ \alpha\colon C \to A') \to (C', f \circ \beta\colon C' \to A')$$

But

$$(f \circ \beta) \circ u = f \circ \alpha$$

implies that u is also qualifying for the pair

$$P = ((C, f \circ \alpha\colon C \to A'), (C', f \circ \beta\colon C' \to A'))$$

so we can take

$$(Ff)(\overline{u}) = \overline{u}$$

where the overbar on the right means give u the domain and codomain in P.

We will leave it to you to show that Ff is indeed a functor and then that F is also a functor! □

Example 30

Let $\mathcal{C}$ be a category with binary products. We define the **squaring functor** as follows. For each object A, fix a product $(A \times A, \rho_1, \rho_2)$ of A with itself. Let $F\colon \mathcal{C} \Rightarrow \mathcal{C}$ send A to $A \times A$.

For a morphism $f\colon A \to B$ in $\mathcal{C}$, we want to define an appropriate morphism

$$Ff\colon (A \times A, \rho_1, \rho_2) \to (B \times B, \sigma_1, \sigma_2)$$

This clearly calls for the mediating morphism trick. So we need a couple of maps: one from $A \times A$ to B_1 and one from $A \times A$ to B_2.

The two compositions $f \circ \rho_i\colon A \times A \to B_i$ for $i = 1, 2$ will do the trick. Specifically, there is a unique mediating morphism

$$\theta_f : A \times A \to B \times B$$

as shown in Figure 21,

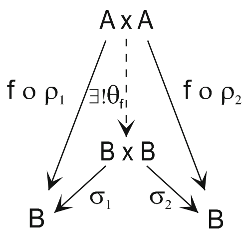

Figure 21

for which

$$\sigma_i \circ \theta_f = f \circ \rho_i$$

Let $Ff = \theta_f$. Then Ff is uniquely defined by the conditions

$$\sigma_i \circ Ff = f \circ \rho_i (i = 1, 2)$$

It is clear that $F1_A = 1_A$, because we have fixed a single product for each pair of objects. Also, if f: $A \to B$, g: $B \to C$ and the product for C is $(C \times C, \tau_1, \tau_2)$, then

$$\tau_i \circ (Fg \circ Ff) = g \circ \sigma_i \circ Ff = g \circ f \circ \rho_i = \tau_i \circ F(g \circ f)$$

for all i and so $Fg \circ Ff = F(g \circ f)$. Thus, F is a covariant functor on $\mathcal{C}$. □

■ **Example 31**

Let $\mathcal{C}$ be a category with binary products. To define a **product functor**

$$F: \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$$

we must assume that for every pair (X, Y) of objects in $\mathcal{C}$, we have selected a product

$$(X \times Y, \zeta_1, \zeta_2)$$

The product functor F takes an object (A_1, A_2) to its chosen product $(A_1 \times A_2, \alpha_1, \alpha_2)$ and a morphism

$$(f_1, f_2): (A_1, A_2) \to (B_1, B_2)$$

to the product morphism

$$f_1 \times f_2: (A_1 \times A_2, \alpha_1, \alpha_2) \to (B_1 \times B_2, \beta_1, \beta_2) \tag{32}$$

recall that $f_1 \times f_2$ is defined as the unique morphism satisfying the conditions

$$\beta_1 \circ (f_1 \times f_2) = f_1 \circ \alpha_1 \quad \text{and} \quad \beta_2 \circ (f_1 \times f_2) = f_2 \circ \alpha_2$$

To see that F is a functor, we must first show that $1_{A_1} \times 1_{A_2}$ is the identity $1_{A_1 \times A_2}$ on $A_1 \times A_2$ and for this, we use (32). Since

$$\alpha_1 \circ 1_{A_1 \times A_2} = 1_{A_1} \circ \alpha_1 \quad \text{and} \quad \alpha_2 \circ 1_{A_1 \times A_2} = 1_{A_2} \circ \alpha_2$$

the uniqueness condition implies that

$$F(1_A, 1_B) = 1_A \times 1_B = 1_{A \times B}$$

As to composition, suppose that

$$g_1 \times g_2 \colon (B_1 \times B_2, \beta_1, \beta_2) \to (C_1 \times C_2, \gamma_1, \gamma_2)$$

Then

$$F[(g_1, g_2) \circ (f_1, f_2)] = F[(g_1 \circ f_1, g_2 \circ f_2)] = (g_1 \circ f_1) \times (g_2 \circ f_2)$$

Hence, by definition, the map $h = F[(g_1, g_2) \circ (f_1, f_2)]$ is the *unique* map for which

$$\gamma_1 \circ h = (g_1 \circ f_1) \circ \alpha_1 \quad \text{and} \quad \gamma_2 \circ h = (g_2 \circ f_2) \circ \alpha_2$$

The uniqueness conditions implies that we need only show that the map

$$k = F[(g_1, g_2)] \circ F[(f_1, f_2)] = (g_1 \times g_2) \circ (f_1 \times f_2)$$

also satisfies these equations, that is, that

$$\gamma_1 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] = (g_1 \circ f_1) \circ \alpha_1$$

and

$$\gamma_2 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] = (g_2 \circ f_2) \circ \alpha_2$$

As to the first of these equations, we have

$$\begin{aligned} \gamma_1 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] &= (g_1 \circ \beta_1) \circ (f_1 \times f_2) \\ &= g_1 \circ (\beta_1 \circ (f_1 \times f_2)) \\ &= g_1 \circ (f_1 \circ \alpha_1) \end{aligned}$$

as desired. The second equation is proved similarly. □

We have saved the most important example of a functor (at least from the perspective of category theory itself) for last.

■ **Example 33**

One of the most important classes of functors are the *hom functors*, shown in Figure 22. Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$. We refer to A as the **source object** for the hom functor.

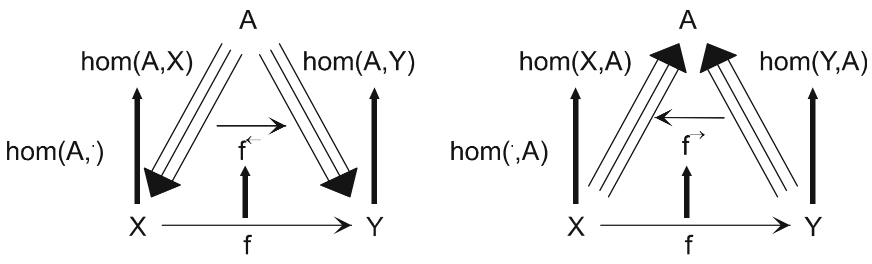

Figure 22

The **covariant hom functor**

$$\hom(A,\ \cdot\ ) : \mathcal{C} \Rightarrow \mathbf{set}$$

sends an object $X \in \mathcal{C}$ to the hom-set of all morphisms from the source object A to X,

$$\hom(A,\ \cdot\ )(X) = \hom(A, X)$$

and it sends a morphism $f \colon X \to Y$ to the "follow by f" map,

$$\hom(A,\ \cdot\ )f = f^{\leftarrow}$$

Thus,

$$f^{\leftarrow} : \hom(A, X) \to \hom(A, Y)$$

is defined by

$$f^{\leftarrow}\tau = f \circ \tau$$

for any $\tau \colon A \to X$. This functor is covariant precisely because

$$(g \circ f)^{\leftarrow} = g^{\leftarrow} \circ f^{\leftarrow}$$

Covariant hom functors are also called **covariant representable functors**.

Dually, the **contravariant hom functor**

$$\hom(\ \cdot\ ,\ A) \colon \mathcal{C} \Rightarrow \mathbf{set}$$

is defined by

$$\hom(\ \cdot\ ,\ A)(X) = \hom(X, A)$$

for all $X \in \mathcal{C}$ and

$$\hom(\ \cdot\ ,\ A)(f) = f^{\rightarrow}$$

where $f^{\rightarrow}$ is the "preceed by f" map,

$$f^{\rightarrow}\tau = \tau \circ f$$

for any $\tau \colon Y \to A$. This functor is contravariant precisely because

$$(g \circ f)^{\rightarrow} = f^{\rightarrow} \circ g^{\rightarrow}$$

Contravariant hom functors are also called **contravariant representable functors**. □

Morphisms of Functors: Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We would like to form a new category, denoted by $\mathcal{D}^{\mathcal{C}}$, whose objects are the *functors* from $\mathcal{C}$ to $\mathcal{D}$. But what about the morphisms between functors?

Consider a pair of parallel covariant functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$, as shown in Figure 23. As discussed earlier in the book, we think of F and G as mapping one-arrow diagrams.

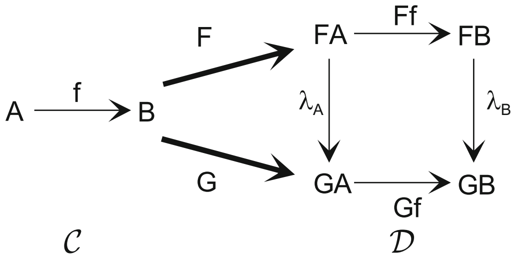

Figure 23 *A natural transformation*

A structure-preserving map between F and G is a "map" between the image one-arrow diagrams

$$FA \xrightarrow{Ff} FB \quad \text{and} \quad Gf\colon GA \xrightarrow{Gf} GB$$

As shown in Figure 23, this is accomplished by a *family* of morphisms in $\mathcal{D}$

$$\lambda = \{\lambda_A\colon FA \to GA \mid A \in \mathcal{D}\}$$

for which the square in Figure 23 commutes, that is,

$$Gf \circ \lambda_A = \lambda_B \circ Ff$$

The family λ is called a *natural transformation* from F to G.

Definition

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be parallel functors of the same type (both covariant or both contravariant). A **natural transformation** *from F to G, denoted by $\lambda\colon F \dot{\to} G$ or $\{\lambda_A\}\colon F \dot{\to} G$ is a family of morphisms in $\mathcal{D}$*

$$\lambda = \{\lambda_A\colon FA \to GA \mid A \in \mathcal{D}\}$$

for which the appropriate square in Figure 24 commutes. Specifically, if F and G are covariant, as shown on the left in Figure 24, then

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

for any $f: A \to B$ *in* $\mathcal{C}$ *and if* F *and* G *are contravariant, as shown on the right in Figure 24, then*

$$\lambda_A \circ Ff = Gf \circ \lambda_B$$

for any $f: A \to B$ *in* $\mathcal{C}$*. Each morphism* λ_A *is called a* **component** *of* λ*. It is customary to say that* λ_A *is* **natural in** A **from** F *to* G*. We denote the class of natural transformations from* F *to* G *by* $\mathrm{Nat}(F, G)$*. We also use the notation* $\lambda(A)$ *for* λ_A *when it is more convenient.* □

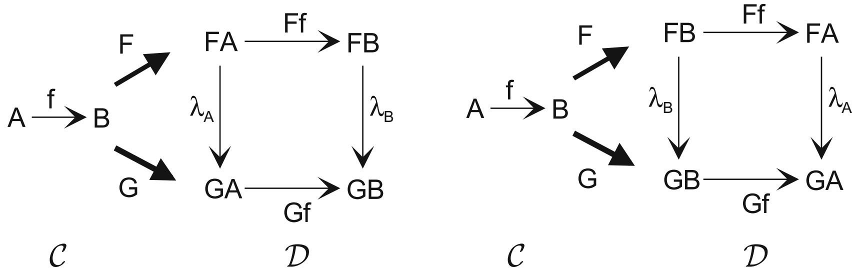

Figure 24 *Natural transformations*

Some authors refer to the function $\lambda: A \mapsto \lambda_A$ that maps an object $A \in \mathcal{C}$ to the component λ_A as a natural transformation as well.

Intuitively Speaking

Intuitively speaking, we can think of a natural transformation as follows. If $f: A \to B$ is a morphism in $\mathcal{C}$, then let us think of Ff and Gf as two different *versions* of f. Then the natural transformation condition

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

is a kind of **commutativity rule**, for it says that we can swap λ (actually, an appropriate component of λ) with one version of f provided we change the version of f.

An Example

Let us do an example.

▪ Example 34 (The determinant)

Fix a positive integer n. As shown in Figure 25, consider two parallel functors G, U: **CRng** $\Rightarrow$ **Grp** defined as follows. The functor G sends a ring R to the general linear group $GL_n(R)$ and a morphism $f: R \to S$ to the map f applied elementwise to the elements of a matrix, which we denote by f_e,

$$G\colon R \mapsto GL_n(R), \quad G\colon (f\colon R \to S) \mapsto (f_e\colon GL_n(R) \to GL_n(S))$$

The functor U sends a ring R to its group R^* of units and a ring map f to the restricted map $f_u\colon R^* \to S^*$, which makes sense since a ring map sends units to units,

$$U\colon R \mapsto R^*, \quad U\colon (f\colon R \to S) \mapsto (\,f\colon R^* \to S^*)$$

So we have two versions of the ring map f: apply f to matrices elementwise and apply f to units. Can you think of some operation $\{\lambda_A\}$ that "commutes" with these two versions of f, that is, for which

$$\lambda_S \circ f_e = f_u \circ \lambda_R$$

for $f\colon R \to S$?

Well, the determinant does not care whether it is applied before or after a ring map f, more precisely, before f_u or after f_e, in symbols,

$$\det(f_e A) = f_u(\det(A))$$

Thus,

$$\det_S \circ f_e = f \circ \det_R$$

which says that $\{\det_R \mid R \in \mathbf{CRng}\}$ is *natural* in R. □

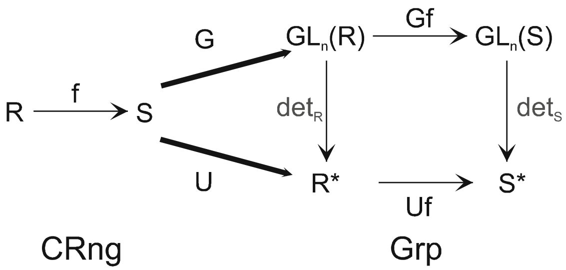

Figure 25 *The determinant is natural*

We will do some additional examples in a moment.

Composition of Natural Transformations

Natural transformations can be composed by composing corresponding components. In particular, if $\lambda\colon F \dot{\to} G$ and $\mu\colon G \dot{\to} H$ are natural transformations, then the composition $\mu \circ \lambda\colon F \dot{\to} H$ is defined by

$$(\mu \circ \lambda)_A = \mu_A \circ \lambda_A$$

We leave it to the reader to show that the composition of natural transformations is a natural transformation. Also, the identity natural transformation $1\colon F \dot{\to} F$ is defined by specifying that

$$1(A) = 1_{FA}$$

Natural Isomorphisms

Suppose that $\lambda\colon F \dot{\to} G$ is a natural transformation from F to G, where each component $\lambda_A\colon FA \approx FB$ is an isomorphism. The condition that λ is natural is

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

Applying λ_B^{-1} on the left and λ_A^{-1} on the right gives

$$Ff \circ \lambda_A^{-1} = \lambda_B^{-1} \circ Gf$$

Thus, the family $\mu = \{\lambda_A^{-1} \mid A \in \mathcal{C}\}$ is a natural transformation from G to F, that is, $\mu\colon G \dot{\to} F$. Moreover, $\mu \circ \lambda = 1$ and $\lambda \circ \mu = 1$ are the respective identity natural isomorphisms (each component is an identity morphism).

Conversely, if $\lambda\colon F \dot{\to} G$ and $\mu\colon G \dot{\to} F$ are natural transformations for which $\mu \circ \lambda = 1$ and $\lambda \circ \mu = 1$, then λ_A is an isomorphism for all $A \in \mathcal{C}$.

■ **Theorem 35**

Let $\lambda\colon F \dot{\to} G$ be a natural transformation. The following are equivalent:
1) Each component of λ is an isomorphism.
2) There is a natural transformation $\mu\colon G \dot{\to} F$ for which

$$\mu \circ \lambda = 1 \quad \textit{and} \quad \lambda \circ \mu = 1$$

where 1 is the appropriate natural isomorphism all of whose components are identity morphisms.

When these statements hold, we say that λ is a **natural isomorphism** *and that F and G are* **naturally isomorphic**, *written $\lambda\colon F \dot{\approx} G$ or $F \dot{\approx} G$. When F and G are set-valued, we use the notation $\dot{\leftrightarrow}$. in place of $\dot{\approx}$, since the components are bijections in this case.* □

More Examples of Natural Transformations

Let us consider some additional examples of natural transformations.

■ **Example 36 (The double-dual)**

Let **Vect** be the category of vector spaces over a field k, with linear maps. We need a little vector space theory for this example. As you probably know, the **dual space** V^* of a vector space V is the family of linear functionals on V. Hence, the **double-dual space** V^{**} is the family of linear functionals on V^*.

For example, if $v \in V$, then the **evaluation** at v map $\overline{v}\colon V^* \to k$ defined by

$$\overline{v}(f) = f(v)$$

for all $f \in V^*$ belongs to the double dual V^{**}. Let us set

$$\epsilon_V\colon V \to V^{**}, \quad \epsilon_V(v) = \overline{v}$$

The **operator adjoint** $\tau^{\to}\colon W^* \to V^*$ of a linear map $\tau\colon V \to W$ is defined by

$$\tau^{\to}(f) = f \circ \tau$$

Therefore, the **second adjoint** $\tau^{\to\to}\colon V^{**} \to W^{**}$ is given by

$$\tau^{\to\to}(\alpha) = \alpha \circ \tau^{\to}$$

for $\alpha \in V^{**}$.

In this case, we would like to find two versions of a linear map $\tau\colon V \to W$ that commute with evaluation.

We begin by looking at $\epsilon_W \circ \tau$. If $v \in V$, then

$$(\epsilon_W \circ \tau)(v) = \epsilon_W(\tau v) = \overline{\tau v}$$

Now we want to massage this until evaluation pops out the front. Applying $\overline{\tau v}$ to $f \in V^*$ gives

$$\overline{\tau v}(f) = f(\tau(v)) = \overline{v}(f \circ \tau) = \overline{v}(\tau^{\to}(f)) = (\overline{v} \circ \tau^{\to})(f)$$

and so

$$\overline{\tau v} = \overline{v} \circ \tau^{\to}$$

Thus,

$$(\epsilon_W \circ \tau)(v) = \overline{\tau v} = \overline{v} \circ \tau^{\to} = \tau^{\to\to}(\overline{v}) = \tau^{\to\to}(\epsilon_V(v)) = (\tau^{\to\to} \circ \epsilon_V)(v)$$

and we finally arrive at

$$\epsilon_W \circ \tau = \tau^{\to\to} \circ \epsilon_V \tag{37}$$

We can now put this in the language of natural transformations. Define a functor $F\colon$ **Vect** $\Rightarrow$ **Vect** that takes a vector space V to its double dual V^{**} and a linear map $\tau\colon V \to W$ to its double adjoint,

$$F\colon V \mapsto V^{**} \quad \text{and} \quad F\colon \tau \mapsto \tau^{\to\to}$$

Then (37) can be written as

$$F\tau \circ \epsilon_V = \epsilon_W \circ I\tau$$

where I is the identity functor on **Vect**. Thus, as shown in Figure 26, the family $\{\epsilon_V \mid V \in \textbf{Vect}\}$ is natural in V.

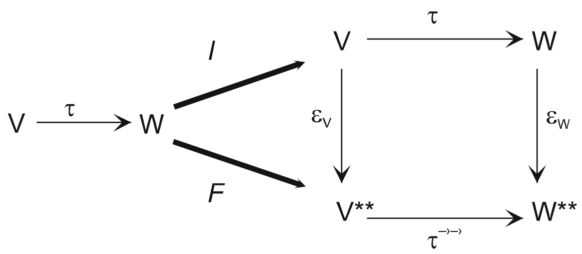

Figure 26

This example is a little more abstruse than the determinant example. The determinant example says that we can apply a ring map either before or after taking the determinant. This example says that we can either follow a linear map τ by evaluation or preceed its *second adjoint* $\tau^{\rightarrow\rightarrow}$ by evaluation. Whoever would have guessed that?

We leave it to you to show that the family is a natural *isomorphism* when restricted to the category of *finite-dimensional* vector spaces. □

■ Example 38 (The Riesz map)

For the category **Vect**, the **dual functor** G is defined by

$$GV = V^* \quad \text{and} \quad G\tau = \tau^{\rightarrow}$$

In examining the relationship between vector spaces and their duals, it is immediately clear that there cannot be a natural transformation from the identity functor on **Vect** to the dual functor on **Vect** because the identity functor is *covariant* but the dual functor is *contravariant*.

On the other hand, there is an important (and basis free) natural transformation for finite-dimensional *inner product* spaces. Let **FinInner** be the category of finite-dimensional real inner product spaces, with unitary transformations. A linear transformation $\sigma: V \to W$ is **unitary** if it is a bijection and

$$\langle \sigma u, v\rangle = \left\langle u, \sigma^{-1} v\right\rangle$$

The background we need here is the Riesz representation theorem. Define the **Riesz map** $R_V: V \to V^*$ by

$$R_V(v)(x) = \langle v, x\rangle$$

In words, $R_V(v)$ is "inner product with v." Because V is finite-dimensional, the Riesz representation theorem says that R_V is an isomorphism and so each element of V^* has the form $R_V(v) = \langle v, \cdot \rangle$ for a unique $v \in V$.

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "R_V"'] & W \arrow[d, "R_W"] \\ V^* \arrow[r, "(\tau^{-1})^{\to}"'] & W^* \end{tikzcd}$$

Figure 27

In an effort to find a commutativity rule involving the Riesz maps, we write for any $v \in V$,

$$\begin{aligned}(R_W \circ \tau)(v) &= R_W(\tau v) \\ &= \langle \tau v, \cdot \rangle \\ &= \langle v, \tau^{-1} \cdot \rangle \\ &= \langle v, \cdot \rangle \circ \tau^{-1} \\ &= R_V(v) \circ \tau^{-1} \\ &= (\tau^{-1})^{\to}(R_V(v)) \\ &= \left((\tau^{-1})^{\to} \circ R_V\right)(v)\end{aligned}$$

and so

$$R_W \circ \tau = \left(\tau^{-1}\right)^{\to} \circ R_V$$

This prompts us to make the following definition. Define the **Riesz functor** G by

$$GV = V^* \quad \text{and} \quad G(\tau) = \left(\tau^{-1}\right)^{\to}$$

where $\tau: V \to W$ is unitary. Then

$$R_W \circ \tau = G\tau \circ R_V$$

and so the family $\{R_V \mid V \in \mathbf{FinInner}\}$ is natural. In words, we can swap the Riesz maps with τ and $(\tau^{-1})^{\to}$.

Example 39 (The coordinate map)

Let k be a field. For each nontrivial vector space V over k, choose an ordered basis $\mathcal{B}_V$. Choose the standard basis $\mathcal{E}_n$ for the vector spaces k^n. Let $\mathbf{FinVectB}^*$ be the category whose objects are the ordered pairs $(V, \mathcal{B}_V)$. We will write V_n to denote the fact that V has dimension n.

The morphisms $\tau: (V_n, \mathcal{B}_V) \to (W_m, \mathcal{B}_W)$ are just the usual linear transformations $\tau: V_n \to W_m$.

Now, the coordinate map is defined by

$$\phi_{(V,\,\mathcal{B}_V)}\colon (V_n, \mathcal{B}_V) \to (k^n, \mathcal{E}_n), \quad \phi_{(V,\,\mathcal{B}_V)}(v) = [v]_{\mathcal{B}_V}$$

where $[v]_{\mathcal{B}_V}$ is the coordinate matrix of v with respect to $\mathcal{B}_V$ is an isomorphism.

The coordinate map can be used to define the matrix representation $[\tau]_{\mathcal{B},\mathcal{C}}$ of a linear map $\tau\colon V \to W$ with respect to a pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively. Recall that this matrix satisfies the equation

$$[\tau v]_{\mathcal{B}_W} = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}[v]_{\mathcal{B}_V}$$

where $[x]_{\mathcal{B}}$ denotes the coordinate matrix of x with respect to $\mathcal{B}$. In terms of coordinate maps, this can be written

$$\phi_{(W,\mathcal{B}_W)}(\tau v) = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}\phi_{(V,\mathcal{B}_V)}(v)$$

or equivalently in terms of the matrix multiplication operator,

$$\phi_{(W,\mathcal{B}_W)} \circ \tau = [\tau]_{\mathcal{B}_V,\mathcal{B}_W} \circ \phi_{(V,\mathcal{B}_V)} \tag{40}$$

Now it is time for some functors, one being the identity functor I on $\mathbf{FinVectB}^*$. The other functor G is the **matrix representation functor** defined by

$$G(V_n, \mathcal{B}_V) = (k^n, \mathcal{E}_n)$$

and

$$G\tau = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}$$

(We leave it to you to check that this is a covariant functor.)

Then (40) becomes

$$\phi_{(W,\mathcal{B}_W)} \circ I\tau = G\tau \circ \phi_{(V,\mathcal{B}_V)}$$

which shows (see Figure 28) that the family $\left\{\phi_{(V,\mathcal{B}_V)} \;\middle|\; V \in \mathbf{FinVectB}^*\right\}$ is natural in V.

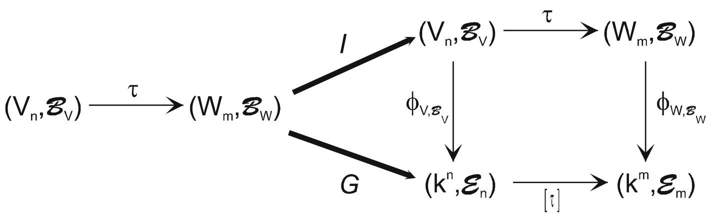

Figure 28 *The coordinate maps are natural*

In words, to swap factors in the composition $\phi_{(W,\mathcal{B}_W)} \circ \tau$, replace τ by its matrix representation. □

Example 41 (Arrow part of functor is natural transformation between hom functors)
Let $F: \mathcal{C} \Rightarrow \mathcal{D}$ be a functor and let $A \in \mathcal{C}$. If $f: X \rightarrow Y$ and $g: A \rightarrow X$ in $\mathcal{C}$, then

$$F(f \circ g) = Ff \circ Fg$$

which can also be written in the form

$$F(f^{\leftarrow}(g)) = \big((Ff)^{\leftarrow} \circ F\big)(g)$$

and so

$$F \circ f^{\leftarrow} = (Ff)^{\leftarrow} \circ F$$

This shows that the square in Figure 29 commutes.

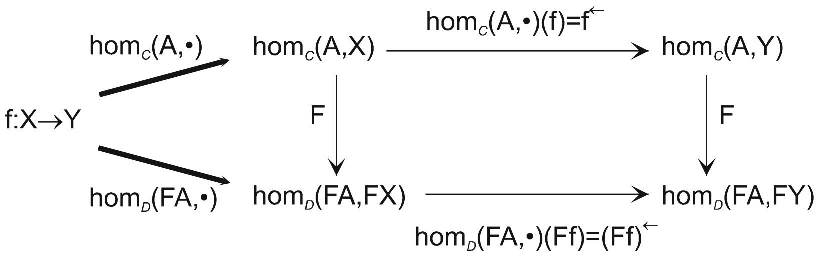

Figure 29

It follows that the arrow parts

$$F_X\colon \mathrm{hom}_{\mathcal{C}}(A, X) \rightarrow \mathrm{hom}_{\mathcal{D}}(A, FX)$$

of the functor F actually form a natural transformation from the hom functor $\mathrm{hom}_{\mathcal{C}}(A, \cdot)$ with source A to the hom functor $\mathrm{hom}_{\mathcal{D}}(FA, F\cdot)$ with source FA, in symbols

$$F\colon \mathrm{hom}_{\mathcal{C}}(A,\ \cdot\ ) \stackrel{\cdot}{\rightarrow} \mathrm{hom}_{\mathcal{D}}(FA, F\ \cdot\ )$$

We will refer to this natural transformation by the name **arrow part** of F. □

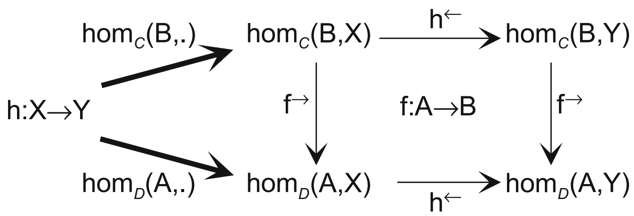

Figure 30

Example 42 (Any morphism defines a natural transformation between hom functors)
Let $\mathcal{C}$ be a category and let f, h and α be morphisms in $\mathcal{C}$ for which the composition exists

$$h \circ \alpha \circ f$$

Because composition is associative, this composition can be written in two ways using the hom functors as follows

$$f^{\rightarrow}(h^{\leftarrow}(\alpha)) = h \circ \alpha \circ f = h^{\leftarrow}(f^{\rightarrow}(\alpha))$$

and so

$$f^{\rightarrow} \circ h^{\leftarrow} = h^{\leftarrow} \circ f^{\rightarrow}$$

Thus, the square in Figure 30 commutes and so the morphism $f\colon A \to B$ defines a natural transformation

$$\{f^{\rightarrow}\}\colon \hom_{\mathcal{C}}(B, \cdot\,) \stackrel{\cdot}{\rightarrow} \hom_{\mathcal{C}}(A, \cdot\,)$$

where each component is $f^{\rightarrow}$ (applied to the appropriate domain). We will see a bit later that all natural transformations between hom functors have this form. □

Natural Isomorphisms and Full Faithfulness

It is not surprising that a natural isomorphism of functors preserves fullness and faithfulness. We leave proof of the following as an exercise.

Theorem 43
1) *Let $F \stackrel{\cdot}{\approx} G$ be naturally isomorphic functors.*
   a) *F is faithful if and only if G is faithful.*
   b) *F is full if and only if G is full.*
   In particular, if $F \stackrel{\cdot}{\approx} I_{\mathcal{C}}$, then F is fully faithful.
2) *Let $F\colon \mathcal{C} \Rightarrow D$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be functors.*
   a) *If $G \circ F$ is faithful, then F is faithful.*
   b) *If $G \circ F$ is full, then G is full.*

In particular, if

$$G \circ F \mathrel{\dot{\approx}} I_{\mathcal{C}} \quad \textit{and} \quad F \circ G \mathrel{\dot{\approx}} I_{\mathcal{D}}$$

then F and G are fully faithful. □

Functor Categories

As mentioned earlier, if $\mathcal{C}$ and $\mathcal{D}$ are categories, we would like to form the category $\mathcal{D}^{\mathcal{C}}$, whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations between functors. The only problem is that our definition of category requires that each hom-set be a *set*, but the class of natural transformations between two functors need not be a set. This issue can be resolved by requiring $\mathcal{C}$ to be a small category, that is, by requiring that $\mathbf{Obj}(\mathcal{C})$ be a set. From now on, when we use the functor category $\mathcal{D}^{\mathcal{C}}$, it is with the tacit assumption that $\mathcal{C}$ is small.

▪ Example 44

Let $\mathbf{2}$ be the category whose objects are 0 and 1 and whose morphisms are 1_0, 1_1 and $01\colon 0 \to 1$. Then each functor $F\colon \mathbf{2} \Rightarrow \mathcal{D}$ essentially just selects an arrow $F(01)\colon F(0) \to F(1)$ of $\mathcal{D}$. Moreover, a natural transformation $\{\lambda_0, \lambda_1\}\colon F \dot{\to} G$ is a pair of morphisms in $\mathcal{D}$, as shown in Figure 31.

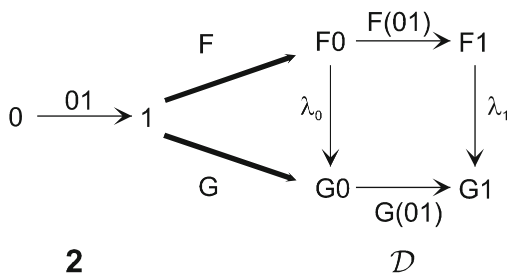

Figure 31

Hence, the functor category $\mathcal{D}^{\mathbf{2}}$, whose objects are the functors $F\colon \mathbf{2} \Rightarrow \mathcal{D}$ and whose morphism are the natural transformations $\{\lambda_0, \lambda_1\}\colon F \dot{\to} G$ between functors is just the category $\mathcal{D}^{\to}$ of arrows of $\mathcal{D}$. □

The Category of Diagrams

If $\mathcal{C}$ is a small category, the family of all diagrams $J\colon \mathcal{J} \Rightarrow \mathcal{C}$ in $\mathcal{C}$ over a particular index category $\mathcal{J}$ form the objects of a new category $\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$. The morphisms $f\colon F \Rightarrow G$ from diagram $J\colon \mathcal{J} \Rightarrow \mathcal{C}$ to diagram $G\colon \mathcal{J} \Rightarrow \mathcal{C}$ are simply the natural transformations from J to G.

We can draw a picture of a morphism

$$\{\lambda_n\} \colon \mathbb{D} \dot{\to} \mathbb{E}$$

from diagram $\mathbb{D}$ to diagram $\mathbb{E}$ as shown in Figure 32. For this to represent a morphism, the square must be commutative.

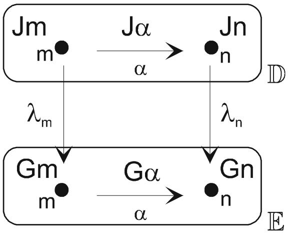

Figure 32

The category of diagrams will prove to be quite useful to us when we discuss universality later in the book.

Natural Equivalence

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *naturally equivalent* if there are antiparallel covariant functors $F \colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G \colon \mathcal{D} \Rightarrow \mathcal{C}$ for which the compositions $F \circ G$ and $G \circ F$ are naturally isomorphic to the corresponding identity functors $I_{\mathcal{D}}$ and $I_{\mathcal{C}}$. There is also a similar concept for contravariant functors.

Definition

1) *Two categories $\mathcal{C}$ and $\mathcal{D}$ are* **naturally equivalent** *if there are covariant functors $F \colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G \colon \mathcal{D} \Rightarrow \mathcal{C}$ for which*

$$F \circ G \dot{\approx} I_{\mathcal{D}} \quad \textit{and} \quad G \circ F \dot{\approx} I_{\mathcal{C}}$$

where $I_{\mathcal{D}}$ and $I_{\mathcal{C}}$ are identity functors.

2) *Two categories $\mathcal{C}$ and $\mathcal{D}$ are* **dually equivalent** (*or* **dual**) *if there are contravariant functors $F \colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G \colon \mathcal{D} \Rightarrow \mathcal{C}$ for which*

$$F \circ G \dot{\approx} I_{\mathcal{D}} \quad \textit{and} \quad G \circ F \dot{\approx} I_{\mathcal{C}}$$

where $I_{\mathcal{D}}$ and $I_{\mathcal{C}}$ are identity functors. □

Note that the functors in this definition are fully faithful.

Example 45

Let us show that the category $\mathbf{FinVect}^*$ of nonzero finite-dimensional vector spaces over a field k and the matrix category $\mathbf{Matr}_k$ are naturally equivalent. We assume that for each vector space

V in **FinVect***, an ordered basis $\mathcal{B}_V$ is chosen and that for the vector spaces k^n, the chosen basis is the standard basis $\mathcal{E}_n$.

The dimension functor dim: **FinVect**$^* \Rightarrow$ **Matr**$_k$ sends V to its dimension and sends each linear transformation $\tau\colon V_n \to W_m$ to the $m \times n$ matrix $[\tau]$ of τ with respect to the chosen ordered bases for V_n and W_m,

$$V_n \xrightarrow{\tau} V_m \quad \overset{\text{dim}}{\Rightarrow} \quad n \xrightarrow{[\tau]} m$$

To see that dim is a functor, note that $\dim(1_V)$ is the identity matrix and if $\tau\colon U \to V$ and $\sigma\colon V \to W$ then

$$\dim(\sigma\tau) = [\sigma\tau]_{\mathcal{B}_U,\mathcal{B}_W} = [\sigma]_{\mathcal{B}_V,\mathcal{B}_W}[\tau]_{\mathcal{B}_U,\mathcal{B}_V} = \dim(\sigma)\dim(\tau)$$

In the other direction, consider the map

$$\exp\colon \mathbf{Matr}_k \Rightarrow \mathbf{FinVect}^*$$

that takes a positive integer n to the vector space k^n and an $m \times n$ matrix $M\colon n \to m$ to the multiplication by M map, denoted by μ_M:

$$n \xrightarrow{M} m \quad \overset{\exp}{\Rightarrow} \quad k^n \xrightarrow{\mu_M} k^m$$

Since $\mu_I = 1_V$ and $\mu_{MN} = \mu_M\,\mu_N$, it follows that exp is also a functor.

The composition dim $\circ$ exp: **Matr**$_k \Rightarrow$ **Matr**$_k$ is the identity functor, since for any positive integer n,

$$\dim \circ \exp(n) = \dim(k^n) = n$$

and for any $m \times n$ matrix M,

$$\dim \circ \exp(M) = \dim(\mu_M) = [\mu_M]_{\mathcal{E}_m,\mathcal{E}_n} = M$$

The composition exp $\circ$ dim is the matrix representation functor, since

$$\exp \circ \dim(V_n) = \exp(n) = k^n$$

and for $\tau\colon V \to W$,

$$\exp \circ \dim(\tau) = exp\left([\tau]_{\mathcal{B}_V,\mathcal{B}_W}\right) = \mu_{[\tau]}$$

Thus, dim $\circ$ exp *is* the identity functor whereas exp $\circ$ dim, while not equal to the identity functor, is naturally isomorphic to the identity functor. Hence, **FinVect*** and **Matr**$_k$ are equivalent categories. $\square$

Natural Transformations Between Hom Functors

Let us speak about natural transformations between hom functors. Let $\mathcal{C}$ be a small category. Recall that for each $A \in \mathcal{C}$, the covariant hom functor

$$\hom_{\mathcal{C}}(A,\ \cdot\ ) : \mathcal{C} \Rightarrow \mathbf{set}$$

with source A is defined by

$$\hom_{\mathcal{C}}(A,\ \cdot\ )(X) = \hom_{\mathcal{C}}(A, X)$$

and for each $f : X \to Y$ in $\mathcal{C}$,

$$\hom(A,\ \cdot\ )f = f_A^{\leftarrow}$$

where we have used subscripts to remind us to which domain the "follow by f" map applies.

Figure 33 shows the diagram for a natural transformation λ between two hom functors.

$$\begin{tikzcd}
 & & \hom_c(A,X) \arrow[r, "f^{\leftarrow}"] \arrow[dd, "\lambda_X"'] & \hom_c(A,Y) \arrow[dd, "\lambda_Y"] \\
X \arrow[r, "f"] & Y \arrow[ur, "\hom_c(A,\bullet)"] \arrow[dr, "\hom_c(B,\bullet)"'] & & \\
 & & \hom_c(B,X) \arrow[r, "f^{\leftarrow}"'] & \hom_c(B,Y)
\end{tikzcd}$$

Figure 33

The naturalness condition is the commutativity of the square, that is,

$$f_B^{\leftarrow} \circ \lambda_X = \lambda_Y \circ f_A^{\leftarrow} \tag{46}$$

Taking $X = A$ and applying this to the identity 1_A gives

$$\left(f_B^{\leftarrow} \circ \lambda_A\right)(1_A) = \left(\lambda_Y \circ f_A^{\leftarrow}\right)(1_A)$$

or

$$f_B^{\leftarrow}(\lambda_A(1_A)) = \lambda_Y(f \circ 1_A)$$

or (replacing Y by X),

$$f \circ (\lambda_A(1_A)) = \lambda_X(f)$$

or finally,

$$\lambda_X(f) = [\lambda_A(1_A)]^{\rightarrow}(f)$$

for all $f\colon A \to X$. Hence,

$$\lambda_X = [\lambda_A(1_A)]^{\rightarrow}$$

for all $f\colon A \to X$. Thus, *all* natural transformations have the form

$$\lambda = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}$$

where

$$h = \lambda_A(1_A) \in \hom_{\mathcal{C}}(B, A)$$

Note that all of the components $h_X^{\rightarrow}$ of λ do the same thing (preceed by h) but to different domains, so they are different morphisms.

Conversely, if $h \in \hom_{\mathcal{C}}(B, A)$, then the family $\{h^{\rightarrow}\}$ is natural from $\hom_{\mathcal{C}}(A, \cdot)$ to $\hom_{\mathcal{C}}(B, \cdot)$ because for any $g\colon A \to X$,

$$f_B^{\leftarrow} \circ h_X^{\leftarrow}(g) = f \circ (g \circ h_X) \quad \text{and} \quad h_X^{\rightarrow} \circ f_A^{\leftarrow}(g) = (f \circ g) \circ h_X$$

which are equal precisely because composition is associative. Thus, we have completely characterized the natural transformations between hom functors.

Theorem 47
Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. Then the natural transformations

$$\lambda\colon \hom_{\mathcal{C}}(A, \cdot) \dot{\to} \hom_{\mathcal{C}}(B, \cdot)$$

between hom functors are precisely the families

$$\lambda = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}$$

as h varies over the set $\hom_{\mathcal{C}}(B, A)$, *where for $X \in \mathcal{C}$, the X-component of λ,*

$$h_X^{\rightarrow}\colon \hom_{\mathcal{C}}(A, X) \dot{\to} \hom_{\mathcal{C}}(B, X)$$

is "preceed by h on $\hom_{\mathcal{C}}(A, X)$." □

The Yoneda Embedding

Now that we understand the nature of natural transformations between hom functors, we can define a rather important *contravariant* functor, called the *Yoneda embedding*

$$y\colon \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}}$$

as follows. To each object $A \in \mathcal{C}$, we associate the covariant hom functor $\hom_{\mathcal{C}}(A, \cdot)$ with source A. Thus, the object part of y is

$$y(A) = \hom_{\mathcal{C}}(A, \ \cdot\ )$$

The arrow part of y maps a morphism $h: B \to A$ to a natural transformation between hom functors and Theorem 47 gives us the "natural" choice

$$y(h) = \{h^{\to}\}: \hom_{\mathcal{C}}(A, \ \cdot\ ) \dot{\to} \hom_{\mathcal{C}}(B, \ \cdot\ )$$

To see that y actually is a contravariant functor, note that

$$y(1_A) = \{1_A^{\to}\}$$

is the identity natural transformation and that

$$y(g \circ h) = (g \circ h)^{\to} = h^{\to} \circ g^{\to} = y(g) \circ y(h)$$

It is customary to view y as a *covariant* functor,

$$y: \mathcal{C}^{op} \Rightarrow \mathbf{Set}^{\mathcal{C}}$$

from the opposite category $\mathcal{C}^{op}$ to the functor category $\mathbf{Set}^{\mathcal{C}}$, or equivalently, as a covariant functor

$$y: \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}^{op}}$$

However, lest all of these opposite categories give you a headache, we will leave the functor y alone and live with its contravariance.

▪ Theorem 48

Let $\mathcal{C}$ be a category. The contravariant functor $y: \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}}$ defined by

$$y(A) = \hom_{\mathcal{C}}(A, \ \cdot\ ) \quad \textit{and} \quad y(h) = \{h^{\to}\}: \hom_{\mathcal{C}}(A, \ \cdot\ ) \dot{\to} \hom_{\mathcal{C}}(B, \ \cdot\ )$$

for all $A \in \mathcal{C}$ and all $h \in \hom_{\mathcal{C}}(B, A)$ is a contravariant embedding of $\mathcal{C}$ into the functor category $\mathbf{Set}^{\mathcal{C}}$, called the **Yoneda embedding** *of $\mathcal{C}$ in $\mathbf{Set}^{\mathcal{C}}$.*

▪ Proof

We must show that y is an embedding, that is, that the object part of y is injective and that the local arrow parts of y are bijective. The object part of y maps A to $\hom_{\mathcal{C}}(A, \cdot)$ and since $\hom_{\mathcal{C}}(A, \cdot)$ and $\hom_{\mathcal{C}}(B, \cdot)$ are distinct for distinct objects A and B, the object part of y is injective.

To get the local arrow part of y, we fix $A, B \in \mathcal{C}$ to get the map

$$y_{A,B}: \hom_{\mathcal{C}}(B, A) \to \mathrm{Nat}(\hom_{\mathcal{C}}(A, \ \cdot\ ), \hom_{\mathcal{C}}(B, \ \cdot\ ))$$

given by

$$y_{A,B}(h) = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \overset{\cdot}{\rightarrow} \hom_{\mathcal{C}}(B,\ \cdot\ )$$

We have already proven that $y_{A,B}$ is surjective, that is, that all natural transformations from $\hom_{\mathcal{C}}(A,\cdot)$ to $\hom_{\mathcal{C}}(B,\cdot)$ have the form $\{h^{\rightarrow}\}$. As to injectivity, if $y_{A,B}(h) = y_{A,B}(k)$ for h,k: $B \rightarrow A$, then

$$\{h_X^{\rightarrow} \mid X \in C\} = \{k_X^{\rightarrow} \mid X \in C\}$$

In particular, for the components associated with $X = A$, we can apply them to 1_A to get

$$1_A \circ h = 1_A \circ k$$

and so $h = k$. Thus, the local arrow parts of *y* are injective and the Yoneda embedding is indeed an embedding. □

The Yoneda embedding states that any category $\mathcal{C}$ can be (contravariantly) embedded in the functor category $\mathbf{Set}^{\mathcal{C}}$ of set-valued functors on $\mathcal{C}$. In other words, each object $A \in \mathcal{C}$ can be *represented* as a hom functor $\hom_{\mathcal{C}}(A,\cdot)$ and each morphism as a natural transformation between hom functors.

To help remember the contravariant Yoneda embedding, we can also think of it as the **source embedding**. Specifically an object $A \in \mathcal{C}$ is used as the *source* of the hom functor $\hom_{\mathcal{C}}(A,\cdot)$ and a morphism h: $B \rightarrow A$ is used to *change* the source from A to B, since the embedding is contravariant. But to change the source from A to B, we must *preceed* by h.

▪ Example 49

It is said that the Yoneda embedding is a vast generalization of Cayley's theorem of group theory. Cayley's theorem says that any group G can be embedded in a permutation group. Specifically, for $a \in G$, **right translation** by a is defined by

$$\rho_a\colon G \rightarrow G, \quad \rho_a(g) = ga$$

Cayley's theorem says that the map

$$\rho\colon G \rightarrow S_G, \quad \rho(a) = \rho_a$$

is an embedding of G into the permutation group S_G.

Now recall that the group G can be thought of as a category $\mathcal{G}$ with just one object, namely G itself. Moreover, each element $a \in G$ is a morphism a: $G \rightarrow G$ and composition of morphisms is the group product of elements.

Since $\mathcal{G}$ has only one object, it has only one hom functor

$$\hom(G,\ \cdot\ )\colon \mathcal{G} \Rightarrow \mathbf{Set}$$

defined by

$$\hom(G,\ \cdot\ )G = \hom(G, G) = U(G)$$

where $U(G)$ is the underlying set of G (that is, G thought of simply as a set) and for $a \in G$,

$$\hom(G, \cdot\,)a = a^{\leftarrow}$$

The contravariant Yoneda embedding $y\colon \mathcal{G} \to \mathbf{Set}^{\mathcal{G}}$ is

$$y(G) = \hom(G, \cdot\,), \quad y(a) = \{a^{\rightarrow}\}$$

where

$$a^{\rightarrow}(b) = ba = \rho_a(b)$$

for all $b \in G$ and so

$$y(G) = \hom(G, \cdot\,), \quad y(a) = \{\rho_a\}$$

for all $a \in G$.

But in this case, since there is only one object G, their is only one component in the family $\{\rho_a\}$ and so the arrow part of y is essentially just the Cayley embedding ρ. $\square$

Yoneda's Lemma

Yoneda's lemma examines the nature of natural transformations from hom-set functors to *arbitrary* set valued functors. With reference to Figure 34, let $A \in \mathcal{C}$ and consider the hom functor with source A,

$$\hom_{\mathcal{C}}(A, \cdot\,)\colon \mathcal{C} \Rightarrow \mathbf{Set}$$

and any set-valued functor $H\colon \mathcal{C} \Rightarrow \mathbf{Set}$.

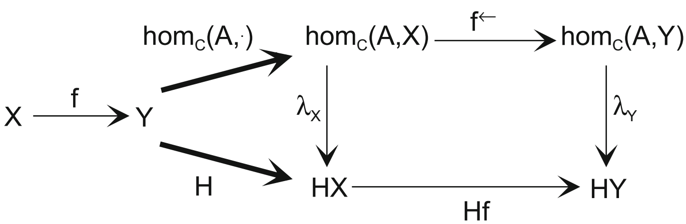

Figure 34

If

$$\lambda\colon \hom_{\mathcal{C}}(A, \cdot\,) \xrightarrow{\cdot} H$$

is a natural transformation, then for any $f\colon X \to Y$,

$$\lambda_Y \circ f^{\leftarrow} = Hf \circ \lambda_X$$

As in the special case we discussed earlier, we take $X = A$ and apply this to 1_A to get (after replacing Y by X),

$$\lambda_X(g) = (Hg)[\lambda_A(1_A)] \tag{50}$$

for any g: $A \to X$. To simplify the notation, let

$$a_A = \lambda_A(1_A) \in H(A)$$

and so

$$\lambda_X(g) = (Hg)a_A$$

Let us refer to the element $a_A \in H(A)$, which *completely characterizes* the natural transformation λ as the **Yoneda representative** of λ.

On the other hand, for any element $a \in HA$, the components λ_X defined by

$$\lambda_X(g) = (Hg)_a \tag{51}$$

for all $X \in \mathcal{C}$ and for all $g \in \hom_{\mathcal{C}}(A, X)$ form a natural transformation $\lambda = \{\lambda_X\}$, because for any $f\colon X \to Y$,

$$\begin{aligned}(\lambda_Y \circ f^{\leftarrow})g &= \lambda_Y(f \circ g)\\ &= H(f \circ g)a\\ &= (Hf \circ Hg)a\\ &= Hf \circ [\lambda_X(g)]\end{aligned}$$

and so

$$\lambda_Y \circ f^{\leftarrow} = Hf \circ \lambda_X$$

Note that by taking $X = A$ and $g = 1_A$ in (51), we get

$$a = \lambda_A(1_A)$$

Thus, any element of $H(A)$ is the Yoneda representative for some natural transformation λ from $\hom_{\mathcal{C}}(A, \cdot)$ to H.

■ **Theorem 52 (Yoneda lemma, part 1)**

*Let $\mathcal{C}$ be a category and let $H\colon \mathcal{C} \Rightarrow$ **Set** be a set-valued functor.*

1) *The natural transformations*

$$\lambda\colon \hom_{\mathcal{C}}(A, \cdot) \stackrel{\cdot}{\to} H$$

are precisely the maps defined by

$$\lambda_X(g) = (Hg)a$$

for all $g: A \to X$*, where* $a \in H(A)$*. The connection between* λ *and its Yoneda representative* a *is given by*

$$a = \lambda_A(1_A)$$

2) *The* **Yoneda representative map**

$$\phi = \phi_{H,A}: \mathrm{Nat}(\hom_{\mathcal{C}}(A, \,\cdot\,), H) \to H(A)$$

defined by

$$\phi(\lambda) = \lambda_A(1_A)$$

is a bijection. It follows that the class $\mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$ *is a set.*

3) *When* $H = \hom_{\mathcal{C}}(B, \cdot)$ *is also a hom functor, the natural transformations*

$$\lambda: \hom_{\mathcal{C}}(A, \,\cdot\,) \stackrel{\cdot}{\to} \hom_{\mathcal{C}}(B, \,\cdot\,)$$

are precisely the families

$$\lambda = \{\lambda^{\to}\}$$

as h *varies over the set* $\hom_{\mathcal{C}}(B, A)$.

■ Proof

We have already proved parts 1) and 3). For part 2), since

$$\lambda_X(g) = (Hg)[\lambda_A(1_A)]$$

it is clear that $\lambda_A(1_A) = \phi(\lambda)$ uniquely determines λ and so ϕ is injective. We have already seen that it is surjective. □

There is another part to Yoneda's lemma, which describes the naturalness of the families $\{\phi_A \mid A \in \mathcal{C}\}$, where H is fixed and $\{\phi_{A,H} \mid H \in \mathbf{Set}^{\mathcal{C}}\}$, where A is fixed

■ Theorem 53 (Yoneda lemma, part 2)

Let $\mathcal{C}$ *be a category. The family of Yoneda representative maps*

$$\{\phi_{H,A}: \mathrm{Nat}(\hom_{\mathcal{C}}(A, \,\cdot\,), H) \approx H(A) \mid A \in \mathcal{C}, H \in \mathbf{Set}^{\mathcal{C}}\}$$

is natural in both H *and* A*, as shown in the commutative diagram of Figure 35.*

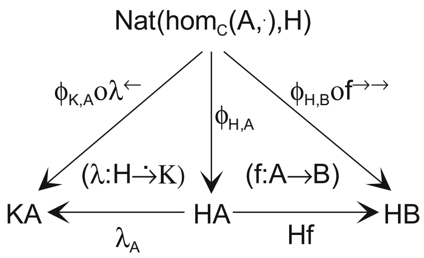

Figure 35

In particular:

1) *For a fixed* $H\colon \mathcal{C} \Rightarrow$ **Set**, *the family of bijections*

$$\{\phi_A\colon \mathrm{Nat}(\mathrm{hom}_{\mathcal{C}}(A,\ \cdot\ ),H) \leftrightarrow HA \mid A \in \mathcal{C}\}$$

is natural in A, as shown in Figure 36.

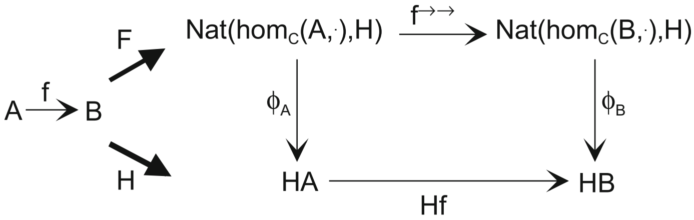

Figure 36

Specifically, define a functor $F\colon \mathcal{C} \Rightarrow$ **Set** *as follows. If* $A \in \mathcal{C}$ *then*

$$F(A) = \mathrm{Nat}(\mathrm{hom}_{\mathcal{C}}(A,\ \cdot\ ),H)$$

Also, if $f\colon A \to B$ *in* $\mathcal{C}$*, then*

$$F(f)\colon \mathrm{Nat}(\mathrm{hom}_{\mathcal{C}}(A,\ \cdot\ ),H) \to \mathrm{Nat}(\mathrm{hom}_{\mathcal{C}}(B,\ \cdot\ ),H)$$

is defined by

$$F(f) = f^{\rightarrow\rightarrow}$$

that is, if $\lambda = \{\lambda_X \mid X \in \mathcal{C}\} \in \mathrm{Nat}(\mathrm{hom}_{\mathcal{C}}(A,\cdot),H)$*, then*

$$f^{\rightarrow\rightarrow}(\lambda) = \{f^{\rightarrow\rightarrow}(\lambda_X) \mid X \in \mathcal{C}\} = \{\lambda_X \circ f^{\rightarrow} \mid X \in \mathcal{C}\}$$

Then

$$\phi_A\colon F \dot{\rightarrow} H$$

2) *If $\mathcal{C}$ is a small category, then for a fixed $A \in \mathcal{C}$, the family of bijections*

$$\{\phi_H\colon \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \leftrightarrow HA \,|\, H \in \mathbf{Set}^{\mathcal{C}}\}$$

is natural in H, as shown in Figure 37.

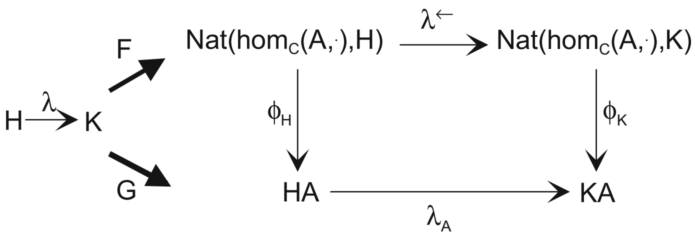

Figure 37

Specifically, define functors F, G: $\mathbf{Set}^{\mathcal{C}} \Rightarrow \mathbf{Set}$ as follows. For any $H \in \mathbf{Set}^{\mathcal{C}}$, let

$$F(H) = \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \quad \textit{and} \quad G(H) = HA$$

and for any $\lambda\colon H \dot{\rightarrow} K$ in $\mathbf{Set}^{\mathcal{C}}$,

$$F(\lambda) = \lambda^{\leftarrow} \quad \textit{and} \quad G(\lambda) = \lambda_A$$

where $\lambda^{\leftarrow}(\mu) = \lambda \circ \mu$. Then

$$\phi_H\colon F \dot{\rightarrow} G$$

▪ Proof

For part 1), we must first show that if

$$\{\lambda_X\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \dot{\rightarrow} H$$

then

$$\{\lambda_X \circ f^{\rightarrow}\}\colon \hom_{\mathcal{C}}(B,\ \cdot\ ) \dot{\rightarrow} H$$

that is, we must show that for all $g\colon X \to Y$,

$$Hg \circ \lambda_X \circ f^{\to} = \lambda_Y \circ f^{\to} \circ g^{\leftarrow}$$

But the condition that λ is natural is

$$Hg \circ \lambda_X = \lambda_Y \circ g^{\leftarrow}$$

and the result follows by applying $f^{\to}$ and noting that $g^{\leftarrow} \circ f^{\to} = f^{\to} \circ g^{\leftarrow}$.

Now, the naturalness of $\phi_A\colon F \overset{\cdot}{\to} H$ is

$$Hf \circ \phi_A = \phi_B \circ Ff$$

for all $f\colon A \to B$ in $\mathcal{C}$. This is equivalent to

$$Hf \circ \phi_A(\lambda) = \phi_B \circ Ff(\lambda)$$

for all $\lambda \in \mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$ and this is equivalent to

$$Hf[\lambda_A(1_A)] = \phi_B(\lambda \circ f^{\to})$$

But

$$\phi_B(\lambda \circ f^{\to}) = (\lambda \circ f^{\to})_B 1_B = \lambda_B \circ f^{\to} 1_B = \lambda_B(f)$$

and the Yoneda lemma implies that

$$\lambda_B(f) = Hf[\lambda_A 1_A]$$

as desired.

For part 2), the naturalness condition we wish to verify is

$$\lambda_A \circ \phi_H = \phi_K \circ \lambda^{\leftarrow}$$

But for $\mu \in \mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$,

$$\lambda_A \circ \phi_H(\mu) = \lambda_A[\mu_A(1_A)]$$

and

$$\phi_K \circ \lambda^{\leftarrow}(\mu) = \phi_K(\lambda \circ \mu) = (\lambda \circ \mu)_A(1_A) = \lambda_A[\mu_A(1_A)] \qquad \square$$

Exercises

1. Show that contravariant functors are also covariant functors.
2. Is the forgetful functor from **Grp** to **Set** full? Is it faithful?

3. Let G be a group and let G' be the commutator subgroup, that is, the subgroup generated by all commutators $aba^{-1}b^{-1}$, where $a, b \in G$. Then G' is a normal subgroup of G and G/G' is abelian. Let F: **Grp** $\Rightarrow$ **AbGrp** send G to G/G' and send $\sigma\colon G \to H$ to the map $F\sigma\colon G/G' \to H/H'$ defined by

$$(F\sigma)(aG') = (\sigma a)H'$$

   a) Show that this defines a functor.
   b) Modify the function F slightly so that it maps into **Grp** and find a natural transformation from the identity functor I: **Grp** $\Rightarrow$ **Grp** to F.
4. Find two distinct covariant functors from **Grp** to **Grp** both of whose object maps are the identity.
5. For the category **Grp**, map each group to its commutator subgroup $C(G)$ and each homomorphism $f\colon G \to H$ to its restriction $Cf\colon C(G) \to C(H)$. Show that C is a functor.
6. Show that the hom functor

$$\hom_{\mathcal{C}}(A, \ \cdot\ )\colon \mathcal{C} \Rightarrow \mathbf{Set}$$

preserves monics, that is, if α: $C \to D$ is monic in $\mathcal{C}$, then

$$\alpha^{\leftarrow}\colon \hom_{\mathcal{C}}(A, C) \to hom_{\mathcal{C}}(A, D)$$

is also monic.
7. Describe the arrow part of the hom functor $\hom_{\mathcal{C}}(B, \cdot)$ and the naturalness condition.
8. Let $\mathcal{C}$ be a category with binary products. Fix a product for each pair of objects in $\mathcal{C}$. If $f\colon A \to A'$ and $g\colon B \to B'$, then define the product $f \times g\colon A \times B \to A' \times B'$ as the unique mediating morphism from the cone

$$\left(A \times B, f \circ \rho_{A\times B,1}\colon A \times B \to A', g \circ \rho_{A\times B,2}\colon A \times B \to B'\right)$$

to the product $A' \times B'$. In other symbols,

$$f \times g = \begin{pmatrix} f \circ \rho_{A\times B,1} \\ g \circ \rho_{A\times B,2} \end{pmatrix}$$

   a) Prove that

$$1_A \times 1_B = 1_{A\times B}$$

   b) If f: $A \to A'$ and g: $B \to B'$ and f': $A' \to A''$ and g': $B' \to B''$ prove that

$$(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \circ g)$$

   or in different notation

$$\begin{pmatrix} f' \circ f \circ \rho_{A\times B,1} \\ g' \circ g \circ \rho_{A\times B,2} \end{pmatrix} = \begin{pmatrix} f' \circ \rho_{A'\times B',1} \\ g' \circ \rho_{A'\times B',2} \end{pmatrix} \circ \begin{pmatrix} f \circ \rho_{A\times B,1} \\ g \circ \rho_{A\times B,2} \end{pmatrix}$$

9. Let $\mathcal{C}$ be a category with binary products. Let A be an object in $\mathcal{C}$ and fix a product $C \times A$ for every object C in $\mathcal{C}$. Define the **product functor**

$$- \times A \colon \mathcal{C} \Rightarrow \mathcal{C}$$

sending C to $C \times A$ and $f\colon C \to D$ to the product morphism

$$f \times 1_A \overset{\text{def}}{=} \begin{pmatrix} f \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix} \colon C \times A \to D \times A$$

where $\begin{pmatrix} x_{XA} \\ y_{XB} \end{pmatrix}$ denotes the mediating morphism for a cone with legs x and y over the diagram $\{A, B\}$. Prove that this does define a functor.

10. Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be categories. A functor $H\colon \mathcal{B} \times \mathcal{C} \Rightarrow \mathcal{D}$ from the product $\mathcal{B} \times \mathcal{C}$ into a category $\mathcal{D}$ is called a **functor of two variables** or a **bifunctor**. If $B \in \mathcal{B}$, the map H_B is defined as follows: H_B takes an object C of $\mathcal{C}$ to $H(B,C)$ and takes a morphism $g\colon C \to C'$ to $H(1_{B,}\, g)$, where 1_B is the identity morphism on B. For any object C in $\mathcal{C}$, the map H_C is defined analogously.
   a) Show that $H_B\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $H_C\colon \mathcal{B} \Rightarrow \mathcal{D}$ are functors.
   b) Show that if $f\colon B \to B'$ and $g\colon C \to C'$, then

$$H_{B'}(g) \circ H_C(f) = H_{C'}(f) \circ H_B(g)$$

   c) Suppose that for each object C in $\mathcal{C}$, there is a functor $G_C\colon \mathcal{B} \Rightarrow \mathcal{D}$ and for each object B in $\mathcal{B}$, there is a functor $F_B\colon \mathcal{C} \Rightarrow \mathcal{D}$. Under what conditions is there a bifunctor $H\colon \mathcal{B} \times \mathcal{D} \Rightarrow \mathcal{D}$ for which $H_B = F_B$ and $H_C = G_C$?

11. Prove the following statements:
   1) Let $F \dot{\approx} G$ be naturally isomorphic functors.
      a) F is faithful if and only if G is faithful.
      b) F is full if and only if G is full.
      In particular, if $F \dot{\approx} I_{\mathcal{C}}$, then F is fully faithful.
   2) Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be functors.
      a) If $G \circ F$ is faithful, then F is faithful.
      b) If $G \circ F$ is full, then G is full.
      In particular, if

$$G \circ F \dot{\approx} I_{\mathcal{C}} \quad \text{and} \quad F \circ G \dot{\approx} I_{\mathcal{D}}$$

      then F and G are fully faithful.

12. Let $F, G: \mathcal{C} \Rightarrow \mathcal{D}$ and let $\lambda(C): F \stackrel{\cdot}{\to} G$ be a natural transformation from F to G.
   a) Let $H: \mathcal{E} \Rightarrow \mathcal{C}$. Let λH: **Obj**$(\mathcal{E}) \to$ **Obj**$(\mathcal{C})$ be defined by

$$\lambda H(E) = \lambda(H(E))$$

   Show that $\lambda H: FH \stackrel{\cdot}{\to} GH$ is a natural transformation.
   b) Let $H\lambda$: **Obj**$(\mathcal{E}) \to$ **Obj**$(\mathcal{C})$ be defined by

$$H\lambda(C) = H(\lambda(E))$$

   Show that $H\lambda: HF \stackrel{\cdot}{\to} HG$ is a natural transformation.
13. Let S be a nonempty set. A group with operators S or an S-group is a group G together with a homomorphism $\sigma: S \to \text{End}(G)$ where End(G) is the group of endomorphisms of G. Let M be a monoid, thought of as a category with one object, where each element of M is a morphism. Show that the objects in the functor category **Grp**^{M} are the groups with operators M.
14. Let A be an abelian group. The **torsion subgroup** A^t of A is the set of elements of A that have finite order. The **torsion functor** G: **AbGrp** $\Rightarrow$ **AbGrp** is defined by

$$GA = A^t$$

and for a group homomorphism $f: A \to B$

$$Gf = f\,\big|_{A^t}: A^t \to B^t$$

which makes sense since a group homomorphism maps torsion elements to torsion elements.
   a) Show that G is indeed a functor.
   b) Find a natural transformation from the torsion functor G to the identity functor I.
15. Let **FinSet** be the category of all finite sets and let **FinOrd** be the category of all finite ordinal numbers. The inclusion map I: **FinOrd** $\Rightarrow$ **FinSet** is a functor, with $I(f) = f$, as a set function. We define another functor Card: **FinSet** $\Rightarrow$ **FinOrd** as follows. Card(S) is the unique finite ordinal that is equipotent to S. For the maps, for each finite set S, we fix a bijection θ_S from S to Card(S), where if n is a finite ordinal then $\theta_n = 1$ (the identity). Then for a set function $f: S \to T$, the map Card(f): Card$(S) \to$ Card(T) is defined as

$$Card(f) = \theta_T \circ f \circ \theta_S^{-1}$$

   a) Show that Card $\circ\, I$: **FinOrd** $\Rightarrow$ **FinOrd** is the identity functor.
   b) Show that $I \circ$ Card: **FinSet** $\Rightarrow$ **FinSet** is not the identity functor, but is naturally isomorphic to the identity functor. Thus, **FinOrd** and **FinSet** are equivalent categories.

16. Let S be a fixed set. Consider the map F that sends each set X to $X^S \times S$, where X^S is the set of all functions from S to a set X. Show that F is the object map of a functor F from **Set** to **Set**. Find a natural transformation from F to the identity functor on **Set**.
17. Let $F, G\colon \mathcal{C} \Rightarrow \mathcal{P}$ be functors from a category $\mathcal{C}$ to a preorder $\mathcal{P}$.
    a) Describe necessary and sufficient conditions under which there is a natural transformation from F to G.
    b) Prove that if $\mathcal{P}$ and $\mathcal{Q}$ are preorders, then the functor category $\mathcal{Q}^{\mathcal{P}}$ is also a preorder.
18. Verify that the functor category $\mathcal{D}^{\mathcal{C}}$ is a category.
19. Prove that the functor category $\mathcal{D}^2$ is essentially the category of arrows $\mathcal{D}^{\rightarrow}$ of $\mathcal{D}$.
20. Show that the map that sends each group to its center cannot be the object map of a functor from **Grp** to **AbGrp**. *Hint*: Consider the triangle formed by $S_2 \to S_3 \to S_2$.
21. A **pointed set** is a pair $S_* = (S, s)$ where $s \in S$. Less formally, a pointed set is just a set that contains a specially designated element. To simplify the notation, we let $*$ denote this element. Let $\mathbf{Set}_*$ be the category whose objects are pointed sets and whose morphisms are all set functions $f\colon A_* \to B_*$ for which $f(*) = *$. These are called **pointed functions**. Let $\mathbf{Set}_0$ be the category whose objects are sets and whose morphisms are *partial* set functions $f\colon A \to B$, that is, the domain of f is a (possibly empty) subset of A. Prove that $\mathbf{Set}_*$ and $\mathbf{Set}_0$ are isomorphic.
22. If S is a set and $s \in S$, we define the ordered pair (S, s) to be a **set with base point** (or a **set with distinguished element**). Let $\mathbf{Set}_*$ be the category of pointed sets (see the previous exercise) and let $\mathcal{C}$ be the category of all sets with base point. Show that the map $F : \mathbf{Set}_* \Rightarrow \mathcal{C}$ sending S_* to $(S_*, *)$ and sending $f\colon S_* \to T_*$ to itself is a functor. Is it an isomorphism?
23. a) Let $F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$ and let

$$\lambda = \{\lambda_C\}\colon F \stackrel{.}{\to} G$$

Then if $H\colon \mathcal{D} \Rightarrow \mathcal{E}$, *then the composition*

$$H \circ \lambda_C\colon FC \to HGC$$

makes sense. Show that the family

$$H\lambda = \left\{H \circ \lambda_C \;\middle|\; C \in \mathcal{C}\right\}$$

is natural from HF to HG.

b) If $K\colon \mathcal{B} \Rightarrow \mathcal{C}$, then for each $B \in \mathcal{B}$,

$$\lambda_{KB}\colon FKB \to GKB$$

This is a form of “composition” of K followed by λ. Show that the family

$$\lambda H = \left\{\lambda_{HB} \;\middle|\; B \in \mathcal{B}\right\}$$

is natural from FH to GH.

c) Let $F, G\colon \mathcal{A} \Rightarrow \mathcal{B}$ and $H, K\colon \mathcal{B} \Rightarrow \mathcal{C}$ and let $\alpha\colon F \dot{\to} G$ and $\beta\colon H \dot{\to} K$. Show that

$$K\alpha_A \circ \beta_{FA} = \beta_{GA} \circ H\alpha_A$$

The **Godement product** $\beta * \alpha$ is defined by

$$(\beta * a)_A := K\alpha_A \circ \beta_{FA} = \beta_{GA} \circ H\alpha A$$

Show that this defines a natural transformation from *HF* to *KG*. Show that the products $H\lambda$ and λH are special cases of the Godement product.

24. Prove the contravariant case of Yoneda's lemma.
25. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a covariant functor. Show that the natural transformations

$$\lambda_{A,B} = \{\lambda_{A,B}(\cdot)\}\colon \hom_{\mathcal{C}}(\,\cdot\,,\ A) \dot{\to} \hom_{\mathcal{D}}(F\,\cdot\,, B)$$

between contravariant functors have the form

$$\lambda_{A,B}(X)f = (Ff)^{\to} g = g \circ Ff$$

for all $f\colon X \to A$, where $g \in \hom_{\mathcal{D}}(FA, B)$. In this case, $g = \lambda_{A,B}(A)1_A$ and so

$$\lambda_{A,B}(X)f = (Ff)^{\to} \lambda_{A,B}(A)1_A = \lambda_{A,B}(A)1_A \circ Ff$$

Universality

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The Universal Mapping Property

Let us recall the definition of a comma category (mid level of generalization). If $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ is a functor and $C \in \mathcal{C}$ is an (anchor) object, then the comma category $(C \to G)$ is the category whose objects are the pairs

$$(U, u\colon C \to GU) \tag{54}$$

for $U \in \mathcal{D}$. Moreover, a morphism

$$\tau\colon (U, u\colon C \to GU) \to (D, f\colon C \to GD) \tag{55}$$

between comma objects is essentially just a morphism $\tau\colon U \to D$ in $\mathcal{D}$ for which

$$G\tau \circ u = f \tag{56}$$

(We have dropped the overbar notation $\overline{\tau}$.)

Therefore, referring to Figure 38, an *initial* comma object (54) has the defining property that for *any* comma object $(D, f\colon C \to GD)$, there is a *unique* morphism (55) for which (56) holds.

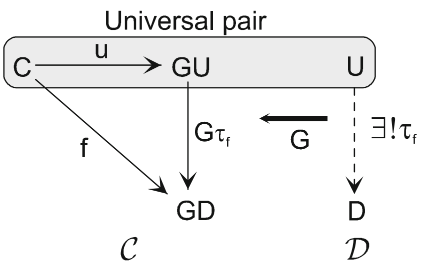

Figure 38

This property (or something equivalent) turns out to be critically important and has a special name.

Universal Mapping Property (UMP): A pair $(U, u_C\colon C \to GU)$ has the universal mapping property if for any morphism $f\colon C \to GD$, there exists a *unique* morphism $\tau_f\colon U \to D$ in $\mathcal{D}$ for which the triangle in Figure 38 commutes, that is, for which

$$G\tau_f \circ u_C = f \tag{57}$$

It is worth mentioning that the presence of the functor G *restricts the codomains* of the maps in question to the image $G\mathcal{D}$. As we will see, this is an extremely valuable feature of these comma categories.

The object U is known as a **universal object**, the map u is a **universal map** and the pair $(U, u_C: C \to GU)$ is a **universal pair** for (C, G). Thus, we can state that

a universal pair for (C, G) is simply an initial comma object in $(C \to G)$.

Note that the universal object U depends on both C and G, as does the universal map u_C.

The Mediating Morphism Maps

For each $D \in \mathcal{D}$, the universal mapping property defines a function

$$\tau_{C,D}: \hom_{\mathcal{C}}(C, GD) \to \hom_{\mathcal{D}}(U, D)$$

by

$$\tau_{C,D}(f) = \tau_f$$

for all $f: C \to GD$. Note that we have indexed τ with both C and D even though C is fixed for now. In the chapter on adjoints, we will allow C to vary as well and it is better to get used to the double indexing now rather than having to deal with it later, when more pressing issues are at hand. For a fixed $C \in \mathcal{C}$, the family of morphisms $\tau_{C,D}$ as D ranges over the objects in $\mathcal{D}$ will be denoted by

$$\{\tau_{C,D}\}_{\mathcal{D}} = \{\tau_{C,D} \mid D \in \mathcal{D}\}$$

Some authors refer to the map τ_f as the **mediating morphism** for f, although many authors do not give τ_f any special name at all. We will use this term and also make the following nonstandard definition.

Definition

Let $G: \mathcal{D} \Rightarrow \mathcal{C}$ be a functor and let

$$\mathcal{U} = (U, u_C: C \to GU)$$

be a universal pair for (C, G). Then the **mediating morphism map** *for $\mathcal{U}$ is the map*

$$\tau_{C,D}: \hom_{\mathcal{C}}(C, GD) \to \hom_{\mathcal{D}}(U, D)$$

defined for all $f\colon C \to GD$ *by*

$$\tau_{C,D}(f) = \tau_f$$

where τ_f *is the unique mediating morphism for* f. □

The definition and uniqueness of mediating morphisms implies that

$$G\tau_{C,D}(f) \circ u_C = f \quad \text{and} \quad \tau_{C,D}(Gh \circ u_C) = h$$

for all $f\colon C \to GD$ and all $h\colon U \to D$ and so the mediating morphism map is bijective, with inverse map

$$\tau^{-1}_{C,D}(h) = Gh \circ u_C \tag{58}$$

for all $h\colon U \to D$. Note also that the universal map is given in terms of the mediating morphism map by

$$u_C = \tau^{-1}_{C,U}(1_U) \tag{59}$$

In fact, we have the following characterization of mediating morphism maps.

Theorem 60

A map

$$\tau_{C,D}\colon \hom_{\mathcal{C}}(C, G, D) \to hom_{\mathcal{D}}(U, D)$$

is the mediating morphism map for a universal pair

$$\mathcal{U} = (U, u_C\colon C \to GU)$$

if and only if $\tau_{C,D}$ *is a bijection and*

$$\tau^{-1}_{C,D}(h_{U,D}) = Gh_{U,D} \circ u_C \tag{61}$$

Proof

If $\tau_{C,D}$ is a bijection given by (61), then for any $f\colon C \to GD$, we have

$$G\tau_{C,D}(f) \circ u_C = f$$

and if

$$Gh \circ u_C = f$$

for any $h: U \to D$, then applying the bijection $\tau_{C,D}$ gives

$$\tau_{C,D}(f) = \tau_{C,D}(Gh \circ u_C) = \tau_{C,D}\big(\tau_{C,D}^{-1}(h)\big) = h$$

Hence, $\tau_{C,D}(f) = \tau_f$ is the unique mediating morphism for f and so $\tau_{C,D}$ is the mediating morphism map. □

A word about notation is in order. We have two choices when it comes to writing formulas such as the one displayed in Theorem 60. We can write

$$\tau_{C,D}^{-1}(h) = Gh \circ u_C$$

with the added condition that this holds for all $h: U \to D$ or we can simply write

$$\tau_{C,D}^{-1}(h_{U,D}) = Gh_{U,D} \circ u_C$$

with the tacit understanding that such a formula holds for all values of $h_{U,D}$.

Frankly, we see virtue in both formats. The first format is a bit easier to read, but it is also a bit harder to follow, especially as the formulas become more complex and contain compositions of maps. In any case, after much deliberation (and some vacillation), we have settled on the more compact second format in most displayed equations, although we will use the first format at times as well.

Naturalness of the Mediating Morphism Maps

One of the main virtues of the mediating morphism maps is that they show how the universal mapping property is actually equivalent to a naturalness property. Another virtue is that they will allow us to obtain a certain "dual symmetry" that leads naturally (both figuratively and literally) to the notion of an adjunction (and left and right adjoints).

As to the naturalness property, let $C \in \mathcal{C}$ and suppose that

$$\{\tau_{C,D}\}_{\mathcal{D}}: \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{D}}(U, D)$$

is a family of *bijections* and set

$$u_C = \tau_{C,U}^{-1}(1_U)$$

Then the maps $\tau_{C,D}$ are mediating morphism maps if and only if

$$\tau_{C,D}^{-1}(\alpha) = G\alpha \circ u_C \tag{62}$$

for all $\alpha: U \to D$. Moreover, (62) implies that for any $h: D \to D'$,

$$\tau_{C,D'}^{-1}(h \circ \alpha) = G(h \circ \alpha) \circ u_C = Gh \circ G\alpha \circ u_C = Gh \circ \tau_{C,D}^{-1}(\alpha)$$

This shows that we may pull the map h outside of $\tau^{-1}_{C,D'}(h \circ \alpha)$ by replacing it with Gh. Note that this formula is *equivalent* to (62), as can be seen by taking $\alpha = u_C$.

Moreover, since each $\tau_{C,D}$ is a bijection, as α varies over $\hom_{\mathcal{D}}(U, D)$, the maps $f = \tau^{-1}_{C,D}(\alpha)$ vary over $\hom_{\mathcal{C}}(C, GD)$ and so (62) is equivalent to

$$\tau^{-1}_{C,D'}(h \circ \tau_{C,D}(f)) = Gh \circ f$$

for all $f\colon C \to GD$. Applying $\tau_{C,D'}$ to both sides, this can be written in the form

$$\left(\tau_{C,D'} \circ (Gh)^{\leftarrow}\right)(f) = (h^{\leftarrow} \circ \tau_{C,D})(f)$$

and so

$$\tau_{C,D'} \circ (Gh)^{\leftarrow} = h^{\leftarrow} \circ \tau_{C,D} \tag{63}$$

on $\hom_{\mathcal{C}}(C, GD)$.

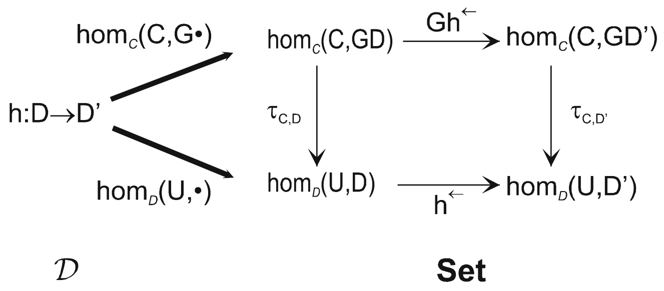

Figure 39

Well, as shown in Figure 39, condition (63) says that the family

$$\{\tau_{C,D}\}_{\mathcal{D}}\colon \hom_{\mathcal{C}}(C, G\,\cdot\,) \dot{\leftrightarrow} \hom_{\mathcal{D}}(U,\ \cdot\,)$$

is a natural *isomorphism* in D from the composite functor

$$\hom_{\mathcal{C}}(C, G\,\cdot\,) = \hom_{\mathcal{C}}(C,\ \cdot\,) \circ G$$

to the hom functor $\hom_{\mathcal{D}}(U,\ \cdot\,)$.

■ Theorem 64

Let $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$ and $U \in \mathcal{D}$. Let

$$\{\tau_{C,D}\}_{\mathcal{D}}\colon \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{C}}(U, D)$$

be a family of bijections and let

$$u_C = \tau^{-1}_{C,U}(1_U)$$

The following are equivalent:

1) **(Mediating morphisms)** *The maps* $\{\tau_{C,D}\}_\mathcal{D}$ *are the mediating morphism maps for the universal pair* $(U, u_C\colon C \to GU)$, *that is,* $\tau_{C,D}(f_{C,GD})$ *is the unique solution to the equation*

$$G\tau_{C,D}\big(f_{C,GD}\big) \circ u_C = f_{C,GD}$$

2) **(Inverse fusion formula)** *The maps* $\{\tau_{C,D}\}_\mathcal{D}$ *satisfy*

$$\tau^{-1}_{C,D}(h_{U,D}) = Gh_{U,D} \circ u_C$$

We will call this formula the **inverse fusion formula**.

3) **(Naturalness in *D*)** $\{\tau_{C,D}\}_\mathcal{D}$ *is a natural isomorphism in* D, *that is,*

$$\tau_{C,D'} \circ \big(Gh_{D,D'}\big)^{\leftarrow} = h^{\leftarrow}_{D,D'} \circ \tau_{C,D}$$

or equivalently (and of more practical use),

$$\tau_{C,D'}\big(Gh_{D,D'} \circ f_{C,GD}\big) = h_{D,D'} \circ \tau_{C,D}\big(f_{C,GD}\big) \qquad \square$$

The term "inverse fusion formula" is not standard. The term "fusion formula" does appear in some literature, but refers to a formula that we will introduce later and call the "direct fusion formula." We use the term "inverse" here because this formula gives an expression for the inverse of the mediating morphisms. The term "fusion" comes from the fact that these formulas can be used to "connect" the concepts of left and right adjoint.

Examples

Let us turn to some examples of universality.

■ Example 65 (Sets)

Perhaps the simplest example of universality comes when G is the identity functor on **Set**.

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[dr, "f"'] & U \arrow[d, dashed, "\exists!\tau_f"] \\ \mathcal{C} & D \end{tikzcd}$$

Figure 40

With reference to Figure 40, a pair $(U, u\colon C \to U)$ is universal for (C, I) if any set function $f\colon C \to D$ can be uniquely factored through u, that is, if there is a unique set function τ_f: $U \to D$ for which

$$\tau_f \circ u = f \tag{66}$$

Now, it is clear that (66) implies

$$f(x) \neq f(y) \ \Rightarrow \ u(x) \neq u(y)$$

for x, $y \in C$ and since this applies to *any* set function f: $C \to D$, it follows that u must be injective.

Moreover, if u is not surjective, then any map τ_f satisfying (66) can be changed by changing the value of τ_f on an element of $U \setminus \mathrm{im}(u)$ without affecting the validity of (66) and so uniqueness will not hold. Therefore, u must be a *bijection*. Conversely, if u: $C \to U$ is a bijection, then (66) will hold for $\tau_f = f \circ u^{-1}$.

Thus, a pair $(U, u\colon C \to U)$ is universal for (C, I) if and only if u is a bijection. Incidentally, this shows that neither universal maps nor universal objects are unique, although in this case any two universal objects are equipollent, that is, isomorphic in **Set**. □

■ Example 67 (Free Groups)

Consider the underlying-set functor G: **Grp** ⇒ **Set**, which sends a group to its underlying set and a morphism to the underlying set function. Let X be a nonempty set. Then a pair $(U, u\colon X \to U)$ where U is a group and u is a set function is universal for (X, G) if for any set function f: $X \to D$ where D is a group, there is a unique *group homomorphism* τ_f: $U \to D$ for which

$$\tau_f \circ u = f$$

as set maps.

Therefore, if $X \subseteq U$ and if j: $X \to U$ is the inclusion map, then to say that the pair $(U, j\colon X \to U)$ is universal is to say that every set function f: $X \to D$ where D is a group can be *uniquely extended* to a group homomorphism from $\overline{f}$: $U \to D$. In this case, we say that such a group U has the **unique extension property** with respect to the subset X.

Now, in treatments of group theory, free groups are defined in one of two ways. The more elementary (or perhaps just less categorical) treatments tend to define the free group F_X as the set of all words over the alphabet $\mathcal{A} = X \cup X^{-1}$, under juxtaposition and the usual rules of exponents. It is then shown that the free group F_X has the unique extension property with respect to X, that is, that the pair $(F_X, j: X \to F_X)$ is universal for (X, G).

In more advanced treatments (or perhaps just more categorical treatments), the free group on X is *defined* as any group that has the unique extension property with respect to X. It is then shown that the set of words over the alphabet $\mathcal{A}$ is one such group. In either case, the end result is that the pair $(F_X, j: X \to F_X)$ is universal for (X, G).

Theorem 68

*Let G: **Grp** $\Rightarrow$ **Set** be the underlying-set functor. Let X be a nonempty set, let F_X be the free group on X and let $j: X \to F_X$ be the inclusion map. Then the pair*

$$(X, j: X \to F_X)$$

is universal for (X, G). □

Example 69 (Vector space bases)

One could say, but never does, that a vector space V with basis $\mathcal{B}$ is a "free vector space" over $\mathcal{B}$. On the other hand, one often does say the following: Any linear map $f: V \to W$ is *uniquely determined* by its values on the basis vectors in $\mathcal{B}$ and these values can be assigned arbitrarily.

But this is the same as saying that the pair

$$(V, j: \mathcal{B} \to V)$$

where j is the inclusion map is universal for $(\mathcal{B}, G)$, where G: **Vect** $\Rightarrow$ **Set** is the underlying set functor. □

Example 70 (Field of quotients)

Let **IntDom** be the category of integral domains, with morphisms being ring embeddings (monomorphisms). Let G: **Field** $\Rightarrow$ **IntDom** be the forgetful functor, forgetting the fact that every nonzero element of a field is a unit. Let R' denote the field of quotients of an integral domain R. Then

$$(R', j: R \to R')$$

where j is the inclusion map, is universal for (R, G). This amounts to saying that any ring embedding $f: R \to F$ where F is a field can be lifted in a unique way to the field of quotients $\overline{f}: R' \to F$. Thus, fields of quotients are universal objects. □

Example 71 (Quotient spaces and canonical projections)

Let $\mathcal{C}$ be the category of all pairs (M, A) of R-modules, where A is a submodule of M. We call $\mathcal{C}$ the category of **modules with distinguished submodules**. A morphism $f: (M, A) \to (N, B)$ is a linear map $f: M \to N$ for which $f(A) \subseteq B$. Composition in $\mathcal{C}$ is composition of linear maps. For if $g: (N, B) \to (P, C)$, then

$$(g \circ f)(A) = g(f(A)) \subseteq g(B) \subseteq C$$

and so

$$g \circ f\colon (M, A) \to (P, C)$$

is a morphism in $\mathcal{C}$. Referring to Figure 41, let $G\colon \mathbf{Mod}_R \Rightarrow \mathcal{C}$ be the functor that maps an R-module M to $(M, \{0\})$ and sends a linear map $f\colon M \to N$ to $f\colon (M, \{0\}) \to (N, \{0\})$.

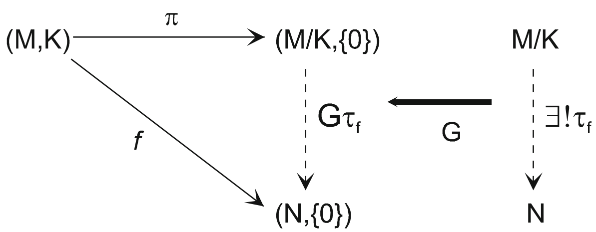

Figure 41

Consider the pair

$$((M/K, \{0\}), \pi\colon (M, K) \to (M/K, \{0\})) \qquad \textbf{(72)}$$

where π is the canonical projection. If $f\colon (M, K) \to (N, \{0\})$, then

$$G\tau_f \circ \pi = f$$

if and only if

$$\tau_f(mK) = f(m)$$

for all $m \in M$ and so this will serve as the definition of τ_f, provided that it is well defined. But

$$mK = nK \quad \Rightarrow \quad m - n \in K \quad \Rightarrow \quad f(m-n) = 0 \quad \Rightarrow \quad f(m) = f(n)$$

and so $\tau_f\colon M/K \to N$ is a well defined module map. Thus, the pair (72) is universal. $\square$

■ Example 73 (The tensor product)

Let $U \times V$ be the cartesian product of two vector spaces over a field k. Let $\mathbf{Vect}^+$ be the category of all vector spaces over F with linear maps, to which is adjoined the object $U \times V$, as a set. The morphisms from $U \times V$ to a vector space W are the bilinear maps and there are no morphisms from a vector space W to $U \times V$. Also, $\hom(U \times V, U \times V)$ consists of just the identity map.

Let $G\colon \mathbf{Vect} \Rightarrow \mathbf{Vect}^+$ be the inclusion functor. Then the tensor product

$$(U \otimes V, t\colon U \times V \to U \otimes V)$$

where $t(u, v) = u \otimes v$, is universal for $(G, U \times V)$. For if W is a vector space and $f\colon U \times V \to W$ is bilinear, then there is a unique linear map $\tau_f\colon U \otimes V \to W$ for which $\tau_f \circ t = f$. Thus, tensor products are universal objects. □

We have just seen that free groups, vector spaces, fields of quotients, quotient modules and tensor products are all universal objects. Thus, all of these constructions are specific cases of *the same categorical concept*—universality.

The Importance of Universality

The traditional method for defining a new mathematical "object" is to first describe what the object *is*, that is, give its *definition* and then describe what it *does*, that is, describe its important *properties*.

$$\text{Traditional perspective: first definition, then properties.}$$

However, a moment's reflection shows that a definition is really only a means to an end and is useless *per se*. The *only* reason to define a mathematical object is because that object has some useful *properties*.

To illustrate, as we have seen, the pair $(F_X, j\colon X \to F_X)$ where F_X is the free group on the set X satisfies the UMP. Moreover, this property characterizes the free group (up to isomorphism). Therefore, one could argue that the sole purpose for defining the free group in the traditional element-based manner is to define an object that has this universal property.

Indeed, the categorical perspective is that the free group should be *defined* as *any pair* that has this universal property.

$$\text{Categorical perspective: definition by (univeral) property.}$$

This reveals the fact that the concept of a free group is really a *categorical* concept, that is, a property of morphisms and not elements, even though the traditional definition of free group is all about elements and ignores maps entirely.

A similar argument can be made for the other examples of universal properties that we have discussed, namely, vector space bases, quotient spaces, fields of fractions, tensor products as well as a host of other common mathematical constructions.

Indeed, there are many other examples of defining an object by a universal property. One example is the direct product of vector spaces described in ► Example 8, which is an example of a **categorical construction**. In fact, categorical constructions are nothing more or less than universal pairs in a category of diagrams. We will discuss these constructions in a later chapter.

Uniqueness of Universal Pairs

Since universal pairs are defined as initial objects in a comma category, they are unique up to isomorphism in that category.

■ **Theorem 74**

Let G: $\mathcal{D} \Rightarrow \mathcal{C}$ and $C \in \mathcal{C}$ and let

$$\mathcal{S} = (S, u: C \to GS)$$

be universal for (C, G). *Then*

$$\mathcal{T} = (T, v: C \to GT)$$

is universal for (C, G) *if and only if there is an isomorphism* α: $S \approx T$ *in* $\mathcal{D}$ *for which*

$$v = G\alpha \circ u \tag{75}$$

■ **Proof**

Suppose first that $\mathcal{S}$ is universal and that α: $S \approx T$ satisfies (75). Then for any f: $C \to GD$, there is a unique mediating morphism τ_f: $S \to D$ for which

$$G\tau_f \circ u = f$$

Then

$$G\left(\tau_f \circ \alpha^{-1}\right) \circ v = G\left(\tau_f \circ \alpha^{-1}\right) \circ G\alpha \circ u = G\tau_f \circ u = f$$

and so $\tau_f \circ \alpha^{-1}$ is a mediating morphism for $\mathcal{T}$. As to uniqueness, if $G\mu_f \circ v = f$ then

$$f = G\mu_f \circ v = G\mu_f \circ G\alpha \circ u = G\left(\mu_f \circ \alpha\right) \circ u$$

and so the uniqueness of mediating morphisms for $\mathcal{S}$ implies that $\mu_f \circ \alpha = \tau_f$ and so $\mu_f = \tau_f \circ \alpha^{-1}$, proving uniqueness. Thus, $\mathcal{T}$ is universal.

For the converse, suppose that $\mathcal{S}$ and $\mathcal{T}$ are both universal. Then since v: $C \to GT$, the universality of $\mathcal{S}$ implies that there exists a unique mediating morphism τ_v: $S \to T$ for which

$$G\tau_v \circ u = v$$

A similar argument shows that there exists a unique mediating morphism τ_u: $T \to S$ for which

$$G\tau_u \circ v = u$$

and so

$$v = G\tau_v \circ u = G\tau_v \circ G\tau_u \circ v = G(\tau_v \circ \tau_u) \circ v$$

Then the uniqueness of mediating morphisms implies that $\tau_v \circ \tau_u = \iota_T$. Similarly, $\tau_u \circ \tau_v = \iota_S$ and so τ_u and τ_v are both isomorphisms. Taking $\alpha = \tau_v$ gives (75). □

Note that universal *morphisms* are not unique in general, even for a single universal object. For if

$$\mathcal{S} = (S, u\colon C \to GS)$$

is universal for (C, G) and if $v\colon C \approx C$, then

$$\mathcal{U} = (S, u \circ v\colon C \to GS)$$

is also universal for (C, G). To see this, the universality of $\mathcal{S}$ applied to

$$u \circ v\colon C \to GS \quad \text{and} \quad u \circ v^{-1}\colon C \to GS$$

implies that there are unique mediating morphisms $\tau_{u\circ v}\colon S \to S$ and $\tau_{u\circ v^{-1}}\colon S \to S$ for which

$$G\tau_{u\circ v} \circ u = u \circ v \quad \text{and} \quad G\tau_{u\circ v^{-1}} \circ u = u \circ v^{-1} \tag{76}$$

Hence,

$$G(\tau_{u\circ v} \circ \tau_{u\circ v^{-1}}) \circ u = G\tau_{u\circ v} \circ G\tau_{u\circ v^{-1}} \circ u = G\tau_{u\circ v} \circ u \circ v^{-1} = u$$

and similarly,

$$G(\tau_{u\circ v^{-1}} \circ \tau_{u\circ v}) \circ u = G\tau_{u\circ v^{-1}} \circ G\tau_{u\circ v} \circ u = G\tau_{u\circ v^{-1}} \circ u \circ v = u$$

and so the uniqueness of mediating morphisms implies that $\tau_{u\circ v}$ is an isomorphism. Therefore (76) and Theorem 74 imply that is universal. □

Couniversality

The dual of the concept of an initial object in a comma category $(C \to G)$ where $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ is the concept of a terminal object in a comma category $(F \to D)$, where $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is a functor. This defines the *couniversal mapping property* (*CMP*) although many authors use the term universal mapping property for this property as well. For later use, we have interchanged the roles of the two categories.

Definition

Referring to Figure 42, let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and let $D \in \mathcal{D}$.

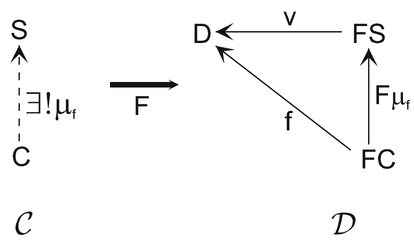

Figure 42

The statement that the object $(S, v\colon FS \to D)$ is terminal in the comma category $(F \to D)$ is the following:

Couniversal mapping property (CMP): *For every $f\colon FC \to D$, there is a unique $\mu_f\colon C \to S$ for which*

$$f = v \circ F\mu_f$$

In this case, the pair

$$\mathcal{V} = (S, v\colon FS \to C)$$

is a **couniversal pair** *for (D, F), the object S is the* **couniversal object**, *the morphism v is the* **couniversal morphism** *and μ_f is the* **mediating morphism** *for f.*

For a fixed $D \in \mathcal{D}$, the map

$$\mu_{C,D}\colon \hom_{\mathcal{D}}(FC, D) \to \hom_{\mathcal{C}}(C, S)$$

that sends each $f\colon FC \to D$ to its unique mediating morphism $\mu_f\colon C \to S$ is called the **comediating-morphism map** *for the pair (D, C) with respect to the couniversal pair $\mathcal{V}$.* □

As before, the comediating-morphism map $\mu_{C,D}$ is a bijection, defined by the condition

$$v \circ F\big(\mu_{C,D}(f)\big) = f$$

for all $f\colon FC \to D$ and with inverse is defined by

$$\mu_{C,D}^{-1}(h) = v \circ Fh$$

for all $h\colon C \to S$ and the couniversal map v can be described in terms of the comediating-morphism map by

$$v = \mu_{S,D}^{-1}(1_S)$$

The dual of Theorem 64 is the following.

Theorem 77

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and let $D \in \mathcal{D}$ and $S \in \mathcal{C}$. Let

$$\mu_{C,D}\colon \hom_{\mathcal{D}}(FC, D) \leftrightarrow \hom_{\mathcal{C}}(C, S)$$

be a family of bijections and let

$$v_D = \mu_{S,D}^{-1}(1_S)$$

The following are equivalent:

1) **(Comediating morphisms)** *The maps $\{\mu_{C,D}\}_C$ are the comediating morphisms for the couniversal pairs $(S, v_D\colon FS \to D)$, that is, $\mu_{C,D}(h_{FC,D})$ is the unique solution to the equation*

$$v_D \circ F\mu_{C,D}(h_{FC,D}) = h_{FC,D}$$

2) **(Fusion formula)** *The maps* $\{\mu_{C,D}\}_{\mathcal{C}}$ *are bijections and*

$$\mu^{-1}_{C,D}(f) = v_D \circ Ff$$

for all $f\colon C \to S$.

3) **(Naturalness in C)** $\{\mu_{C,D}\}_{\mathcal{C}}$ *is a natural isomorphism in C, that is,*

$$g^{\to} \circ \mu_{C,D} = \mu_{C',D} \circ (Fg)^{\to}$$

for each $g\colon C' \to C$. □

Our plan a bit later is to combine Theorem 77 with its dual Theorem 64 and for this purpose, we recast Theorem 77 by setting $\mu_{C,D} = \tau^{-1}_{C,D}$.

▪ Theorem 78

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ *and let* $D \in \mathcal{D}$ *and* $S \in \mathcal{C}$. *Let*

$$\tau_{C,D}\colon \hom_{\mathcal{C}}(C,S) \leftrightarrow \hom_{\mathcal{D}}(FC,D)$$

be a family of bijections and let

$$v_D = \tau_{S,D}(1_S)$$

The following are equivalent:

1) **(Comediating morphisms)** *The maps* $\{\tau^{-1}_{C,D}\}_{\mathcal{C}}$ *are the comediating morphisms for a couniversal pair* $(S, v_D\colon FS \to D)$, *that is,*

$$v_D \circ F\tau^{-1}_{C,D}(h_{FC,D}) = h_{FC,D}$$

2) **(Direct fusion formula)** *The maps* $\{\tau_{C,D}\}_{\mathcal{C}}$ *satisfy*

$$\tau_{C,D}(f_{C,S}) = v_D \circ Ff_{C,S}$$

We call this formula the **direct fusion formula**.

3) **(Naturalness in C)** $\{\tau_{C,D}\}_{\mathcal{C}}$ *is a natural isomorphism in C, that is,*

$$\tau_{C',D} \circ g^{\to}_{C,C'} = (Fg_{C,C'})^{\to} \circ \tau_{C,D}$$

or equivalently (and of more practical use),

$$\tau_{C',D}(f_{C,GD} \circ g_{C',C}) = \tau_{C,D}(f_{C,GD}) \circ Fg_{C',C}$$

Exercises

1. Let $F < E$ be a field extension and let $X \subseteq E$ be a set of algebraically independent elements over F. Let U: **Field** $\Rightarrow$ **Set** be the forgetful functor. Find a universal pair for X and U.
2. Let Δ: $\mathcal{C} \Rightarrow \mathcal{C} \times \mathcal{C}$ be the **diagonal functor**, which sends an object C of $\mathcal{C}$ to the ordered pair (C, C) and a morphism f: $C \to C'$ to the morphism (f, f): $(C, C) \to (C', C')$. Show that we can view the product as couniversal for this functor.
3. Show that the completion M' of a metric space M, along with the inclusion j: $M \to M'$, describes a universal pair.
4. Let G be a group and let R be a commutative ring with identity. The **group ring** RG is the R-algebra of formal finite sums $\sum r_i g_i$, where $r_i \in R$ and $g_i \in G$. Multiplication is done using the product in G and linearity over R. Show that RG is a universal object.
5. Show that the polynomial ring $F[x]$ is a universal object.
6. Show that the first isomorphism theorem of R-modules follows directly from the universal property of quotients.
7. Let U: $\mathbf{Vect}_k \Rightarrow$ **Set** be the underlying set functor. Which sets S have couniversal pairs?
8. Let $\mathcal{D}$ be a category and let G: $\mathcal{D} \Rightarrow$ **Set** be a *contravariant* functor. A **universal element** for G is a pair (S, u), where $S \in \mathcal{D}$ and $u \in GS$, with the property that if (X, x) is another such pair, then there is a unique morphism τ: $X \to S$ of $\mathcal{D}$ for which $(G\tau)x = u$.
   a) Let $\mathcal{P}'$: **Set** $\Rightarrow$ **Set** be the contravariant power set functor defined as follows: $\mathcal{P}'$ takes a set X to its power set $\mathcal{P}X$ and $\mathcal{P}'$ takes a function f: $A \to B$ to the induced inverse function f^{-1}: $\mathcal{P}B \to \mathcal{P}A$. Does $\mathcal{P}'$ have a universal element?
   b) Consider the power set functor $\mathcal{P}$: **Set** $\Rightarrow$ **Set** that sends a set A to its power set and a function f: $A \to B$ to the induced function from $\mathcal{P}A$ to $\mathcal{P}B$. Does $\mathcal{P}$ have a universal element?
9. Let G: $\mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$ and $U \in \mathcal{D}$. Let

$$\{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}\colon \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{C}}(U, D)$$

   be a family of bijections and

$$u = \tau^{-1}_{C,U}(1_U)$$

   Show that the following are equivalent.
   a) $\{\tau_{C,D}\}$ is natural in D.
   b) $\{\tau_{C,D}\}_{\mathcal{D}}$ satisfy the formula

$$\tau^{-1}_{C,D'}(g \circ h) = Gg \circ \tau^{-1}_{C,D}(h)$$

   for all h: $U \to D$ and g: $D \to D'$.
   c)

$$\tau^{-1}_{C,D'}(g) = Gg \circ \tau^{-1}_{C,D}(1_U)$$

   for all g: $D \to D'$.

Representable Functors

Definition

A set-valued functor $F\colon \mathcal{C} \Rightarrow \textbf{Set}$ *is* **representable** *by an object* $A \in \mathcal{C}$ *if there is a natural isomorphism*

$$\{\lambda_C\}\colon \hom_{\mathcal{C}}(A, \cdot\,) \stackrel{\cdot}{\approx} F$$

for some hom functor $\hom_{\mathcal{C}}(A, \cdot\,)$. □

10. Let $F, G\colon \mathcal{C} \Rightarrow \textbf{Set}$ be representable functors, with

$$\{\lambda_C\}\colon \hom_{\mathcal{C}}(A, \cdot\,) \stackrel{\cdot}{\approx} F$$

and

$$\{\mu_C\}\colon \hom_{\mathcal{C}}(B, \cdot\,) \stackrel{\cdot}{\approx} G$$

Prove that for any $\tau\colon F \stackrel{\cdot}{\to} G$, there is a unique $h\colon B \to A$ for which

$$\tau \circ \lambda = \mu \circ h^{\to}\colon \hom_{\mathcal{C}}(A, \cdot\,) \stackrel{\cdot}{\to} G$$

11. Let $F\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$. Prove that the functor

$$\hom_{\mathcal{C}}(C, F\,\cdot\,)\colon \mathcal{D} \Rightarrow \textbf{Set}$$

is representable if and only if there is a universal pair for (C, F).

12. Let $F\colon \mathcal{C} \Rightarrow \textbf{Set}$ be representable by A and let $\{*\}$ be a one-element set.
    a) Prove that if a pair of the form

$$(A, u\colon \{*\} \to FA)$$

is universal for A and F, then F is representable.

    b) Conversely, prove that if F is representable by $\{\lambda_C\}$, then

$$(A, u\colon \{*\} \to FA)$$

is universal for A and F for some u.

Cones and Limits

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We wish to continue our exploration of universality with some additional examples. For this, we need to define a few more categorical concepts.

Cones and Cocones

Limits and colimits are key concepts in category theory. These concepts are based in turn on the concepts of cones and cocones.

Definition

Referring to Figure 43, let $\mathbb{D}(J\colon \mathcal{J} \Rightarrow \mathcal{C})$ be a diagram in a category $\mathcal{C}$ and let $V \in \mathcal{C}$.

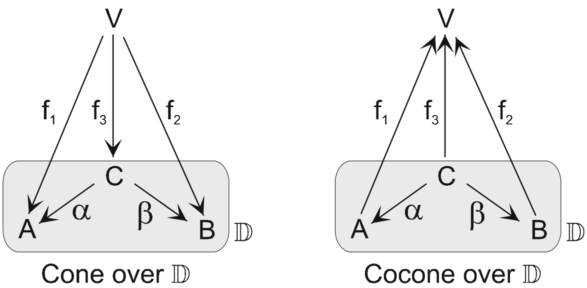

Figure 43 *Cones and cocones over* $\mathbb{D}$

1) *A* **cone** $(V, \mathbb{D})$ *from V to $\mathbb{D}$ with* **vertex** *V and* **base** *$\mathbb{D}$ consists of the object V, together with one morphism of $\mathcal{C}$ from V to each node in $\mathbb{D}$. We call these morphisms the* **legs** *of the cone. Moreover, all triangles involving the vertex V and any two legs must commute. Thus, in the left half of Figure 43, we must have*

$$\alpha \circ f_3 = f_1 \quad \text{and} \quad \beta \circ f_3 = f_2$$

A **cone over** *$\mathbb{D}$ is a cone from some vertex V to $\mathbb{D}$. To specify the vertex and legs of a cone $(V, \mathbb{D})$, we will write*

$$\mathcal{K} = \left(V, \left\{ f_n\colon V \to Jn \,\middle|\, n \in \mathcal{J} \right\}\right)$$

or if we do not care about the functor J,

$$\mathcal{K} = \big(V, \{\, f_i \colon V \to A_i \mid i \in I \}\big)$$

for an appropriate index set I.

2) *The dual of a cone is a* **cocone***, as shown in the right half of Figure 43. The definition of cocone is obtained from the definition of cone by "reversing the arrows", that is, by replacing the phrase*

together with one morphism of $\mathcal{C}$ from V to each node in $\mathbb{D}$

with the phrase

together with one morphism of $\mathcal{C}$ from each node in $\mathbb{D}$ to V □

We phrase the following comments in the language of cones, but they apply equally well to cocones.

Often a cone is pictured without drawing all of its legs. For instance, Figure 44 shows a cone over a base diagram $\mathbb{D}$ (the shaded portion). The cone also contains a leg λ from P to C, but the cone condition requires that

$$\lambda = g \circ \alpha = f \circ \beta$$

and so it need not be explicitly drawn.

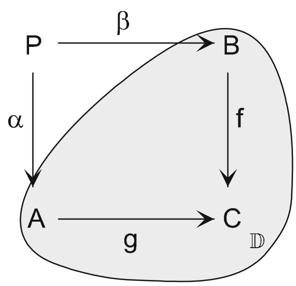

Figure 44 *A cone*

Note that the presence or absence of identity morphisms in the base diagram $\mathbb{D}$ does not enter into the cone condition.

Cone and Cocone Categories

The cones over a diagram $\mathbb{D}(J \colon \mathcal{J} \Rightarrow \mathcal{C})$ form the objects of a **cone category** $\mathbf{cone}_{\mathcal{C}}(\mathbb{D})$. As shown in Figure 45, a **cone morphism** $h \colon \mathcal{K} \to \mathcal{L}$ from a cone

$$\mathcal{K} = \big(V, \{\, f_n \colon V \to Jn \mid n \in \mathcal{J} \}\big)$$

to a cone

$$\mathcal{L} = \left(W, \{g_n\colon W \to Jn \middle| n \in \mathcal{J}\}\right)$$

is a morphism $h\colon V \to W$ of $\mathcal{C}$ between the *vertices* of the cones with the property that any triangle involving h and one leg from each cone must commute, that is,

$$g_n \circ h = f_n$$

for all $n \in \mathcal{J}$.

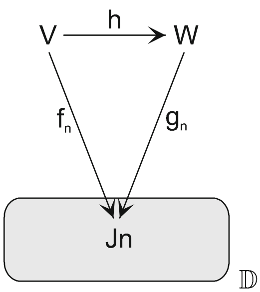

Figure 45 *A cone morphism h*

We leave it to the reader to verify that this does define a category, where composition of cone morphisms is ordinary composition of morphisms in $\mathcal{C}$.

The notion of a **cocone category** $\mathbf{cocone}_{\mathcal{C}}(\mathbb{D})$ is dual to that of a cone category and we leave the details to the reader.

Any Category Is a Cone Category: Objects Are One-Legged Cones

Actually, any category $\mathcal{C}$ can be thought of as a cone category (or cocone category), where each object in $\mathcal{C}$ is the vertex of a one-legged cone.

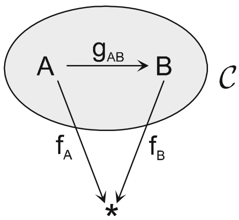

Figure 46 *Any category is a cone category*

As shown in Figure 46, let $*$ be a symbol that does not represent any object or morphism in $\mathcal{C}$ and consider the category $\mathcal{C}^*$ whose objects are $\mathbf{Obj}(\mathcal{C}) \cup \{*\}$ and whose morphisms are the morphisms in $\mathcal{C}$, along with one morphism $f_A\colon A \to *$ for each object A of $\mathcal{C}$ and an identity morphism 1_* for $*$. Composition is composition in $\mathcal{C}$, along with the rule

$$f_B \circ g_{AB} = f_A$$

Let $\mathbf{cone}_{\mathcal{C}^*}(\{*\})$ denote the category of all cones in $\mathcal{C}^*$ over the diagram $\{*\}$. Since the cones over $\{*\}$ are precisely the sets

$$\mathcal{K}_A = (A, f_A\colon A \to *)$$

for $A \in \mathcal{C}$, we can identify each object A in $\mathcal{C}$ with the *one-legged* cone $\mathcal{K}_A$. Also, each morphism $g\colon A \to B$ in $\mathcal{C}$ can be identified with the corresponding cone morphism in $\mathbf{cone}_{\mathcal{C}^*}(\{*\})$. In this way, we can think of $\mathcal{C}$ as the cone category $\mathbf{cone}_{\mathcal{C}^*}(\{*\})$.

Limits and Colimits

A terminal object in a cone category $\mathbf{cone}_{\mathcal{C}}(\mathbb{D})$, that is, a terminal cone is also called a *limit* of $\mathbb{D}$.

Definition

Let $\mathbb{D}(J\colon \mathcal{J} \Rightarrow \mathcal{C})$ be a diagram in a category $\mathcal{C}$.

1) *A* **limit** *of $\mathbb{D}$ is a terminal cone over $\mathbb{D}$, that is, a cone*

$$\mathcal{K} = \left(V, \{f_n\colon V \to Jn \mid n \in \mathcal{J}\}\right)$$

with the property that given any cone with vertex W,

$$\mathcal{L} = \left(W, \{g_n\colon W \to Jn \mid n \in \mathcal{J}\}\right)$$

there is a unique cone morphism $\theta\colon \mathcal{L} \to \mathcal{K}$, that is, a unique morphism $\theta\colon W \to V$ for which

$$f_n \circ \theta = g_n$$

for all $n \in \mathcal{J}$. A limit (or its vertex) is often denoted by $\varprojlim \mathbb{D}$.

2) *Dually, a* **colimit** *of $\mathbb{D}$ is an initial cocone over $\mathbb{D}$, that is, a cocone*

$$\mathcal{K} = \left(V, \{f_n\colon Jn \to V \mid n \in \mathcal{J}\}\right)$$

with the property that given any cocone

$$\mathcal{L} = \left(W, \{g_n\colon n \in \mathcal{J} \to W \mid n \in \mathcal{J}\}\right)$$

there is a unique cocone morphism $\theta\colon \mathcal{L} \to \mathcal{K}$, that is, a unique morphism $\theta\colon V \to W$ for which

$$\theta \circ f_n = g_n$$

for all $n \in \mathcal{J}$. A colimit (or its vertex) is often denoted by $\varinjlim \mathbb{D}$. □

Theorem 79

Limits are determined up to isomorphism, that is, if a diagram $\mathbb{D}$ has a limit $\mathcal{K}$, then the limits of $\mathbb{D}$ are precisely the cones that are isomorphic to $\mathcal{K}$. The dual statement holds for cocones. □

Terminal Cones and Couniversality

Let us fix an index category $\mathcal{J}$, a category $\mathcal{C}$ and a diagram

$$\mathbb{D}(J\colon \mathcal{J} \Rightarrow \mathcal{C})$$

Our goal is to show that terminal cones (limits) over $\mathbb{D}$ are essentially just special types of couniversal pairs in an appropriate comma category.

The first "issue" with a cone is that it mixes apples and oranges, that is, objects in $\mathcal{C}$ (the vertex of the cone) with objects in the diagram category $\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$ (the base of the cone).

To fix this, we define the **constant diagram** $\mathbb{V}(J_{\mathbb{V}}\colon \mathcal{J} \Rightarrow \mathcal{C})$ with index set $\mathcal{J}$ and vertex $V \in \mathcal{C}$ to be the constant functor defined by

$$J_V(n) = V \quad \text{and} \quad J_V(f) = 1_V$$

As pictured in Figure 47, this is the diagram for which all nodes in the underlying graph are labeled V and all arcs are labeled 1_V.

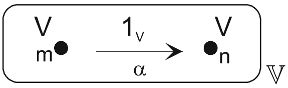

Figure 47

Since the categories $\mathcal{J}$ and $\mathcal{C}$ are fixed, it is clear that there is a bijection between objects $V \in \mathcal{C}$ and constant diagrams $\mathbb{V}(J_{\mathbb{V}}\colon \mathcal{J} \Rightarrow \mathcal{C})$ in the diagram category $\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$ and so these concepts are equivalent.

Now, Figure 48 shows a cone

$$\mathcal{K} = \big(V, \{\lambda_n\colon V \to Jn \mid n \in \mathcal{J}\}\big)$$

over the base diagram $\mathbb{D}(J\colon \mathcal{J} \Rightarrow \mathcal{C})$ alongside the equivalent picture obtained by replacing the vertex V of the cone with the constant diagram $\mathbb{V}$.

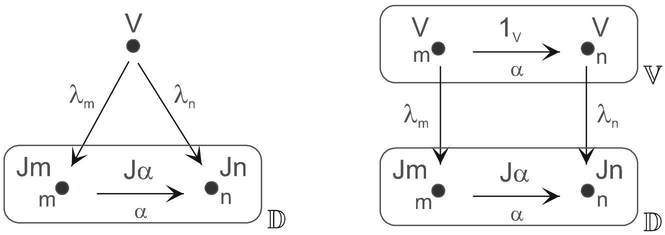

Figure 48

Note that the cone's leg set $\{\lambda_n \mid n \in \mathcal{J}\}$ corresponds to the *natural transformation*

$$\lambda = \{\lambda_n \mid n \in \mathcal{J}\}: J_V \dot{\to} J \quad \textbf{(80)}$$

between the $\mathcal{J}$-diagrams (functors) on the right in Figure 48. Thus, we may think of the cone

$$\mathcal{K} = (V, \{\lambda_n: V \to Jn\})$$

$\mathcal{K}$ with vertex V over the base diagram $\mathbb{D}(J: \mathcal{J}: \Rightarrow \mathcal{C})$ as a pair

$$\left(V, \{\lambda_n\}: J_V \dot{\to} J\right) \quad \text{or} \quad \left(V, \{\lambda_n\}: \mathbb{V} \dot{\to} \mathbb{D}\right) \quad \textbf{(81)}$$

consisting of the vertex V and the natural transformation $\{\lambda_n\}$ that makes up the leg set for the cone. But this looks a lot like a member of a comma category with anchor object J, where the domains of the arrows are restricted to constant diagrams (functors).

So, as shown in Figure 49, we define the **constant-diagram functor** $G_{\mathcal{J}}: \mathcal{C} \Rightarrow \mathrm{dia}_{\mathcal{J}}(\mathcal{C})$ that sends an object V to the constant diagram $\mathbb{V}(J_V: \mathcal{J} \Rightarrow \mathcal{C})$ and a morphism $f: V \to W$ to the "constant" natural transformation $\{f\}: \mathbb{V} \dot{\to} \mathbb{W}$ all of whose components are f. Then the cone $\mathcal{K}$ in (81) can be written in the form

$$\left(V, \{\lambda_n\}: G_{\mathcal{J}} V \dot{\to} J\right)$$

which belongs to the comma category $(G_{\mathcal{J}} \to J)$ with anchor object J.

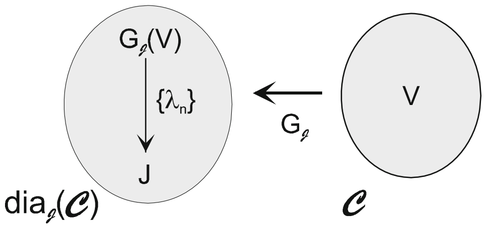

Figure 49

Moreover, a cone morphism $f: \mathcal{K} \to \mathcal{L}$ between cones corresponds to a morphism between the corresponding comma pairs. Thus, we can state that a terminal cone is equivalent to a terminal object in the comma category $(G_{\mathcal{J}} \to J)$, that is, to a *couniversal* pair for $(G_{\mathcal{J}}, J)$.

Categorical Constructions

Categorical constructions provide more evidence that universality is, well, universal. As mentioned earlier, a categorical construction is nothing more or less than an example of a universal or couniversal pair in the context of a category of diagrams. However, we will use the traditional terminology of limits and terminal cones (and colimits and initial cocones), rather than universal pairs (and couniversal pairs).

We will concentrate on three important categorical constructions and their duals: equalizers and coequalizers, products and coproducts, and pullbacks and pushouts.

Equalizers and Coequalizers

We begin with a definition.

Definition

Let $\mathcal{C}$ be a category and let $f, g\colon A \to B$ be parallel morphisms.

1) *A morphism $h\colon C \to A$* **right-equalizes** *f and g if*

$$f \circ h = g \circ h$$

2) *Dually, a morphism $k\colon B \to C$* **left-equalizes** *f and g if*

$$k \circ f = k \circ g$$

Now we can define the *equalizer* of f and g. □

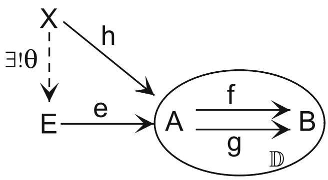

Figure 50

Definition

Let $\mathcal{C}$ be a category. As shown in Figure 50, an **equalizer** *for a diagram consisting of two parallel morphisms*

$$\mathbb{D} = \{ f\colon A \to B, g\colon A \to B\}$$

is a limit (terminal cone)

$$\mathcal{E} = (E, e\colon E \to A)$$

over $\mathbb{D}$, that is, a pair $\mathcal{E}$ for which e right equalizes f and g and for any $h\colon X \to A$ that right-equalizes f and g, there is a unique mediating morphism $\theta\colon X \to E$ for which

$$e \circ \theta = h$$ □

As alluded to earlier, we often omit legs of a cone that are uniquely determined by other legs, which is why we have denoted the cone $\mathcal{E}$ by

$$\mathcal{E} = (E, e\colon E \to A)$$

rather than

$$\mathcal{E} = (E, e\colon E \to A, f \circ e\colon E \to B)$$

(By the definition of cone, $f \circ e = g \circ e$.)

It is not hard to see that equalizer morphisms are monic.

▪ Theorem 82

If $\mathcal{E} = (E, e\colon E \to A)$ is an equalizer of the pair $f, g\colon A \to B$, then e is monic.

▪ Proof

A common technique for showing that two morphisms are equal is to show that they are both mediating morphisms for the same limit or colimit. Recalling that monic means left-cancellable, we want to show that if $\alpha, \beta\colon X \to E$ then

$$e \circ \alpha = e \circ \beta \Rightarrow \alpha = \beta$$

To this end, we draw the diagram in Figure 51.

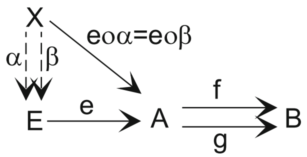

Figure 51

Then the cone condition $f \circ e = g \circ e$ implies that

$$f \circ e \circ \alpha = g \circ e \circ \alpha$$

and so

$$\mathcal{X} = (X, e \circ \alpha\colon X \to A)$$

is a cone over $\{f, g\colon A \to B\}$. But each of α and β is the mediating morphism for $\mathcal{X}$ and so $\alpha = \beta$. Thus, e is left-cancellable. □

Note that there is no requirement in the definition of equalizer that f and g be distinct. We leave it to the reader to show that a pair (E, e) is an equalizer of (f, f) if and only if $e\colon E \to A$ is an isomorphism.

Example 83

For the category **Set** and for many common "set-based" categories, such as **Grp**, **Rng** and **Mod**, the equalizer of $f, g\colon A \to B$ is easy to describe. In fact, the name "equalizer" gives away the description: It is the largest object (group, ring, module) contained in A upon which f and g are equal:

$$E = \{a \in A \mid f(a) = g(a)\}$$

along with the inclusion map $e\colon E \to A$. This can be seen by noting that f and g are right-equalized by $h\colon X \to A$, that is,

$$f \circ h = g \circ h$$

if and only if f and g agree on the image $h(X)$, that is, if and only if $h(X) \subseteq E$. Hence, the map $h'\colon X \to E$ defined simply by restricting the *range* of h to E is a mediating morphism, since

$$e \circ h' = h$$

As to uniqueness, if $\theta\colon X \to E$ also satisfies $e \circ \theta = h$, then $e \circ \theta = e \circ h'$ and so $\theta = h'$. □

Coequalizers

The dual to the equalizer is the *coequalizer*.

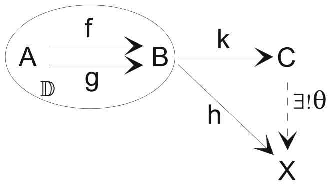

Figure 52

Definition

Let $\mathcal{C}$ be a category. Referring to Figure 52, the **coequalizer** *of the diagram*

$$\mathbb{D} = \{f\colon A \to B, g\colon A \to B\}$$

is an initial cocone

$$\mathcal{I} = (C, k\colon B \to C)$$

under $\mathbb{D}$, that is, a pair $\mathcal{I}$ for which k left-equalizes f and g and for every $k\colon B \to C$ that left-equalizes f and g, there is a unique mediating morphism $\theta\colon C \to X$ for which

$$\theta \circ k = h$$

□

In general, even though coequalizers are obtained from equalizers by "reversing all arrows," specific coequalizers seem rather more complex, or at least rather less intuitive than equalizers.

In certain contexts, the coequalizer describes one of the most important constructions of modern algebra, namely, *quotient structures*. We remind the reader of the bijective correspondence between partitions of a set and equivalence relations on the set. We will use both concepts interchangeably in these examples.

Example 84

Consider the coequalizer in **Set**. The condition that $h\colon B \to C$ left-equalizes f and g is

$$h \circ f(a) = h \circ g(a)$$

and this is equivalent to the statement that h is constant on all sets of the form $\{f(a), g(a)\}$ for $a \in A$.

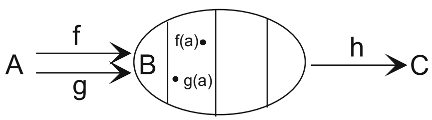

Figure 53

It is well known that any set function $h\colon B \to C$ induces a *partition* of its domain B whose blocks are the nonempty inverse images $h^{-1}(c)$, for $c \in C$. Moreover, as shown in Figure 53, h left-equalizes f and g if and only if for all $a \in A$, the elements $f(a)$ and $g(a)$ belong to the same block of this h-induced partition of B.

As we will see, the most "universal" such partition is the *finest* partition $\mathcal{P}$ of B for which each set $\{f(a), g(a)\}$ lies in a single block. Note, however, that the fact that $f(a)$ and $g(a)$ lie in a single block K of $\mathcal{P}$ may force other elements of B to also lie in that block. For example, if $g(a) = f(a')$ for some $a' \neq a$, then of course $f(a') \in K$ and so $g(a')$ must also lie in K.

Perhaps the best way to get a handle on the partition $\mathcal{P}$ is to consider the corresponding equivalence relation $\equiv$ defined by $\mathcal{P}$. Thus, we begin by defining a binary relation on B by

$$b \approx b' \quad \text{if} \quad b = b' \text{ or } \{b, b'\} = \{f(a), g(a)\} \text{ for some } a \in A$$

This relation is both reflexive and symmetric, but it need not be transitive, so we must pass to the transitive closure $\equiv$. Thus, $b_1 \equiv b_n$ if there is a finite sequence

$$b_1, b_2, \ldots, b_n$$

of elements of B for which $b_i \approx b_{i+1}$ for $i = 1, \ldots, n-1$.

Now, the function h left-equalizes f and g if and only if h is constant on the *equivalence classes* of this equivalence relation. Moreover, the most "universal" choice for h is any function that assigns *different* values to these equivalence classes. Perhaps the simplest way to define such a function, which we denote by π is to send each element of a particular equivalence class E to the equivalence class E itself, that is, $\pi(b) = [b]$, where $[b]$ is the equivalence class containing b.

The function π is called the **canonical projection** associated with the equivalence relation or the partition and $\mathcal{P}$ is denoted by $B/\equiv$. Hence, $\pi\colon B \to B/\equiv$ is defined by

$$\pi(b) = [b]$$

To see that π is indeed a coequalizer of f and g, if $h\colon B \to X$ left-equalizes f and g, then h is constant on the aforementioned equivalence classes and so the function $\theta\colon (B/\equiv) \to X$ defined by

$$\theta([b]) = h(b)$$

is well-defined. Moreover, $h = \theta \circ \pi$. As to uniqueness, if $h = \tau \circ \pi$ for some $\tau\colon B/\equiv \to X$, then

$$\theta([b]) = h(b) = \tau([b])$$

for all $b \in B$ and so $\tau = \theta$. Hence,

$$(B/\equiv, \pi\colon B \to B/\equiv)$$

is a coequalizer of f and g. □

■ **Example 85**

We can perform a similar analysis to determine the coequalizer in other categories. For example, consider the coequalizer in **Grp**. As with **Set**, a group homomorphism $h\colon B \to C$ left-equalizes f and g if, for all $a \in A$, the values $f(a)$ and $g(a)$ belong to the same equivalence class of the equivalence relation $\equiv$ generated by h. However, since h is a group homomorphism, this equivalence relation is a *congruence relation*, that is,

$$x \equiv y \quad \text{and} \quad u \equiv v \quad \Rightarrow \quad x^{-1} \equiv y^{-1} \quad \text{and} \quad xu \equiv yv$$

In the language of partitions, the blocks of the associated partition are called *congruence classes*. Thus, we seek the congruence relation $\equiv$ on B that has the finest partition among all congruence relations $\equiv$ for which $f(a) \equiv g(a)$, for all $a \in A$.

The congruence relations on B correspond bijectively to the *normal* subgroups of B. (This is a critical fact that seems often to be played down in elementary algebra books.) This correspondence sends a congruence relation $\equiv$ to the normal subgroup $[1]_\equiv$ and a normal subgroup $N \trianglelefteq B$ to the congruence relation

$$a \equiv_N b \quad \text{if} \quad b^{-1}a \in N$$

Moreover, $\equiv_N$ has a finer partition than $\equiv_M$ if and only if $N \subseteq M$. Thus, we may deal with normal subgroups instead of congruence relations.

To determine the normal subgroup $N = [1]_\equiv$ associated with the desired congruence relation $\equiv$, note that $f(a) \equiv g(a)$ if and only if $g(a)^{-1} f(a) \equiv 1$, that is, if and only if $g(a)^{-1}f(a) \in N$. Hence, if N is the **normal closure** $\langle S \rangle_{\text{nor}}$ of the set

$$S = \left\{ g(a)^{-1} f(a) \mid a \in A \right\}$$

that is, if N is the smallest normal subgroup of G containing S, then our candidate for the coequalizer of f and g is the projection map

$$\pi\colon B \to \frac{B}{\langle S\rangle_{\text{nor}}}$$

First, it is clear that

$$\pi \circ f = \pi \circ g$$

Also, if $h\colon B \to X$ has the property that $h \circ f = g \circ f$, then the mediating morphism condition is

$$\theta \circ \pi = h$$

that is,

$$\theta([b]) = h(b)$$

But this uniquely defines a group homomorphism $\theta\colon B/\langle S\rangle_{\text{nor}} \to X$. Thus,

$$\left(B/\langle S\rangle_{\text{nor}}, \pi\colon B \to B/\langle S\rangle_{\text{nor}}\right)$$

is a coequalizer of f and g.

Thus, for groups (and similarly for other algebraic structures) the dual of restricting to the substructure

$$E = \left\{a \in A \mid f(a) = g(a)\right\} = \left\{a \in A \mid g(a)^{-1} f(a) = 1\right\}$$

is factoring out by the "normal" substructure generated by the set

$$S = \left\{g(a)^{-1} f(a) \mid a \in A\right\}$$

□

Products and Coproducts

The direct product is a familiar construction in many categories. Here is the formal definition for general categories.

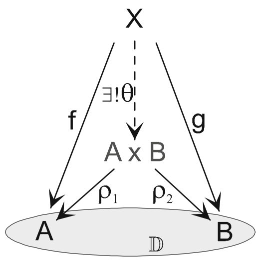

Figure 54

Definition

Let $\mathcal{C}$ be a category. Referring to Figure 54, let $\mathbb{D}$ be the diagram consisting of two objects A and B, with no morphisms. Any limit of $\mathbb{D}$ is called a **product** *of A and B. Thus, a product of A and B is a triple*

$$(A \times B, \rho_1 \colon A \times B \to A, \rho_2 \colon A \times B \to B)$$

with the property that for any object X and morphisms $f \colon X \to A$ and $G \colon X \to B$, there exists a unique mediating morphism $\theta \colon X \to A \times B$ for which the diagram in Figure 54 commutes, that is, for which

$$\rho_1 \circ \theta = f \ \ \textit{and} \ \ \rho_2 \circ \theta = g$$

The maps ρ_1 and ρ_2 are called the **projection maps**. □

Although it is quite misleading, it is common to denote a product simply as $A \times B$, without explicit mention of the projection maps. Since the product is a limit, we know immediately that all products of A and B are isomorphic.

Note that two morphisms $\alpha, \beta \colon X \to A \times B$ are equal if and only if

$$\rho_1 \circ \alpha = \rho_1 \circ \beta \ \text{ and } \ \rho_2 \circ \alpha = \rho_2 \circ \beta$$

This is a common application of the uniqueness of the mediating morphism.

Once the product of two objects is defined, it is not hard to generalize to the product of any nonempty family $\mathcal{F} = \{A_i \mid i \in I\}$ of objects in $\mathcal{C}$. Let $\mathbb{D}$ be the diagram consisting of the objects in $\mathcal{F}$, with no morphisms. A **product** of $\mathcal{F}$ is a limit in the category of cones over $\mathbb{D}$. In other words, the product is a pair

$$\left(\prod_{i \in I} A_i, \left\{\rho_i \colon \prod_{i \in I} A_i \to A_i\right\}\right)$$

where

$$\prod_{i \in I} A_i = \prod \mathcal{F}$$

is an object and the morphisms ρ_i are called projections, with the following property: If X is any object of $\mathcal{C}$ with morphisms $g_i \colon X \to A_i$ then there is a unique mediating morphism $\theta \colon X \to \prod \mathcal{F}$ for which $\rho_i \circ \theta = g_i$. If any pair of objects has a product, then $\mathcal{C}$ **has binary**

products. If any finite family of objects has a product, then $\mathcal{C}$ **has finite products**. If any family of objects has a product, then $\mathcal{C}$ **has products**.

The dual notion to product is **coproduct**, denoted by +, whose defining digraphs are the duals of the diagrams in Figure 54. We leave it to the reader to formulate the precise definition of coproduct.

One note on notation: In older literature, the product is denoted by $\sqcap$ and the coproduct by $\sqcup$.

A careful look at the definition of product shows that the product of the empty family is an object T with the property that there is exactly one morphism entering T from any object, that is, T is terminal. In short, the product of the empty family is a terminal object, if one exists. Otherwise, the product does not exist.

Similarly, the coproduct of the empty family is an initial object in $\mathcal{C}$, if it exists. As a result, the statement that a category has finite products implies that it also has a terminal object and similarly for coproducts. If $\mathcal{C}$ has both finite products and finite coproducts, then $\mathcal{C}$ has a zero object.

Since products are limits, they are defined up to isomorphism. This is also true of the projection maps. For if

$$(A \times B, \rho_1 \colon A \times B \to A, \rho_2 \colon A \times B \to B)$$

is a product of A and B and if $\lambda \colon A \times B \approx A \times B$ is an isomorphism, then

$$(A \times B, \rho_1 \circ \lambda \colon A \times B \to A, \rho_2 \circ \lambda \colon A \times B \to B)$$

is also a product of A and B.

We leave it as an exercise to show that, in general, the projection morphisms of a product need not be epic (right-cancellable). However, it is often the case that projections are not only epic, but right-invertible.

Theorem 86

Let $\mathcal{C}$ be a category. Assume that the product $(A \times B, \rho_1, \rho_2)$ exists. If there is at least one morphism $f \colon A \to B$, then ρ_1 is right-invertible. If there is at least one morphism $f \colon B \to A$, then ρ_2 is right-invertible.

Proof

The diagrams in Figure 55 tell the whole story. They show the product of A and B along with two cones, one with vertex A and the other with vertex B. Here f_{AB} is any morphism from A to B and f_{BA} is any morphism from B to A.

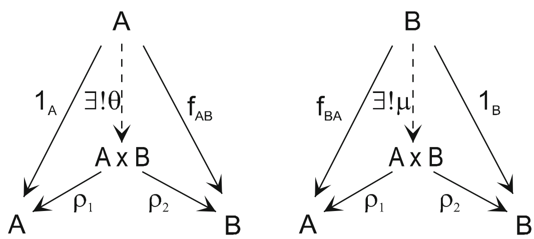

Figure 55

Since $\rho_1 \circ \theta = 1_A$, it follows that ρ_1 is right-invertible. Similarly, $\rho_2 \circ \mu = 1_B$ implies that ρ_2 is right-invertible. □

Example 87

1) In **Set**, the product is the cartesian product, with the usual projections.
2) In **Grp**, **Mod** and **Rng**, the product is the usual direct product of groups, modules or rings, defined coordinatewise.
3) In **Poset**(P), the product is the least upper bound.
4) In **Poset**, the category of all posets, with monotone maps, there are two potential candidates for the product

$$(P \times Q, \rho_1, \rho_2)$$

of posets P and Q. The **product order** on the cartesian product $P \times Q$ is defined by

$$(p_1, q_1) \leq (p_2, q_2) \quad \text{if} \quad p_1 \leq p_2 \quad \text{and} \quad q_1 \leq q_2$$

The set $P \times Q$ with this order is called the **product** of P and Q. Also, the **lexicographic order** on the cartesian product $P \times Q$ is defined by

$$(p_1, q_1) \leq (p_2, q_2) \quad \text{if} \quad p_1 < p_2 \quad \text{or} \quad (p_1 = p_2 \quad \text{and} \quad q_1 \leq q_2)$$

This is also a partial order on $P \times Q$. We leave it as an exercise to determine which of these candidates gives the categorical product. (Or is it both?)
5) In the exercises, we ask the reader to show that the category **Field** does not have products. □

Example 88

In **Set**, the coproduct is the *disjoint union* $A \sqcup B$ of the sets (with the obvious inclusions). Formally, the disjoint union is

$$A \sqcup B = \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$$

In **Grp**, the coproduct is the *free product* of groups, in fact, this is often taken as the *definition* of the free product. In **AbGrp**, **Mod**$_R$ and **Vect**$_F$, the coproduct is the usual direct *sum* of abelian groups, modules and vector spaces, respectively. Thus, *finite* coproducts are the same as finite products, but infinite coproducts differ from their product counterparts. In **CRng**, the coproduct of commutative rings is their tensor product. In **Poset**(P), the coproduct is the greatest lower bound. □

Pullbacks and Pushouts

We next look at pullbacks and pushouts.

Definition

Let $\mathcal{C}$ be a category. As shown in Figure 56, a **pullback** *of the pair $(f\colon A \to C, g\colon B \to C)$ is a terminal cone over the diagram $\mathbb{D}$.*

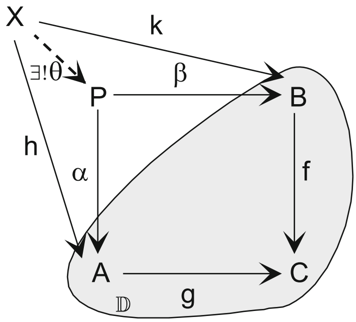

Figure 56

Thus, a pullback is a triple

$$(P, \alpha\colon P \to A, \beta\colon P \to B)$$

(*along with the determined third leg from P to C*) *for which*

$$g \circ \alpha = f \circ \beta$$

and with the property that for any object X and morphisms h: $X \to A$ and k: $X \to B$ for which

$$f \circ k = g \circ h$$

there exists a unique mediating morphism θ: $X \to P$ for which

$$\alpha \circ \theta \quad \text{and} \quad \beta \circ \theta = k$$

Referring to Figure 56, α is called the **pullback** *of f along g and β is called the* **pullback** *of g along f.* □

In any category $\mathcal{C}$, pullbacks, equalizers and products are related as follows:

$$\begin{array}{lcl} \text{pullbacks and terminal objects} & \Rightarrow & \text{finite products} \\ \text{binary products and equalizers} & \Rightarrow & \text{pullbacks} \end{array}$$

We leave proof of the first statement as an exercise and prove the second.

■ Theorem 89

If a category $\mathcal{C}$ has binary products and equalizers, then it has pullbacks. In fact, a pullback of $\mathbb{D} = \{f\colon A \to C, g\colon B \to C\}$ can be obtained as follows: As shown in Figure 57, let

$$\mathcal{P} = (A \times B, \rho_1, \rho_2)$$

be a product of A and B and let

$$\mathcal{E} = (E, e\colon E \to A \times B)$$

be an equalizer of the lower and upper diagonal maps

$$f \circ \rho_1 \colon A \times B \to C \quad \textit{and} \quad g \circ \rho_2 \colon A \times B \to C$$

Then the triple

$$\mathcal{U} = (E, \rho_1 \circ e \colon E \to A, \rho_2 \circ e \colon E \to B)$$

is a pullback of $\mathbb{D}$.

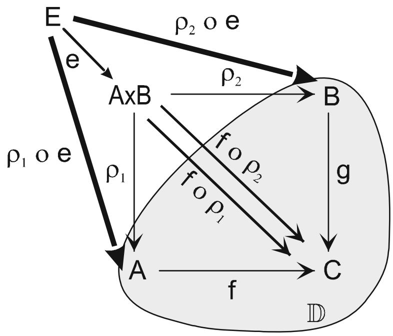

Figure 57 *A pullback*

■ Proof

As shown in Figure 58, if $\mathcal{X} = (X, h, k)$ is a cone over $\mathbb{D}$, that is, if

$$f \circ h = g \circ k \tag{90}$$

then we must show that there is a unique mediating morphism $\alpha \colon X \to E$ for which

$$(\rho_1 \circ e) \circ \alpha = h \quad \text{and} \quad (\rho_2 \circ e) \circ \alpha = k \tag{91}$$

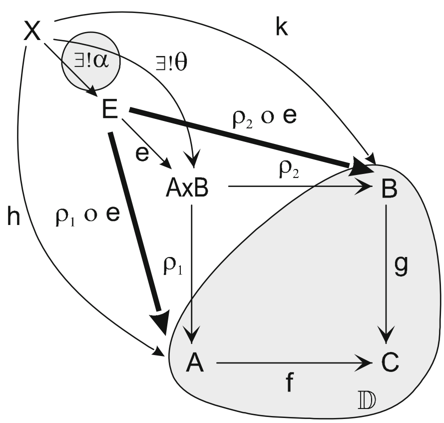

Figure 58

Now, by definition of the product, there exists a unique $\theta\colon X \to P$ for which

$$\rho_1 \circ \theta = h \quad \text{and} \quad \rho_2 \circ \theta = k \tag{92}$$

Substituting these expressions for h and k in (90) gives

$$f \circ \rho_1 \circ \theta = g \circ \rho_2 \circ \theta$$

Thus, θ also equalizes the upper and lower paths and so there is a unique $\alpha\colon X \to E$ for which

$$e \circ \alpha = \theta$$

Substituting into (92) gives

$$(\rho_1 \circ e) \circ \alpha = h \quad \text{and} \quad (\rho_2 \circ e) \circ \alpha = k$$

which is (91) as desired.

As to uniqueness, if $\mu\colon X \to E$ is another such mediating morphism, that is, if

$$(\rho_1 \circ e) \circ \mu = h \quad \text{and} \quad (\rho_2 \circ e) \circ \mu = k$$

then the uniqueness of θ satisfying (92) implies that

$$e \circ \mu = \theta = e \circ \alpha$$

But the equalizer arrow e is monic and so $\mu = \alpha$. □

Example 93

The previous theorem gives us a strong sense of what the pullback looks like in many familiar categories. For instance, in **Set**, we equalize the upper and lower diagonals

$$f \circ \rho_1\colon A \times B \to C \quad \text{and} \quad g \circ \rho_2\colon A \times B \to C$$

to get

$$E = \{(a, b) \in A \times B \mid g(a) = f(b)\}$$

with legs equal to the restriction of the projection maps ρ_1 and ρ_2 to the set E. This shows that pullbacks are a form of "super diagonalization" based on the functions f and g. □

The pullback of a monic is monic. However, the pullback of an epic need not be epic.

Theorem 94

Let C be a category.

1) *If a monic in C has a pullback, then the pullback is also monic.*
2) *The pullback of an epic need not be epic.*
3) *The pullback of a right-invertible morphism is right-invertible.*
4) *The pullback of an isomorphism is an isomorphism.*

Proof

For part 1), Figure 59 shows a pullback $\mathcal{P} = (P, \alpha, \beta)$ for $\{f, g\}$, where g is monic.

$$\begin{tikzcd}
X \arrow[dr, shift left, "k"] \arrow[dr, shift right, "h"'] \arrow[drr, bend left, "\beta \circ h = \beta \circ k"] \arrow[ddr, bend right, "\alpha \circ h = \alpha \circ k"'] & & \\
& P \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, hook, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Figure 59

To show that α is also monic, let

$$\alpha \circ h = \alpha \circ k$$

To show that $h = k$, we show that h and k are mediating arrows for $\mathcal{P}$. First note that

$$\beta \circ h = \beta \circ k$$

since

$$g \circ \beta \circ h = f \circ \alpha \circ h = f \circ \alpha \circ k = g \circ \beta \circ k$$

and since g is left-cancellable. Thus, all paths from X to either A, B or C commute. It follows that

$$\mathcal{X} = (X, \alpha \circ h = \alpha \circ k, \beta \circ h = \beta \circ k)$$

is a cone over $\{f: A \to C, g: B \to C\}$ and that both h and k are mediating arrows from $\mathcal{X}$ to $\mathcal{P}$, whence $h = k$. We leave proof of the remaining statements as an exercise. □

One can easily generalize the pullback to families $\mathcal{F} = \{f_i: A_i \to A\}$ of morphisms with the same codomain A: A pullback of $\mathcal{F}$ is a terminal cone over the cocone with vertex A and legs $f_i: A_i \to A$.

The dual construction to the pullback is the *pushout*. We leave it to the reader to formulate the definition and to show that the pushout of two set functions $f: C \to A$ and $g: C \to B$ is the disjoint union of A and B, but with elements that have the same preimage under f and g identified.

Exponentials

Let us begin with some motivation. Let A, B and C be sets. The exponential notation C^B is often used to denote the family of set functions from B to C. Now consider a set function $f\colon A \times B \to C$ of two variables. We can turn f into a function $\tau_f\colon A \to C^B$ of one variable by letting $\tau_f(a)$ be the function defined by

$$\tau_f(a)(b) = f(a, b) \tag{95}$$

It is clear that f and τ_f determine each other uniquely, that is, if $\sigma\colon A \to C^B$ satisfies

$$\sigma(a)(b) = f(a, b)$$

then $\sigma = \tau_f$. This is a technique that computer scientists refer to as *currying*. (Specifically, **currying** is the technique of transforming a function of several arguments into a function of the first argument that returns a function of the remaining arguments. The technique was named after the logician Haskell Curry.)

To make this notion categorical, we need to remove mention of elements. To this end, define the **evaluation function** ϵ to be the map that takes a function $\alpha \in C^B$ and an element b in the domain B and gives $\alpha(b) \in C$, that is,

$$\epsilon\colon C^B \times B \to C, \quad \epsilon(\alpha, b) = \alpha(b)$$

Then (95) can be written in the form

$$\epsilon\big(\tau_f(a), b\big) = f(a, b)$$

or

$$\epsilon \circ \big(\tau_f \times 1_B\big)(a, b) = f(a, b)$$

or finally,

$$\epsilon \circ \big(\tau_f \times 1_B\big) = f$$

This looks a lot like a couniversal mapping property.

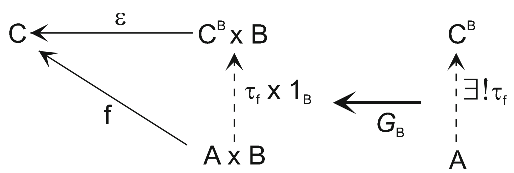

Figure 60

To this end, define the "product by B" functor $G_B\colon \mathcal{C} \Rightarrow \mathcal{C}$ by

$$G_B(A) = A \times B, \quad G_B(f) = f \times 1_B$$

for $A \in \mathcal{C}$ and $f \in \mathbf{Mor}(\mathcal{C})$. Then with reference to Figure 60, for any $f\colon A \times B \to C$, there is a unique $\tau_f\colon A \to C^B$ for which

$$f = \epsilon \circ G_B \tau_f \tag{96}$$

Let us generalize this construction.

Definition

Let $\mathcal{C}$ be a category with binary products and let $B, C \in \mathcal{C}$. As shown in Figure 60, an **exponential** *of B with C consists of a pair*

$$\left(C^B, \epsilon\colon C^B \times B \to C\right)$$

where C^B is an object of $\mathcal{C}$ and where the following property holds: For any object $X \in \mathcal{C}$ and morphism

$$f\colon X \times B \to C$$

there is a unique morphism $\tau_f\colon X \to C^B$ for which

$$\epsilon \circ \left(\tau_f \times 1_B\right) = f$$

The map ϵ is called **evaluation** *and the morphism τ_f is called the (exponential)* **transpose** *of f.*□

Example 97

Consider the category **Poset** of all posets, with monotone maps. Let $P, Q \in$ **Poset**. Using the example of **Set**, we guess that the exponential

$$\left(Q^P, \epsilon\colon Q^P \times P \to Q\right)$$

is given as follows. Let Q^P be the set of all monotone maps from P to Q and let ϵ be defined by

$$\epsilon(g, p) = g(p)$$

To see if this is an exponential, suppose that $f\colon X \times P \to Q$ is a monotone map. Define $\tau_f\colon X \to Q^P$ as the curried version of f, that is,

$$\tau_j(x) = f(x, \cdot)$$

To see that τ_f is monotone, if $x \leq y$, then for all $p \in P$, we have $(x, p) \leq (y, p)$ in $X \times P$ and so

$$\tau_f(x)(p) = f(x, p) \leq f(y, p) = \tau_f(y)(p)$$

which implies that $\tau_f(x) \leq \tau_f(y)$. Then we have

$$\big(\tau_f \times 1_P\big)(x, p) = \big(\tau_f(x), p\big)$$

and so

$$\epsilon \circ \big(\tau_f \times 1_P\big)(x, p) = \epsilon\big(\tau_f(x), p\big) = \tau_f(x)(p) = f(x, p)$$

as desired. As to uniqueness, if $\mu\colon X \to Q^P$ satisfies

$$\epsilon \circ (\mu \times 1_P)(x, p) = f(x, p)$$

then

$$\mu(x)(p) = f(x, p) = \tau_f(x)(p)$$

for all $p \in P$ and all $x \in X$ and so $\mu = \tau_f$. □

Example 98

Consider the category **Grp** and let G be a group. Using the example of **Set**, we guess that the exponential

$$\big(G^G, \epsilon\colon G^G \times G \to G\big)$$

is given as follows. Let G^G be the set of all group homomorphisms from G to itself and let ϵ be defined by

$$\epsilon(f, a) = f(a)$$

To see if this is an exponential, suppose that $f\colon X \times G \to G$ is a group homomorphism. Define $\tau_f\colon X \to G^G$ by

$$\tau_f(x) = f(x, \cdot)$$

To see if τ_f is a group homomorphism, let $x, y \in X$. Then we want to show that

$$\tau_f(xy) = \tau_f(x) \circ \tau_f(y)$$

This can be written

$$f(xy, \cdot) = f(x, \cdot) \circ f(y, \cdot)$$

or applying it to $a \in G$,

$$f(xy, a) = f(x, f(y, a))$$

which looks a bit untrue in general, thus putting the existence of group exponentials in serious doubt.

In fact, if we take $X = \{1\}$, then the existence of exponentials would imply that there is a bijection between group homomorphisms $f\colon \{1\} \times G \to G$ and mediating morphisms $\tau_f\colon \{1\} \to G^G$. But there are in general many of the former and only one of the latter! Thus, the category **Grp** does not have exponentials. □

Existence of Limits

Let us now consider the question of the existence of arbitrary limits in a category. In general, it is too much to expect a category to have arbitrary limits. For example, in **Set**, the diagram consisting of all sets cannot have a product, since this would be the Cartesian product of all sets, which is not a set. Thus, we restrict attention to *small diagrams*.

Definition

1) *A diagram* $\mathbb{D}$ *in a category* $\mathcal{C}$ *is called a* **small diagram** *if the class of objects of* $\mathbb{D}$ *is a set and a* **finite diagram** *if the class of objects of* $\mathbb{D}$ *is a finite set.*
2) *A category* $\mathcal{C}$*. is* **complete** *if every small diagram has a limit and* **finitely complete** *if every finite diagram has a limit. Dually,* $\mathcal{C}$ *is* **cocomplete** *if every small diagram has a colimit and* **finitely cocomplete** *if every finite diagram has a colimit.* □

We assume in the following discussion that all diagrams are **small diagrams** and use the term **small product** to denote the product of a small diagram. Our main theorem says that if $\mathbb{D}$ is a small diagram in a category $\mathcal{C}$ that has equalizers and if certain small products related to $\mathbb{D}$ exist, then $\mathbb{D}$ has a limit in $\mathcal{C}$.

Note that since the presence or absence of loops labeled with an identity morphism in a diagram $\mathbb{D}$ does not change the cone category over $\mathbb{D}$, this has no effect on the existence of a limit. Therefore, in order to ensure that every node in $\mathbb{D}$ has in-degree at least 1, we may assume that every node has a loop labeled with the appropriate identity morphism.

Now let $\mathbb{D}$ be a small diagram in $\mathcal{C}$, as shown on the left in Figure 61. For clarity, we have omitted the loops at each node labeled with identity morphisms. Let

$$\{D_k \mid k \in K\}$$

be the multiset of objects that label the nodes of $\mathbb{D}$, where the index set K indexes the *nodes* of $\mathbb{D}$. (A **multiset** is a "set" in which each element may appear more than once.) Thus, D_k and D_j need not be distinct objects for $k \neq j$.

As shown in Figure 61, we make two new diagrams from $\mathbb{D}$. The diagram $\mathbb{D}^-$ consists of just the nodes of $\mathbb{D}$. To make the diagram $\mathbb{D}^+$, if a node of $\mathbb{D}$ has in-degree m_k in $\mathbb{D}$, then $\mathbb{D}^+$ has m_k nodes, each labeled with D_k.

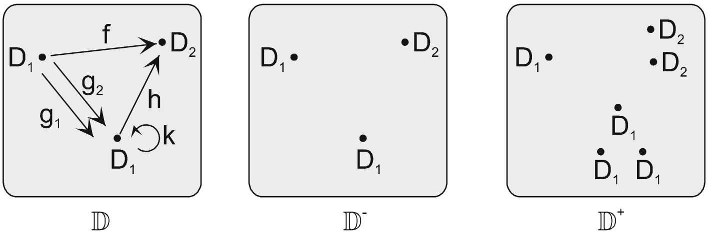

Figure 61

Now we can state the general result.

▪ Theorem 99

Let $\mathcal{C}$ be a category with equalizers. Let $\mathbb{D}$ be a diagram in $\mathcal{C}$ and let

$$\{D_k \mid k \in K\}$$

be the multiset of objects labeling the nodes of $\mathbb{D}$. Assume the following:

1) *The product*

$$\mathcal{P} = \left(P, \{\rho_k \mid k \in K\}\right)$$

of the diagram $\mathbb{D}^-$ defined above exists.

2) *The product*

$$\mathcal{Q} = \left(Q, \{\pi_{k,i} \colon Q \to D_k \mid k \in K, 1 \le i \le m_k\}\right)$$

of the diagram $\mathbb{D}^+$ defined above exists.

Then $\mathbb{D}^+$ *has a limit in $\mathcal{C}$. In particular, if $\mathcal{C}$ has equalizers and small products, then $\mathcal{C}$ is complete and if $\mathcal{C}$ has equalizers and finite products, then $\mathcal{C}$ is finitely complete.*

▪ Proof

The proof is illustrated in Figure 62.

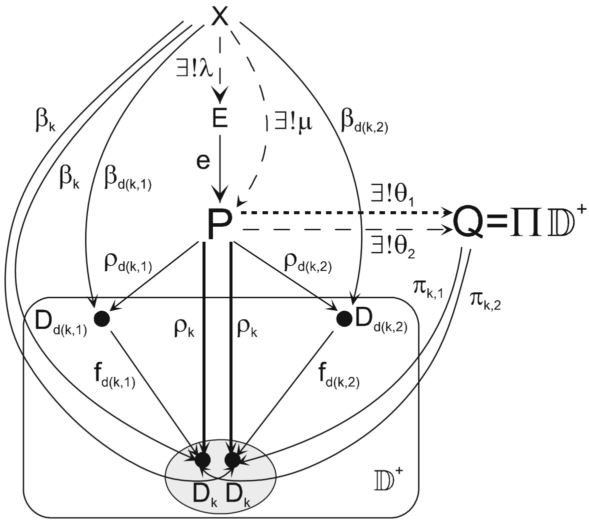

Figure 62

The figure shows a node of $\mathbb{D}$ with in-degree $m_k = 2$ and object label D_k, which manifests itself in the diagram $\mathbb{D}^+$ as a pair of nodes labeled D_k. Two morphisms

$$f_{d(k,1)}: D_{d(k,1)} \to D_k \quad \text{and} \quad f_{d(k,2)}: D_{d(k,2)} \to D_k$$

in $\mathbb{D}$ that enter D_k are also shown. The figure shows the vertices P and Q of the products of $\mathbb{D}^-$ and $\mathbb{D}^+$,

$$\mathcal{P} = (P = \prod \mathbb{D}^-, \rho_k: P \to D_k)$$
$$\mathcal{Q} = (Q = \prod \mathbb{D}^+, \pi_{k,i}: Q \to D_k)$$

Notice that the projections $\pi_{k,i}$ for the product $\mathcal{Q}$ are doubly indexed because each object in $\mathbb{D}$ may be duplicated in $\mathbb{D}^+$. For example, since the object D_k appears twice in $\mathbb{D}^+$, we need two projections from Q to D_k, labeled $\pi_{k,1}$ and $\pi_{k,2}$.

The projections ρ_k of the product $\mathcal{P}$ of $\mathbb{D}^-$ can be used to define two cones over $\mathbb{D}^+$, both with vertex P. The first cone is obtained by simply duplicating each projection ρ_k a total of m_k times, giving the cone

$$\mathcal{K}_1 = \left(P, \left\{\rho_{k,i}: P \to D_k \mid k \in K, 1 \le i \le m_k\right\}\right)$$

where $\rho_{k,i} = \rho_k$.

The second cone over $\mathbb{D}^+$ uses the morphisms of $\mathbb{D}$. Specifically, each leg of this cone is a path of length 2 that consists of a projection to the domain of *each* morphism $f_{d(k,i)}$ of $\mathbb{D}$, followed by the morphism $f_{d(k,i)}$ itself. Thus,

$$\mathcal{K}_2 = \left(P, \left\{f_{d(k,i)} \circ \rho_{d(k,i)}: P \to D_k \mid k \in K, 1 \le i \le m_k\right\}\right)$$

Now, since $\mathcal{Q}$ is a product of $\mathbb{D}^+$, there are unique mediating morphisms θ_1, θ_2: $P \to Q$, corresponding to the two cones $\mathcal{K}_1$. and $\mathcal{K}_2$, for which

$$\pi_{k,i} \circ \theta_1 = \rho_k$$

and

$$\pi_{k,i} \circ \theta_2 = f_{d(k,i)} \circ \rho_{d(k,i)}$$

If (E, e: $E \to P$) is the equalizer of θ_1 and θ_2, then applying e to the right gives the two equations

$$\pi_{k,i} \circ \theta_1 \circ e = \rho_k \circ e$$

and

$$\pi_{k,i} \circ \theta_2 \circ e = f_{d(k,i)} \circ \rho_{d(k,i)} \circ e$$

and since $\theta_1 \circ e = \theta_2 \circ e$, the left sides are equal and therefore so are the right sides, whence

$$\rho_k \circ e = f_{d(k,i)} \circ \rho_{d(k,i)} \circ e$$

This says that the pair

$$\mathcal{E} = \left(E, \left\{\rho_k \circ e\colon E \to D_k \middle| k \in K\right\}\right)$$

is a cone over $\mathbb{D}$, since the morphisms $\{f_{d(k,i)}\}$ constitute a complete set of morphisms in $\mathbb{D}$.

To see that $\mathcal{E}$ is terminal, consider an arbitrary cone over $\mathbb{D}$,

$$\mathcal{X} = (X, \{\beta_k\colon X \to D_k\})$$

where

$$f_{d(k,i)} \circ \beta_{d(k,i)} = \beta_k$$

Since $\mathcal{X}$ is also a cone over $\mathbb{D}^-$, there is a unique mediating morphism μ: $X \to P$ for which

$$\rho_k \circ \mu = \beta_k$$

The map μ right-equalizes θ_1 and θ_2, since for all projections $\pi_{k,i}$, we have

$$\pi_{k,i} \circ (\theta_1 \circ \mu) = \rho_k \circ \mu = \beta_k$$

and

$$\pi_{k,i} \circ (\theta_2 \circ \mu) = f_{d(k,i)} \circ \rho_{d(k,i)} \circ \mu = f_{d(k,i)} \circ \beta_{d(k,i)} = \beta_k$$

and so

$$\theta_1 \circ \mu = \theta_2 \circ \mu$$

Hence, there is a unique mediating morphism $\lambda\colon X \to E$ for which

$$e \circ \lambda = \mu$$

But then

$$(\rho_k \circ e) \circ \lambda = \rho_k \circ \mu = \beta_k$$

which shows that $\lambda\colon X \to E$ is a cone morphism from $\mathcal{X}$ to $\mathcal{E}$. As to uniqueness, if

$$(\rho_k \circ e) \circ \alpha = \beta_k$$

then

$$(\rho_k \circ e) \circ \alpha = (\rho_k \circ e) \circ \lambda$$

for all ρ_k and so $e \circ \alpha = \mu = e \circ \lambda$, which implies that $\alpha = \lambda$. □

Completeness

We have mentioned that if $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has finite products. Therefore, since an equalizer is a special type of pullback, we have the following result.

Theorem 100

Let $\mathcal{C}$ be a category. The following are equivalent:

1) *$\mathcal{C}$ is complete.*
2) *$\mathcal{C}$ has equalizers and small products.*

The following are also equivalent:

3) *$\mathcal{C}$ is finitely complete.*
4) *$\mathcal{C}$ has equalizers and finite products.*
5) *$\mathcal{C}$ has pullbacks and a terminal object.* □

Exercises

1. Let $\mathcal{C}$ be a category. Show that if $\mathcal{K} = \left(K, \left\{f_n \mid n \in \mathcal{J}\right\}\right)$ is a terminal cone in $\mathcal{C}$ and $\theta : L \approx K$ is an isomorphism in $\mathcal{C}$, then

$$\mathcal{L} = \left(L, \left\{f_n \circ \theta \mid n \in \mathcal{J}\right\}\right)$$

is also a terminal cone.
2. Prove that there is at most one cone morphism $\alpha\colon \mathcal{K} \to \mathcal{L}$ between cones when $\mathcal{L}$ has only monic legs.
3. Show that the pair (E, e) is an equalizer of (f, f) if and only if $e\colon E \to A$ is an isomorphism.
4. Let (E, e) be an equalizer. Prove that if e is epic, then it is an isomorphism.
5. Find the limits of a diagram consisting of a single morphism $f\colon A \to B$.

6. If a category $\mathcal{C}$ has finite products, prove that any $f: A \to B$ can be factored into a product $f = \rho \circ \theta$, where θ is right-invertible. Also, if there is a map from B to A in $\mathcal{C}$, then ρ is left-invertible.
7. Prove that the category **Field** does not have products.
8. Let $\mathcal{C}$ be a category. Is it true that if the product of a family $\mathcal{F} = \{A_i \mid i \in I\}$ exists in $\mathcal{C}$, then the product of any nonempty subfamily of $\mathcal{F}$ also exists? *Hint*: Consider the category of all sets of size 2 or less.
9. a) Find the product in **Poset**.
   b) Find the coproduct in **Poset**.
10. Find the coproduct of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.
11. Let

$$\mathcal{P} = (A \times B, \rho_1: A \times B \to A, \rho_2: A \times B \to B)$$

be the product of A and B. Show that if α, β: $X \to A \times B$ are parallel morphisms satisfying

$$\rho_1 \circ \alpha = \rho_1 \circ \beta$$

and

$$\rho_2 \circ \alpha = \rho_2 \circ \beta$$

then $\alpha = \beta$.
12. Show that if a category $\mathcal{C}$ has the property that every pair of objects has a product, then any finite family of objects has a product. In fact, the product $(A \times B) \times C$ is equal to $A \times B \times C$.
13. Show that the projection morphisms of a product need not be epic.
14. Let A, B and C be objects in a category $\mathcal{C}$. Assuming that all coproducts exist, show that if A and B are isomorphic, then $A + C$ and $B + C$ are also isomorphic.
15. a) Find the coequalizer in **Mod**.
    b) Find the coequalizer in **Rng.**
16. Prove that in **Vect**, a morphism is monic if and only if it is an equalizer. State and prove the dual.
17. Let G: $\mathcal{D} \Rightarrow$ **Set**. Suppose that there is a universal pair

$$(S_x, u_x: \{x\} \to GS_x)$$

for every one-element set $\{x\}$. Let X be a nonempty set. Suppose that the coproduct

$$(C := +_{x \in X} S_x, \{\kappa_x: S_x \to C\})$$

exists. Prove that

$$(C, u: X \to GC)$$

is universal for (C, G), where

$$u(x) = (G\kappa_x \circ u_x)(x)$$

18. Let $\mathcal{C}$ be a category. Show that if a morphism in $\mathcal{C}$ has one of the following properties, then so does its pullback (if it exists).
    a) right-invertible
    b) an equalizer.
19. Find a category in which the pullback of a left-invertible morphism need not be left-invertible. *Hint*: Try a subcategory of **Set**.
20. Find a category in which the pullback of an epic need not be epic. *Hint*: Try a subcategory of **Set**.
21. Let $\mathcal{C}$ be a category with pullbacks. Show that if $\mathcal{C}$ has a terminal object T, then $\mathcal{C}$ has finite products.
22. In **Set**, let $A \subseteq C$ and let $j\colon A \to C$ be the inclusion map. Show that the pullback object P of $(j\colon A \to C,\ f\colon B \to C)$ is isomorphic to the inverse image $f^{-1}(A)$.
23. Consider the diagram in Figure 69. Show that if the two small squares are pullbacks, then the entire rectangle is a pullback.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & B_1 \arrow[r, "g_1"] \arrow[d, "h_2"'] & C_1 \arrow[d, "h_3"'] \\
A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, "g_2"'] & C_2
\end{tikzcd}$$

Figure 63

24. Describe the limit of any diagram in **Set**, starting with the product of the objects in the diagram. In particular, describe the limit in **Set** of the diagram consisting of the following three objects and two arrows:

$$1\colon \{a, b\} \to \{1, 2\};\ 1(x) = 1$$
$$2\colon \{c, d\} \to \{1, 2\};\ 2(x) = 2$$

25. As shown in Figure 64,

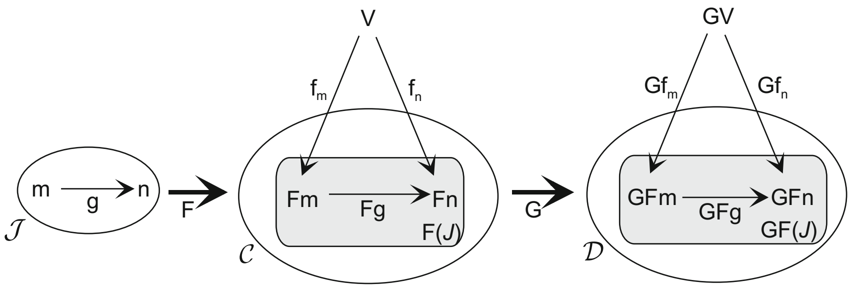

Figure 64

a functor $G\colon \mathcal{C} \Rightarrow \mathcal{D}$ **preserves limits** if the following holds: For any limit

$$\mathcal{K} = \left(V, \{ f_n \mid n \in \mathcal{J}\}\right)$$

of a functor $F\colon \mathcal{J} \Rightarrow \mathcal{C}$, the cone

$$G\mathcal{K} = \left(GV, \{Gf_n \mid n \in \mathcal{J}\}\right)$$

is a limit for the functor $G \circ F$. For $A \in \mathcal{C}$, prove that the hom functor

$$\hom_{\mathcal{C}}(A, \cdot) : \mathcal{C} \Rightarrow \textbf{Set}$$

preserves limits. State and explain the dual.

26. A functor $G\colon \mathcal{C} \Rightarrow \mathcal{D}$ **reflects limits** if whenever

$$\mathcal{K} = \left(V, \{ f_n \mid n \in \mathcal{J}\}\right)$$

is a cone over $F\colon \mathcal{J} \Rightarrow \mathcal{C}$ for which the image

$$G\mathcal{K} = \left(GV, \{Gf_n \mid n \in \mathcal{J}\}\right)$$

is a limit of $G \circ F$, then $\mathcal{K}$ is a limit of F. Prove that if $\mathcal{C}$ is small and complete and if G preserves limits and reflects isomorphisms (isomorphism are limits over a single-object diagram), then G reflects limits.

27. Prove that any fully faithful functor $G\colon \mathcal{C} \Rightarrow \mathcal{D}$ reflects limits.

Inverse and Direct Systems

A partially ordered set N is **directed** if for any i, $j \in N$, there is a $k \in N$ for which $k > i$ and $k > j$. We make the following definitions, which are standard for specific cases (such as the category of groups or modules) but perhaps not standard in the general setting of a category.

Definition

Let $\mathcal{C}$ be a category. An **inverse system** *is a diagram $\mathbb{G}$ in $\mathcal{C}$ with the following properties:*

1) *The objects of $\mathbb{G}$ are indexed by a directed partially ordered set N.*
2) *There is exactly one morphism*

$$f_{i,j}\colon A_j \to A_i$$

between each pair (A_i, A_j) of objects of $\mathbb{G}$ with $i \le j$ (note the direction of the morphisms).

3) $f_{i,i} = 1_{A_i}$ *is the identity for all $i \in N$*
4) $f_{k,i} \circ f_{i,j} = f_{k,j}$ *for all $k \le i \le j$ in N* □

28. Show that the limit of an inverse system

$$\mathbb{G} = \left(\{G_i\},\ f_{i,j}\colon G_j \to G_i\}\right)$$

of groups is

$$\lim_{\leftarrow}(\mathbb{G}) = \left\{ \alpha \in \prod_{i \in I} G_i \;\middle|\; f_{i,j}(\alpha_j) = \alpha_i, \text{ for all } i \le j \right\}$$

where we have written $\alpha(k)$ as α_k. Hence, in this case, the elements of $\lim\limits_{\leftarrow}(\mathbb{G})$ are elements of the product whose coordinates are related as specified by the morphisms of $\mathbb{G}$.

29. Show that when the partial order on N is equality, the limit of an inverse system is just the direct product.

We can dualize the notions of inverse system and limit.

Definition

Let $\mathcal{C}$ be a category. A **direct system** *is a diagram $\mathbb{G}$ in $\mathcal{C}$ with the following properties:*

1) *The objects of $\mathbb{G}$ are indexed by a directed partially ordered set N.*
2) *There is exactly one morphism*

$$f_{i,j}: A_i \to A_j$$

between each pair (A_i, A_j) of objects of $\mathbb{G}$ with $i \le j$.

3) $f_{i,i} = 1_{A_i}$ *is the identity for all $k \in N$*
4) $f_{j,k} \circ f_{i,j} = f_{i,k}$ *for all $i \le j \le k$ in N* □

30. Prove that the colimit of a direct system $\mathcal{M}$ of modules is a quotient module of the coproduct (direct sum)

$$+ M_i = \sum_{i \in I} M_i$$

of the family of modules, defined as follows. For $a_i \in M_i$, write

$$[a_i]_i = \kappa_i(a_i)$$

where $\kappa_k: M_k \to \sum M_i$ is the canonical injection map. Also, if $j > i$, write

$$[a_i]_{i,j} = [a_i]_i - \left[f_{i,j}(a_i) \right]_j$$

Let N be the submodule generated by the $[a]_{i,j}$ and define the colimit as the quotient

$$\operatorname{dirlim}(\mathcal{M}) = \lim_{\rightarrow} \mathcal{M} = +M_i / N$$

Adjoints

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It has been said that the notion of an *adjoint* is the pinnacle notion in category theory; that it is the culmination and perhaps even the *raison d'être* of all of the theory that comes before it. It has also been said that adjoints are both unifying and ubiquitous in mathematics and have a strong and powerful presence in other disciplines as well, such as computer science.

There are many approaches to the study of adjoints and it seems as though every source uses a somewhat different tack. We will approach the subject through the concept of (bi)naturalness.

To improve readibility, we will often write $G_{\mathcal{D}\Rightarrow\mathcal{C}}$ in place of $G\colon \mathcal{D} \Rightarrow \mathcal{C}$.

Binaturalness

Suppose that $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ are antiparallel functors and let

$$\{\tau_{C,D}\} = \{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}\colon \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{C}}(FC, D)$$

be a doubly indexed family of maps. Thus, C ranges over the objects of $\mathcal{C}$ and D ranges over the objects of $\mathcal{D}$. Then we can combine Theorem 64 and Theorem 78 as follows. Note that $U = FC$ in Theorem 64 and $S = GD$ in Theorem 78.

Theorem 101

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be antiparallel functors and let

$$\{\tau_{C,D}\} = \{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}\colon \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{C}}(FC, D)$$

be a doubly-indexed family of bijections. Let

$$u_C = \tau^{-1}_{C,FC}(1_{FC}) \quad \textit{and} \quad v_D = \tau_{GD,D}(1_{GD})$$

1) *The following are equivalent:*
   a) **(Mediating morphisms)** *The maps $\{\tau_{C,D}\}_{\mathcal{D}}$ are the mediating morphism maps for the universal pairs $(FC, u_C : C \to GFC)$, that is, $\tau_{C,D}(f_{C,GD})$ is the unique solution to the equation*

$$G\tau_{C,D}\big(f_{C,GD}\big) \circ u_C = f_{C,GD}$$

b) **(Inverse Fusion formula)** *The maps* $\{\tau_{C,D}\}_{\mathcal{D}}$ *satisfy*

$$\tau^{-1}_{C,D}(h_{U,D}) = Gh_{U,D} \circ u_C$$

c) **(Naturalness in D)** $\{\tau_{C,D}\}_{\mathcal{D}}$ *is a natural isomorphism in* D*, that is,*

$$\tau_{C,D'} \circ \left(Gh_{D,D'}\right)^{\leftarrow} = h^{\leftarrow}_{D,D'} \circ \tau_{C,D}$$

or equivalently,

$$\tau_{C,D'}\left(Gh_{D,D'} \circ f_{C,GD}\right) = h_{D,D'} \circ \tau_{C,D}\left(f_{C,GD}\right)$$

2) *The following are equivalent:*
a) **(Comediating morphisms)** *The maps* $\{\tau^{-1}_{C,D}\}_{\mathcal{C}}$ *are the comediating morphisms for the couniversal pairs* $(GD, \upsilon_D\colon FGD \to D)$, that is, $\tau^{-1}_{C,D}(h_{FC,D})$ *is the unique solution to the equation*

$$\upsilon_D \circ F\tau^{-1}_{C,D}(h_{FC,D}) = h_{FC,D}$$

b) **(Direct Fusion formula)** *The maps* $\{\tau^{-1}_{C,D}\}_{\mathcal{C}}$ satisfy

$$\tau_{C,D}(f) = \upsilon_D \circ Ff$$

c) **(Naturalness in C)** $\{\tau^{-1}_{C,D}\}_{\mathcal{C}}$ *is a natural isomorphism in* C*, that is,*

$$g^{\rightarrow}_{C,C'} \circ \tau^{-1}_{C,D} = \tau^{-1}_{C',D} \circ \left(Fg_{C,C'}\right)^{\rightarrow}$$

or equivalently,

$$\tau_{C',D}\left(f_{C,GD} \circ g_{C',C}\right) = \tau_{C,D}\left(f_{C,GD}\right) \circ Fg_{C',C} \qquad \square$$

■ Definition

In the context of Theorem 101, if any one (and therefore all) condition holds from each of section 1) and section 2), then

1) *the family* $\{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}$ *is* **binatural** *in* C *and* D*,*
2) *the triple*

$$\mathcal{A} = \left(F_{\mathcal{C}\Rightarrow\mathcal{D}}, G_{\mathcal{D}\Rightarrow\mathcal{C}}, \{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}\right)$$

is called an **adjunction** *from* $\mathcal{C}$ *to* $\mathcal{D}$*,*

3) *the functor* F *is called a* **left adjoint** *of* G*, denoted by* $F \dashv G$ *and the functor* G *is called a* **right adjoint** *of* F*, denoted by* $F \vdash G$*.* $\square$

The Unit-Counit Structure

Theorem 101 is centered on the families $\{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}$ of bijections. We can also take the point of view of the families of maps

$$\{u_C: C \to GFC\}_{\mathcal{C}} \quad \text{and} \quad \{v_D: FGD \to D\}_{\mathcal{D}}$$

To begin, we note a special case of the fusion formulas that involve only these two families. Specifically, setting $C = GD$ and $h_{FGD,D} = v_D$ in the first fusion formula gives

$$\tau^{-1}_{GD,D}(v_D) = Gv_D \circ u_C$$

But $v_D = \tau_{GD,D}(1_{GD})$ and so

$$Gv_D \circ u_C = 1_{GD}$$

Similarly,

$$v_{FC} \circ Fu_C = 1_{FC}$$

It will be convenient to call these two formulas the **basic fusion formulas**.

Now, in the setting of Theorem 101, if $f: C \to C'$, then

$$uC' \circ f: C \to GFC'$$

and so the direct fusion formula gives

$$\begin{aligned}
\tau_{C,FC'}(u_{C'} \circ f) &= v_{FC'} \circ F(u_{C'} \circ f) \\
&= v_{FC'} \circ Fu_{C'} \circ Ff \\
&= v_{FC'} \circ F\tau^{-1}_{C',FC'}(1_{FC'}) \circ Ff \\
&= 1_{FC'} \circ Ff \\
&= Ff
\end{aligned}$$

Hence,

$$\tau^{-1}_{C,FC'}(Ff_{C,C'}) = u_{C'} \circ f_{C,C'} \tag{102}$$

and the inverse fusion formula gives

$$GF f_{C,C'} \circ u_C = u_{C'} \circ f_{C,C'}$$

This says that the family $\{u_C\}_{\mathcal{C}}$ is a natural transformation from the identity functor to the composite functor GF,

$$\{u_C\}_{\mathcal{C}}: I_{\mathcal{C}} \xrightarrow{\cdot} GF$$

Dually, we leave it to the reader to show that the fusion formulas also imply that

$$v_D \circ FGh_{D',D} = h_{D',D} \circ v_{D'}$$

and so

$$\{v_D\}: FG \dot{\to} I_{\mathcal{D}}$$

Thus, the units and counits of an adjunction are natural transformations. Of course, they also satisfy the basic fusion formulas.

Conversely, suppose that we start with natural transformations

$$\{u_C\}_{\mathcal{C}}: I_{\mathcal{C}} \dot{\to} GF \quad \text{and} \quad \{v_D\}_{\mathcal{D}}: FG \dot{\to} I_{\mathcal{D}}$$

that satisfy the basic fusion formulas. Then we define a family of maps

$$\tau_{C,D}: \hom_{\mathcal{C}}(C, GD) \to \hom_{\mathcal{D}}(FC, D)$$

by what would be the direct fusion formula

$$\tau_{C,D}(f_{C,GD}) = v_D \circ Ff_{C,GD} \tag{103}$$

The naturalness of $\{u_C\}$ and the basic fusion formulas give for any $f: C \to GD$,

$$\begin{aligned} G(v_D \circ Ff) \circ u_C &= Gv_D \circ GFf \circ u_C \\ &= Gv_D \circ u_{GD} \circ f \\ &= 1_{GD} \circ f \\ &= f \end{aligned}$$

and so

$$G\tau_{C,D}(f_{C,GD}) \circ u_C = f_{C,GD}$$

Hence, $\tau_{C,D}$ has *left* inverse

$$\mu_{C,D}(h_{FC,D}) = G(h_{FC,D}) \circ u_C \tag{104}$$

Moreover, the naturalness of $\{v_D\}$ and the basic fusion formulas imply that for any $h: FC \to D$

$$\begin{aligned} \tau_{C,D}(\mu_{C,D}(h_{FC,D})) &= v_D \circ F(Gh_{FC,D} \circ u_C) \\ &= v_D \circ FGh_{FC,D} \circ Fu_C \\ &= h_{FC,D} \circ v_{FC} \circ Fu_C \\ &= h_{FC,D} \circ 1_{FC} \\ &= h_{FC,D} \end{aligned}$$

Thus, the maps $\tau_{C,D}$ are bijections, $\mu_{C,D} = \tau^{-1}_{C,D}$ and so (103) and (104) are the two fusion formulas. In summary, the fusion formulas imply the naturalness of the families $\{u_C\}_{\mathcal{C}}$ and $\{v_D\}_{\mathcal{D}}$ (and the basic fusion formulas) and conversely, the naturalness of these two families and the basic fusion formulas imply the fusion formulas.

This calls for a definition followed by a theorem.

Definition

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be antiparallel functors. The triple

$$\mathcal{N} = \left(F_{\mathcal{C}\Rightarrow\mathcal{D}}, G_{\mathcal{D}\Rightarrow\mathcal{C}}, \{u_C\}_{\mathcal{C}}, \{v_D\}_{\mathcal{D}}\right)$$

where $u_C\colon C \to GFC$ and $v_D\colon FGD \to D$ is called a **unit-counit structure** *if*

$$\{u_C\}_{\mathcal{C}}\colon I_{\mathcal{C}} \stackrel{\cdot}{\to} GF \quad \textit{and} \quad \{v_D\}_{\mathcal{D}}\colon FG \stackrel{\cdot}{\to} I_{\mathcal{D}}$$

and if these families satisfy the basic fusion formulas

$$Gv_D \circ u_C = 1_{GD} \quad \textit{and} \quad v_{FC} \circ Fu_C = 1_{FC}$$

In this case, the maps u_C are called **units** *and the maps v_D are called* **counits**. *The family $\{u_C\}_{\mathcal{C}}$ is also called a unit and the family $\{v_C\}_{\mathcal{D}}$ is called a counit.* □

Theorem 105

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$. Then the following are equivalent.

1) *The triple*

$$\mathcal{A} = \left(F_{\mathcal{C}\Rightarrow\mathcal{D}}, G_{\mathcal{D}\Rightarrow\mathcal{C}}, \{u_C\}_{\mathcal{C}}, \{v_D\}_{\mathcal{D}}\right)$$

is an adjunction and

$$u_C = \tau^{-1}_{C,FC}(1_{FC}) \quad \textit{and} \quad v_D = \tau_{GD,D}(1_{GD})$$

2) *The triple*

$$\mathcal{N} = \left(F_{\mathcal{C}\Rightarrow\mathcal{D}}, G_{\mathcal{D}\Rightarrow\mathcal{C}},, \{u_C\}_{\mathcal{C}},, \{v_D\}_{\mathcal{D}}\right)$$

is a unit-counit structure and

$$\tau_{C,D}\left(f_{C,GD}\right) = v_D \circ Ff_{C,GD}$$

We should point out that unit-counit structures are often notated quite differently in the literature. In particular, the unit maps u_C is often denoted by η_C and the unit family $\{u_C\}_{\mathcal{C}}$ by η. Also, the counit maps v_D is often denoted by ϵ_D and the counit family $\{v_D\}_{\mathcal{D}}$ by ϵ. Further, the composition $Gv_D \circ u_{GD}$ is denoted by $(G\epsilon \circ \eta G)(D)$ (as if category theory wasn't difficult enough without such notation) and the composition $v_{FC} \circ Fu_C$ by $(\epsilon F \circ F\eta)(C)$. Finally, the unit-counit conditions are written

$$G\epsilon \circ \eta G = 1_{\mathcal{D}} \quad \text{and} \quad \epsilon F \circ F\eta = 1_{\mathcal{C}}$$

and one often sees the diagrams

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G \quad \text{and} \quad F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

Uniqueness of Adjoints

Adjoints are not unique, but they are unique up to natural isomorphism. Here is the formulation for left adjoints.

Theorem 106

The left adjoints of a functor $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ are unique up to natural isomorphism, that is, if $F \dashv G$, then

$$\overline{F} \dashv G \quad \Leftrightarrow \quad \overline{F} \overset{\cdot}{\approx} F$$

Proof

Since $F \dashv G$, we have

$$GFf \circ u_C = u_{C'} \circ f \tag{107}$$

Moreover, $\overline{F} \dashv G$ if and only if

$$G\overline{F}f \circ \overline{u}_C = \overline{u}_{C'} \circ f \tag{108}$$

Assume first that (108) holds. We want to show that there is a natural isomorphism $\{\lambda_C\}\colon F \overset{\cdot}{\approx} \overline{F}$, that is, that

$$\overline{F}f = \lambda_{C'} \circ Ff \circ \lambda_C^{-1} \tag{109}$$

for all $f\colon C \to C'$. But since $(FC, u_C\colon C \to GFC)$ and $\left(\overline{F}C, \overline{u}_C\colon C \to G\overline{F}C\right)$ are universal pairs for (C, G), Theorem 74 implies that there are isomorphisms $\lambda_C\colon FC \approx \overline{F}C$ for which

$$\overline{u}_C = G\lambda_C \circ u_C$$

Hence, the properties of the units imply that

$$\begin{aligned} G\left(\lambda_{C'} \circ Ff \circ \lambda_C^{-1}\right) \circ \overline{u}_C &= G(\lambda_{C'}) \circ G(Ff) \circ G\left(\lambda_C^{-1}\right) \circ G\lambda_C \circ u_C \\ &= G(\lambda_{C'}) \circ G(Ff) \circ u_C \\ &= G(\lambda_{C'}) \circ u_{C'} \circ f \\ &= \overline{u}_{C'} \circ f \\ &= G\overline{F}f \circ \overline{u}_C \end{aligned}$$

and since the map $\xi \mapsto G\xi \circ v_C$ is injective, we get (109).

For the converse, suppose that (109) holds. Then $\lambda_C\colon FC \to \overline{F}C$ for each $C \in \mathcal{C}$. Let

$$\overline{u}_C = G\lambda_C \circ u_C$$

Then

$$
\begin{aligned}
G\overline{F}f \circ \overline{u}_C &= G\left(\lambda_{C'} \circ Ff \circ \lambda_C^{-1}\right) \circ \overline{u}_C \\
&= G(\lambda_{C'}) \circ G(Ff) \circ G\left(\lambda_C^{-1}\right) \circ G\lambda_C \circ \overline{u}_C \\
&= G(\lambda_{C'}) \circ G(Ff) \circ u_C \\
&= G(\lambda_{C'}) \circ u_{C'} \circ f \\
&= \overline{u}_{C'} \circ f
\end{aligned}
$$

which is (108). □

Summary

Let us summarize the definition and characterizations of adjoints.

Theorem 110

Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be antiparallel functors and let

$$\{\tau_{C,D}\} = \{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}\colon \hom_{\mathcal{C}}(C, GD) \leftrightarrow \hom_{\mathcal{C}}(FC, D)$$

be a family of bijections. Let

$$u_C = \tau_{C,FC}^{-1}(1_{FC}) \quad \textit{and} \quad v_D = \tau_{GD,D}(1_{GD})$$

1) *The following are equivalent:*
   a) **(Mediating morphisms)** *The maps $\{\tau_{C,D}\}_{\mathcal{D}}$ are the mediating morphism maps for the universal pairs $(FC, u_C\colon C \to GFC)$, that is, $\tau_{C,D}(f_{C,GD})$ is the unique solution to the equation*

   $$G\tau_{C,D}\left(f_{C,GD}\right) \circ u_C = f_{C,GD}$$

   b) **(Right Fusion formula)** *The maps $\{\tau_{C,D}\}_{\mathcal{D}}$ satisfy*

   $$\tau_{C,D}^{-1}(h_{U,D}) = Gh_{U,D} \circ u_C$$

   c) **(Naturalness in D)** $\{\tau_{C,D}\}_{\mathcal{D}}$ is a natural isomorphism in D, that is,

   $$\tau_{C,D'} \circ \left(Gh_{D,D'}\right)^{\leftarrow} = h_{D,D'}^{\leftarrow} \circ \tau_{C,D}$$

   or equivalently,

   $$\tau_{C,D'}\left(Gh_{D,D'} \circ f_{C,GD}\right) = h_{D,D'} \circ \tau_{C,D}\left(f_{C,GD}\right)$$

2) *The following are equivalent:*
   a) (Comediating morphisms) *The maps* $\{\tau^{-1}_{C,D}\}_C$ *are the comediating morphisms for the couniversal pairs* $(GD, v_D : FGD \to D)$*, that is,* $\tau^{-1}_{C,D}(h_{FC,D})$ *is the unique solution to the equation*

$$v_D \circ F\tau^{-1}_{C,D}(h_{FC,D}) = h_{FC,D}$$

   b) **(Left Fusion formula)** *The maps* $\{\tau^{-1}_{C,D}\}_C$ *satisfy*

$$\tau_{C,D}(f) = v_D \circ Ff$$

   c) **(Naturalness in** $\boldsymbol{D}$**)** $\{\tau^{-1}_{C,D}\}_C$ *is a natural isomorphism in* C*, that is,*

$$g^{\rightarrow}_{C,C'} \circ \tau^{-1}_{C,D} = \tau^{-1}_{C',D} \circ \left(Fg_{C,C'}\right)^{\rightarrow}$$

   or equivalently,

$$\tau_{C',D}\Big(f_{C,GD} \circ g_{C',C}\Big) = \tau_{C,D}\big(f_{C,GD}\big) \circ Fg_{C',C}$$

3) *If one (and therefore all) conditions hold from each of section 1) and section 2), then the family* $\{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}$ *is binatural and* F *is a left adjoint of* G *and* G *is a right adjoint of* F*.*
4) **(Unit-counit structure)** *The family* $\{\tau_{C,D}\}_{\mathcal{C},\mathcal{D}}$ *is binatural if and only if the 4-tuple*

$$\mathcal{N} = \left(F_{\mathcal{C}\Rightarrow\mathcal{D}}, G_{\mathcal{D}\Rightarrow\mathcal{C}}, \{u_C\}_{\mathcal{C}}, \{v_D\}_{\mathcal{D}}\right)$$

   is a unit-counit structure, that is, if and only if
   a) $u_C \colon I_{\mathcal{C}} \dot{\to} GF$*, that is,*

$$GFf_{C,C'} \circ u_C = u_{C'} \circ f_{C,C'}$$

   b) $v_D \colon FG \dot{\to} I_{\mathcal{D}}$*, that is,*

$$v_D \circ FGf_{D',D} = f_{D',D} \circ v_{D'}$$

   c) *The basic fusion formulas hold,*

$$Gv_D \circ u_C = 1_{GD} \quad \textit{and} \quad v_{FC} \circ Fu_C = 1_{FC}$$

   d) *Also,*

$$\tau_{C,D}\big(f_{C,GD}\big) = v_D \circ Ff_{C,GD}$$

Examples of Adjoints

Now it is time to consider some examples. Speaking in general terms, suppose that $G: \mathcal{D} \Rightarrow \mathcal{C}$ is a functor and that for every $C \in \mathcal{C}$, we have identified a universal pair

$$(U, u_C: C \to GU)$$

along with the mediating morphism map $\tau_{C,D}$. One possible strategy for finding a left adjoint F of G is as follows.

First, we set $FC = U$ for all $C \in \mathcal{C}$. As to the arrow part of F, equation (102)

$$\tau^{-1}_{C,FC'}(Ff_{C,C'}) = u_{C'} \circ f_{C,C'}$$

can be written as

$$Ff_{C,C'} = \tau_{C,FC'}(u_{C'} \circ f_{C,C'})$$

and this completes the definition of the functor F, but it remains to show that $F \dashv G$. This is where Theorem 110 provides several alternatives (that is, pick one from section 1 and one from section 2).

Example 111 (Free Groups)

First a few observations about free groups. If F_X is the free group on a nonempty set X, then a set function $f: X \to A$ from X to a group A induces a group homomorphism $\overline{f}: F_X \to A$ defined by

$$\overline{f}(x_1^{e_1} \cdots x_n^{e_n}) = (fx_1)^{e_1} \cdots (fx_n)^{e_n}$$

where $x_i \in X$. More generally, a set function $f: X \to Y$ induces a group homomorphism $\overline{f}: F_X \to F_Y$ between free groups.

Now let **Set*** be the category of nonempty sets and let G: **Grp** $\Rightarrow$ **Set*** be the underlying-set functor, which forgets all algebraic structure and sends group homomorphisms to set functions. We have seen that the family

$$\{(F_X, u_X: X \to GF_X)\}_{X \in \mathbf{Set}}$$

where u_X is set inclusion is universal. Now, the mediating morphisms

$$\tau_{X,A}: \hom_{\mathbf{Set}^*}(X, GA) \to \hom_{\mathbf{Grp}}(F_X, A)$$

are given by the equation

$$G(\tau_{X,A}\, f) \circ u_X = f$$

for all set functions $f\colon X \to GA$ and so if $x \in X$, then

$$\tau_{X,A}\, f(x) = f(x)$$

which means that $\tau_{X,A}(f)\colon F_X \to A$ is the group homomorphism that agrees with f on X, that is,

$$\tau_{X,A}(f) = \overline{f}$$

A candidate for left adjoint $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is the functor with object part $FX = F_X$ and arrow part satisfying

5

$$Ff_{X,Y} = \tau_{X,A}\left(u_Y \circ f_{X,Y}\right) = \overline{u_Y \circ f_{X,Y}} = \overline{f_{X,Y}}$$

To verify that $F \dashv G$, we check the binaturalness of $\tau_{X,A}$. The naturalness of $\tau_{X,A}$ in X is

$$\tau_{X,A}(g \circ f) = \tau_{X',A}(g) \circ Ff$$

that is,

$$\overline{g \circ f} = \overline{g} \circ \overline{f}$$

for all $f\colon X \to X'$ and $g\colon X' \to GA$, which is true. Naturalness in A is

$$\tau_{X,A'}(Gg \circ f) = g \circ \tau_{X,A}(f)$$

that is,

$$\overline{g \circ f} = g \circ \overline{f}$$

for all $g\colon A \to A'$ and $f\colon X \to GA$, which holds since g is a group homomorphism. Thus, the free group functor is the left adjoint the forgetful functor. □

■ **Example 112**

Let $U\colon \mathbf{Vect}_k \Rightarrow \mathbf{Set}^*$ be the underlying-set functor. In a manner similar to that of Example 111, one can show that the functor that sends each nonempty set X to the vector space k_X over k with basis X and sends each set function $f\colon X \to Y$ to the unique linear extension $\tau\colon k_X \to k_Y$ is a left adjoint for U. □

■ **Example 113**

Let X be a set. The set of all functions from X to $\mathbb{Z}$ is denoted by $\mathbb{Z}^X$. The *support* of a function $\alpha\colon X \to \mathbb{Z}$ is the set

$$\text{supp}(g) = \left\{x \in X \middle| (x) \neq 0\right\}$$

Let $\mathbb{Z}_0^X$ denote the set of all functions in $\mathbb{Z}^X$ with finite support. For example, for any $x_0 \in X$, the *indicator function* $\epsilon_{x_0}\colon X \to \mathbb{Z}$ defined by

$$\epsilon_{x_0}(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

has finite support. Note that any$\in \mathbb{Z}_0^X$ has a unique expression as a finite sum

$$\alpha = \sum_{x \in \text{supp}(\alpha)} \alpha(x)\epsilon_x$$

Now let G: **AbGrp** $\Rightarrow$ **Set** be the underlying-set functor. For each nonempty set X, the pair

$$(\mathbb{Z}_0^X, u_X\colon X \to \mathbb{Z}_0^X)$$

is universal, where $u_X(x) = \epsilon_x$. To see this, if f: $X \to GA$, where A is an abelian group, then the mediating morphism τ_f (if it exists) is the unique map for which

$$\tau_f \circ u_X = f$$

Applying this to $x \in X$ gives

$$\tau_f(\epsilon_x) = f(x)$$

and so τ_f exists and is uniquely defined by

$$\tau_f(\alpha) = \sum_{x \in \text{supp}(\alpha)} \alpha(x) f(x)$$

for any $\alpha \in \mathbb{Z}_0^X$. Thus,

$$\tau_{X,A}(f)(\alpha) = \sum_{x \in \text{supp}(\alpha)} \alpha(x) f(x)$$

If F: **Set** $\Rightarrow$ **AbGrp** is a right adjoint for G, then

$$FX = \mathbb{Z}_0^X$$

and

$$Ff_{X,X'} = \tau_{X,A}(u_{X'} \circ f_{X,X'})$$

Applying this to $\alpha \in \mathbb{Z}_0^X$ gives

$$F(f)(\alpha) = \sum_{x \in \text{supp}(\alpha)} \alpha(x)(u_{X'} \circ f)(x) = \sum_{x \in \text{supp}(\alpha)} \alpha(x)\epsilon_{f(x)}$$

for f: $X \to X'$.

Again we check binaturalness. The naturalness of $\tau_{X,A}$ in X is

$$\tau_{X,A}(g \circ f) = \tau_{X',A}(g) \circ Ff$$

for all $f\colon X \to X'$ and $g\colon X' \to GA$. To see if this holds, note that for $\alpha \in \mathbb{Z}_0^X$, the left side is

$$\tau_{X,A}(g \circ f)(\alpha) = \sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)(g \circ f)(x)$$

and the right side is

$$\tau_{X',A}(g)\left(\sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)\epsilon_{f(x)}\right) = \sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)\tau_{X',A}(g)\big(\epsilon_{f(x)}\big) = \sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)(g(f(x)))$$

and so the naturalness in X condition is met.

Naturalness in A is

$$\tau_{X,A'}(g \circ f) = g \circ \tau_{X,A}(f)$$

for all $g\colon A \to A'$ and $f\colon X \to GA$. Applying this to $\alpha \in \mathbb{Z}_0^X$ gives

$$\tau_{X,A'}(g \circ f)(\alpha) = \sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)(g \circ f)(x)$$

and

$$g \circ \tau_{X,A}(f)(\alpha) = g\left(\sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)f(x)\right) = \sum_{x \in \mathrm{supp}(\alpha)} \alpha(x)(g \circ f)(x)$$

and so the naturalness in A condition is met. Thus F is a right adjoint of G. □

Example 114

Let $\mathcal{C}$ be a category with finite products. For a product

$$\mathcal{P} = (C \times D, \rho_1\colon C \times D \to C, \rho_2\colon C \times D \to D)$$

we denote the unique mediating morphism from a cone

$$(X, f\colon X \to C, g\colon X \to D)$$

to $\mathcal{P}$ by the 2×1 matrix

$$\begin{pmatrix} f \\ g \end{pmatrix}\colon X \to C \times D$$

Thus, $\begin{pmatrix} f \\ g \end{pmatrix}$ is uniquely determined by the conditions

$$\rho_1 \circ \begin{pmatrix} f \\ g \end{pmatrix} = f \quad \text{and} \quad \rho_2 \circ \begin{pmatrix} f \\ g \end{pmatrix} = g$$

Recall that if $f\colon D \to D'$ and $g\colon E \to E'$ are morphisms in $\mathcal{C}$, then their product is defined to be the unique mediating morphism for the composite maps

$$f \times g = \begin{pmatrix} f \circ \rho_1^{D\times E} \\ g \circ \rho_2^{D\times E} \end{pmatrix} \colon D \times E \to D' \times E'$$

Hence, $f \times g$ is the unique map for which

$$\rho_i^{D'\times E'} \circ (f \times g) = f \circ \rho_i^{D\times E}$$

for $i = 1, 2$.

Now, the whole story of this example is shown in the rather ghastly diagram in Figure 65.

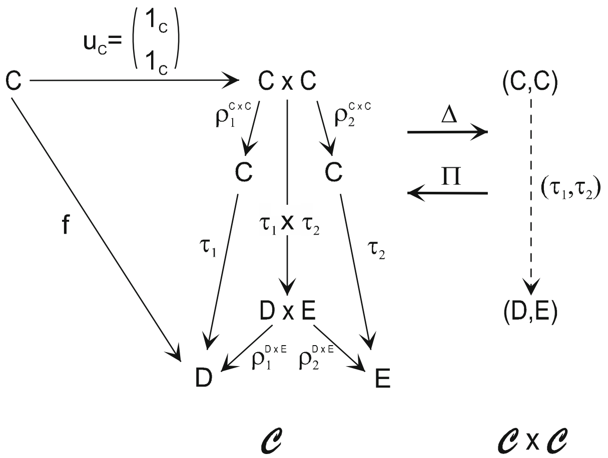

Figure 65

Let $\mathcal{C} \times \mathcal{C}$ be the product category. Define the **diagonal functor** $\Delta\colon \mathcal{C} \Rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\Delta C = (C, C) \quad \text{and} \quad \Delta f = (f, f)$$

where $(f, f)\colon (C, C) \to (D, D)$ acts coordinatewise. In the other direction, define the **product functor** $\Pi\colon \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$ by

$$\prod(D, E) = D \times E \quad \text{and} \quad \prod(f, g) = f \times g = \begin{pmatrix} f \circ \rho_1^{D\times E} \\ g \circ \rho_2^{D\times E} \end{pmatrix}$$

To see that Δ is a left adjoint of Π, we first show that the pair

$$U = \left((C,C), \left\{u_C = \begin{pmatrix} 1_C \\ 1_C \end{pmatrix} : C \to C \times C\right\}\right)$$

is universal. Note first that u_C is the unique mediating morphism satisfying

$$\rho_1^{C\times C} \circ u_C = 1_C \quad \text{and} \quad \rho_2^{C\times C} \circ u_C = 1_C$$

The universal condition is that for all f: $C \to D \times E$, there is a unique morphism (τ_1, τ_2): $(C, C) \to (D, E)$ for which

$$\begin{pmatrix} \tau_1 \circ \rho_1^{C\times C} \\ \tau_2 \circ \rho_2^{C\times C} \end{pmatrix} \circ \begin{pmatrix} 1_C \\ 1_C \end{pmatrix} = f$$

Applying $\rho_i^{D\times E}$ for $i = 1, 2$ gives

$$\tau_i \circ \rho_i^{C\times C} \circ \begin{pmatrix} 1_C \\ 1_C \end{pmatrix} = \rho_i^{D\times E} \circ f$$

that is,

$$\tau_i = \rho_i^{D\times E} \circ f$$

which uniquely defines τ_i. Hence, U is universal and

$$\tau_{C,\,(D,E)}\left(f_{C,D\times E}\right) = \left(\rho_1^{D\times E} \circ f_{C,D\times E}, \rho_2^{D\times E} \circ f_{C,D\times E}\right)$$

The naturalness of $\tau_{C,\,(D,E)}$ in C is

$$\tau_{C',\,(D,E)}\left(f_{C,D\times E} \circ g_{C',C}\right) = \tau_{C,(D,E)}\left(f_{C,D\times E}\right) \circ \left(g_{C',C}, g_{C',C}\right)$$

But

$$\begin{aligned} \tau_{C',(D,E)}\left(f_{C,D\times E} \circ g_{C',C}\right) &= \left(\rho_1^{D\times E} \circ f_{C,D\times E} \circ g_{C',C}, \rho_2^{D\times E} \circ f_{C,D\times E} \circ g_{C',C}\right) \\ &= \left(\rho_1^{D\times E} \circ f_{C,D\times E}, \rho_2^{D\times E} \circ f_{C,D\times E}\right) \circ \left(g_{C',C}, g_{C',C}\right) \\ &= \tau_{C,(D,E)}\left(f_{C,D\times E}\right) \circ \left(g_{C',C}, g_{C',C}\right) \end{aligned}$$

and so this holds.

The naturalness in D is

$$\tau_{C,\,(D',E')}\left(\left[h_{D,D'} \times k_{E,E'}\right] \circ f_{C,D\times E}\right) = \left(h_{D,D'}, k_{E,E'}\right) \circ \tau_{C,(D,E)}\left(f_{C,D\times E}\right)$$

But,

$$\begin{aligned}
&\tau_{C,(D',E')}\left(\left(h_{D,D'} \times k_{E,E'}\right) \circ f_{C,D\times E}\right) \\
&\qquad = \left(\rho_1^{D'\times E'} \circ \left(h_{D,D'} \times k_{E,E'}\right) \circ f_{C,D\times E}, \rho_2^{D'\times E'} \circ \left(h_{D,D'} \times k_{E,E'}\right) \circ f_{C,D\times E}\right) \\
&\qquad = \left(h_{D,D'} \circ \rho_1^{D\times E} \circ f_{C,D\times E}, k_{E,E'} \circ \rho_2^{D\times E} \circ f_{C,D\times E}\right) \\
&\qquad = \left(h_{D,D'} \times k_{E,E'}\right) \circ \left(\rho_1^{D\times E} \circ f_{C,D\times E}, \rho_2^{D\times E} \circ f_{C,D\times E}\right)
\end{aligned}$$

Thus, both naturalness conditions hold and $\Delta \dashv \Pi$. □

Adjoints and the Preservation of Limits

As shown in Figure 66, a functor $G: \mathcal{D} \Rightarrow \mathcal{C}$ **preserves limits** if for any limit

$$\mathcal{K} = \left(V, \left\{f_n: V \to A_n \middle| n \in \mathcal{J}\right\}\right)$$

over a diagram $\mathbb{D}$ in $\mathcal{D}$, the image cone

$$F\mathcal{K} = \left(FV, \left\{Ff_n: FV \to FA_n \middle| n \in \mathcal{J}\right\}\right)$$

is a limit over the diagram $G\mathbb{D}$ in $\mathcal{C}$.

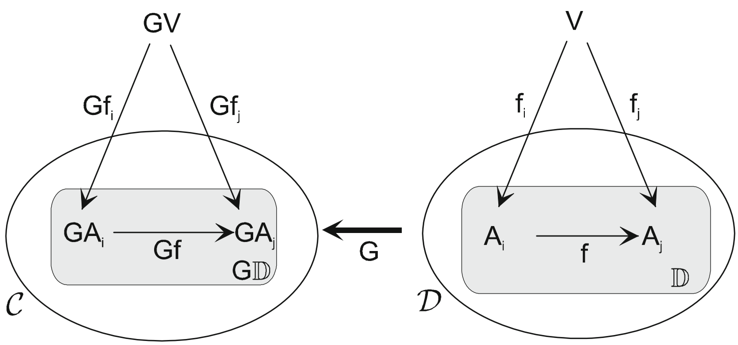

Figure 66

In an earlier exercise, we asked the reader to show that for any $A \in \mathcal{C}$, the hom functor $\hom_{\mathcal{C}}(A, \cdot)$ preserves limits.

■ **Theorem 115**

If a functor $G: \mathcal{D} \Rightarrow \mathcal{C}$ has a left adjoint $F: \mathcal{C} \Rightarrow \mathcal{D}$, then G preserves limits.

▪ Proof

With reference to Figure 67, let

$$\mathcal{L} = \big(V, \{f_D: V \to D \mid D \in J(\mathcal{J})\}\big)$$

be a limit of the diagram $\mathbb{D}(J: \mathcal{J} \Rightarrow \mathcal{C})$ in $\mathcal{D}$ and let

$$G\mathcal{L} = \big(GV, \{Gf_D: GV \to GD \mid D \in J(\mathcal{J})\}\big)$$

be the image cone in $\mathcal{C}$.

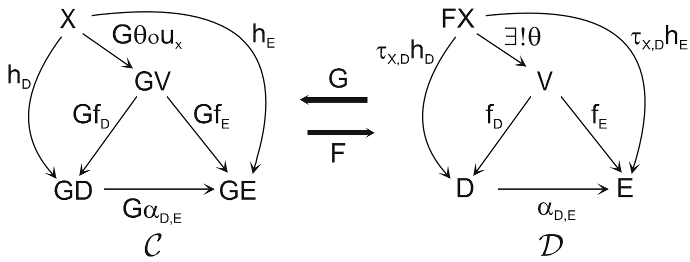

Figure 67

To see that $G\mathcal{L}$ is a limit of $G\mathbb{D}$, let

$$\mathcal{X} = \big(X, \{h_D: X \to GD \mid D \in J(\mathcal{J})\}\big)$$

be a cone over $G\mathbb{D}$ in $\mathcal{C}$. Taking the image of $\mathcal{X}$ under the bijective mediating-morphism maps

$$\tau_{X,D}: \hom_{\mathcal{C}}(X, GD) \leftrightarrow \hom_{\mathcal{D}}(FX, D)$$

gives

$$\tau_{X,D}\mathcal{X} = \big(FX, \{\tau_{X,D}h_D: FX \to D \mid D \in J(\mathcal{J})\}\big)$$

To see that $\tau\mathcal{X}$ is a cone over $\mathbb{D}$, we must show that

$$\tau_{X,D}h_E = \alpha_{D,E} \circ \tau_{X,D}h_D \tag{116}$$

But the naturalness of $\tau_{X,D}$ (see Theorem 64) implies that for each $\alpha: D \to E$ and $\beta: X \to GD$,

$$\tau^{-1}_{X,E}(\alpha_{D,E} \circ \tau_{X,D}(\beta)) = G\alpha_{D,E} \circ h_D = h_E$$

and so (116) follows.

Since $\tau\mathcal{X}$ is a cone, there is a unique mediating morphism $\theta: FX \to V$ for which

$$f_D \circ \theta = \tau_{X,D}h_D$$

Moreover, the map

$$\tau^{-1}_{X,D}\theta = G\theta \circ u_X$$

is a cone morphism from $\mathcal{X}$ to $G\mathcal{L}$, since

$$Gf_D \circ (G\theta \circ u_X) = G(f_D \circ \theta) \circ u_X = G(\tau_{X,D}h_D) \circ u_X = h_D$$

It remains to show uniqueness. If $\lambda\colon X \to GV$ satisfies

$$Gf_D \circ \lambda = h_D$$

then

$$\tau_{X,D}h_D = \tau_{X,D}(Gf_D \circ \lambda) = f_D \circ \tau_{X,D}\lambda$$

since

$$G(f_D \circ \tau_{X,D}\lambda) \circ u_X = Gf_D \circ G(\tau_{X,D}\lambda) \circ u_X = Gf_D \circ \lambda$$

and so the uniqueness of θ implies that $\lambda = \tau^{-1}_{X,D}\theta$. □

The Existence of Adjoints

We now come to one of the most important theorems in category theory, called *the adjoint functor theorem*, which characterizes the existence of adjoints. Let us state the theorem now. The proof will follow shortly.

Theorem 117 (Adjoint functor theorem)
Let $\mathcal{D}$ be a complete category. A functor $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ has a left adjoint $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ if and only if the following hold:
1) *G preserves limits*
2) *G satisfies the solution set condition for all $C \in \mathcal{C}$.* □

The Solution Set Condition

To understand the solution set condition, let us start from the bottom and work up. Let $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$. We refer to a map of the form $f\colon C \to GD$, for $D \in \mathcal{D}$ as a **source morphism**. For any source morphism f, we refer to a factorization of the form

$$G\tau \circ u = f \tag{118}$$

where

$$u: C \to GS \quad \text{and} \quad \tau: S \to D$$

as an **(S, u)-factorization** for f with mediating morphism τ. The pair

$$\mathcal{U} = (S, u: C \to GS) \tag{119}$$

for which there is an (S, u)-factorization of f is known as a **solution** for f (although it may seem more reasonable to refer to the mediating morphism τ as a *solution* for the pair (S, u)). We call the object S a **solution object** and the map u in a solution (S, u) a **solution map**.

Thus, a pair $(S, u: C \to GS)$ is universal if and only if *every* source morphism has a *unique* (S, u)-factorization. We can weaken the universality condition by removing the requirement that mediating morphisms must be unique. Let us say that a pair $(S, u: C \to GS)$ is **quasiuniversal** (a nonstandard term) if every source morphism has a (perhaps nonunique) (S, u)-factorization. Put another way, a pair $(S, u: C \to GS)$ is quasiuniversal if it is a solution for all source morphisms.

Actually, *any* morphism $f: C \to GD$ has at least one (S, u)-factorization, namely,

$$G1_D \circ f = f$$

that is, any morphism $f: C \to GD$ has at least one solution

$$(D, f: C \to GD) \tag{120}$$

So it might seem now that this definition is useless, but stay tuned.

A **solution class** S_C for $G: \mathcal{D} \Rightarrow \mathcal{C}$ and $C \in \mathcal{C}$ is any class that contains a solution (S, u) for *every* possible source morphism $f: C \to GD$ and every $D \in \mathcal{D}$. Any class that contains the pairs (120) is a solution class. However, in general, solution classes are *proper* classes, that is, they are not sets. We can now define the solution *set* condition.

Definition

A functor $G: \mathcal{D} \Rightarrow \mathcal{C}$ satisfies the **solution set condition** *for a given $C \in \mathcal{C}$ if there is a solution class that is actually a* set. □

In more colloquial terms, $G: \mathcal{D} \Rightarrow \mathcal{C}$ satisfies the solution set condition for a given $C \in \mathcal{C}$ if there is a set worth of pairs $(S, u: C \to GS)$ that provide a solution for all source morphisms.

Note that a solution $\mathcal{U}$ is quasiuniversal if and only if the set $\{\mathcal{U}\}$ is a solution set and so if a quasiuniversal pair exists for (C, G), then G satisfies the solution set condition for C.

One half of the adjoint functor theorem now follows quite easily. For if $\mathcal{U} = (U, u: C \to GU)$ is a universal pair for (C, G), then G has the solution set condition for C. Therefore, if G has a left adjoint, then G satisfies the solution set condition for all $C \in \mathcal{C}$. Since we have already proven that G also preserves limits, we have proved one half of the adjoint functor theorem. To prove the converse, we must take a closer look at factorizations and solutions.

A Closer Look at Factorizations and Solutions

Solutions have a form of "absorption property" described as follows. Suppose that a source morphism $f\colon C \to GS$ has a solution (S, u) with factorization

$$G\tau \circ u = f$$

If (T, v) is a solution for u with factorization

$$G\sigma \circ v = u$$

then

$$G(\tau \circ \sigma) \circ v = G\tau \circ G\sigma \circ v = G\tau \circ u = f$$

and so (T, v) is also a solution for f. Therefore, if (T, v) is a common solution for all solution maps in a solution set $\mathcal{S}_C$, then (T, v) is a solution for every source morphism. Let us record these facts in a theorem.

Theorem 121

Let $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$.

1) *If (S, u) is a solution for $f\colon C \to GD$ then any solution (T, v) for the solution map u is also a solution for f.*
2) *If $\mathcal{S}_C$ is a solution set for C and if (T, v) is a common solution for every solution map u in $\mathcal{S}_C$, then (T, v) is a quasiuniversal pair for (C, G).*
3) *If (S, u) is a quasiuniversal pair for (C, G) and if (T, v) is a solution for u, then (T, v) is also a quasiuniversal pair for (C, G).* □

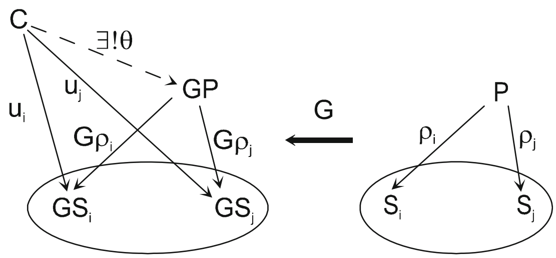

Figure 68

Theorem 122

As shown in Figure 68, let $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $C \in \mathcal{C}$ have solution set $\mathcal{S}_C$. Assume that the product

$$\mathcal{P} = \left(P, \left\{\rho_i\colon P \to S_i \;\middle|\; (S_i, u_i) \in \mathcal{S}_C\right\}\right)$$

of the solution objects S_i for $(S_i, u_i) \in \mathcal{S}_C$ exists and that G preserves this product, that is, that

$$G\mathcal{P} = \left(GP, \left\{G\rho_i\colon GP \to GS_i \;\middle|\; (S_i, u_i) \in \mathcal{S}_C\right\}\right)$$

is a product of the objects GS_i for $(S_i, u_i) \in \mathcal{S}_C$. If $\theta\colon C \to GP$ is the mediating morphism for the cone

$$\mathcal{K} = \left(C, \left\{u_i\colon C \to GS_i \;\middle|\; (S_i, u_i) \in \mathcal{S}_C\right\}\right)$$

with vertex C and whose legs are the solution maps u_i, then the pair (P, θ) is quasiuniversal for (C, G).

5

Proof

Since

$$G\rho_i \circ \theta = u_i$$

for all i, θ is a common solution map for all of the solution maps u_i in the solution set $\mathcal{S}_C$ and so the pair (P, θ) is quasiuniversal for (C, G). □

The Adjoint Functor Theorem

We can now prove the adjoint functor theorem.

Theorem 123 (Adjoint functor theorem–Freyd 1964)

Let $\mathcal{D}$ be a complete category. A functor $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ has a left adjoint $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ if and only if the following hold:

1) *G preserves limits*
2) *G satisfies the solution set condition for all $C \in \mathcal{C}$.*

Proof

One direction has been proved. Assume that 1) and 2) hold. Fix $C \in \mathcal{C}$. Since C has a solution set, Theorem 122 implies that there is a *quasiuniversal* pair (S, u) for (C, G). The story of the proof that there is a *universal* pair for (C, G) is told in Figure 69.

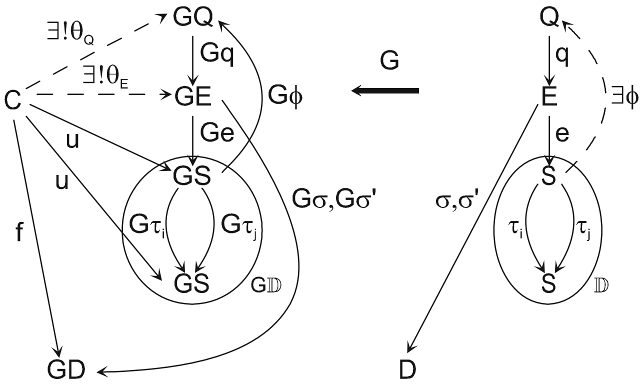

Figure 69

By way of motivation, since (S, u) is quasiuniversal and $u: C \to GS$, there is a mediating morphism $\tau: S \to S$ for u itself, that is,

$$G\tau \circ u = u$$

Of course, the identity is one such mediating morphism, but there may be others. Suppose that τ is one such nonidentity mediating morphism. If $f: C \to GD$ is a source morphism, with mediating morphism σ, that is, if

$$G\sigma \circ u = f$$

then Theorem 121 implies that $\sigma \circ \tau$ is also a mediating morphism for f. Therefore, it is unlikely that (S, u) will be universal.

This gives us the idea of trying to find a quasiuniversal pair (E, θ) for which θ has only one mediating morphism (which will be the identity). One way to achieve this goal is to equalize all of the mediating morphisms for u with respect to (S, u).

So let

$$\mathcal{T} = \{\tau: S \to S \mid G\tau \circ u = u\}$$

The right side of Figure 69 shows two such maps τ_i and τ_j. Let

$$\mathcal{E} = (E, e: E \to S)$$

be the equalizer of the set $\mathcal{T}$. On the left side of the figure, we find the G-image of this structure, along with the cone

$$\mathcal{K} = (C, \{u: C \to GS\})$$

over $G\mathbb{D}$. The cone condition for $\mathcal{K}$ is that

$$G\tau_i \circ u = u$$

which is precisely the condition that $\tau_i \in \mathcal{T}$. Now, since $G\mathcal{E}$ is an equalizer of $G\mathcal{T}$, there is a unique mediating morphism $\theta_E: C \to GE$ for which

$$Ge \circ \theta_E = u \tag{124}$$

We wish to show that (E, θ_E) is a universal pair for (C, G).

Equation (124) says that (E, θ_E) is a solution for u and so Theorem 121 implies that (E, θ_E) is a quasiuniversal pair for (C, G). It remains to establish uniqueness of mediating morphisms. So let $f\colon C \to GD$ and suppose that

$$G\sigma \circ \theta_E = f = G\sigma' \circ \theta_E$$

for $\sigma, \sigma'\colon E \to D$. Let $(Q, q\colon Q \to E)$ be the equalizer of σ and σ'. Then $(GQ, Gq\colon GQ \to GE)$ is the equalizer of $G\sigma$ and $G\sigma'$ and so there exists a unique $\theta_Q\colon C \to GE$ for which

$$Gq \circ \theta_Q = \theta_E$$

Since $\theta_Q : C \to GE$, there is a $\phi\colon S \to Q$ for which

$$G\phi \circ u = \theta_Q$$

Now, if

$$\sigma = e \circ q \circ \phi$$

then

$$\begin{aligned} G\sigma \circ u &= Ge \circ Gq \circ G\phi \circ u \\ &= Ge \circ Gq \circ \theta_Q \\ &= Ge \circ \theta_E \\ &= u \end{aligned}$$

and so $\sigma \in \mathcal{T}$. It follows that $\sigma \circ e = \tau \circ e$ for all $\tau \in \mathcal{T}$, including the identity $1_S \in \mathcal{T}$. Hence,

$$e = \sigma \circ e = e \circ q \circ \phi \circ e$$

Since e is monic, this gives

$$q \circ \phi \circ e = 1_S$$

which shows that the monic q is also right-invertible and is therefore an isomorphism. Thus, $\sigma \circ q = \sigma' \circ q$ implies that $\sigma = \sigma'$. This establishes uniqueness and shows that (E, θ_E) is universal. Hence, G has a left adjoint. □

■ Example 125

The adjoint functor theorem can be used to prove that the forgetful functor U: **Grp** $\Rightarrow$ **Set** has a left adjoint F. This means that free groups exist for every generating set X!

To see this, suppose that $X \in$ **Set**. We must show that there is a set worth of pairs $(H, u\colon X \to H)$ that provide solutions for every set function $f\colon X \to A$, where A is a group. Let B be the subgroup of A generated by the elements of $f\,X$. The generating set $W = \{fx_i\} \cup \{(fx_i)^{-1}\}$

has the same or smaller cardinality as X and so the cardinality of B is bounded by the cardinality of X, say $|B| \leq \alpha(|X|)$.

Now, consider a particular cardinality $k \leq \alpha(|X|)$. The family of all sets of cardinality k is not a set. However, we can construct all groups of cardinality k up to isomorphism by taking a single set S of cardinality k and scanning all of the k functions $f: S \times S \to S$, picking out the legal group operations. This shows that a family consisting of one group from each isomorphism class from among the groups of cardinality $\max\{\kappa, \aleph_0\}$ is a set. Indeed, a family $\mathcal{F}$ consisting of one group from each isomorphism class from among the groups of cardinality $\max\{k, \alpha(|X|)\}$ is a set.

Finally, the family of all pairs $(H, u: X \to H)$ for $H \in \mathcal{F}$ and all maps u is a solution set for X. Hence, the solution set condition holds for every $X \in$ **Set** and so the forgetful functor U: **Grp** $\Rightarrow$ **Set** has a left adjoint. This is a proof that free groups exist. □

Exercises

1. Let $F: \mathcal{C} \Rightarrow \mathcal{D}$ and $G: \mathcal{D} \Rightarrow \mathcal{C}$ satisfy $F \dashv G$. Let $F': \mathcal{E} \Rightarrow \mathcal{C}$ and $G': \mathcal{C} \Rightarrow \mathcal{E}$ satisfy $F' \dashv G'$. Prove that $F \circ F' \dashv G' \circ G$.
2. Let **Set**$_*$ be the category of pointed sets and pointed functions. Let U: **Set**$_*$ $\Rightarrow$ **Set** be the underlying set functor. Find a left adjoint for U.
3. Find a left adjoint of the underlying-set functor U: **Rng** $\Rightarrow$ **Set**.
4. Let **1** be the category with a single object 0 and a single morphism 1_0. Let $\mathcal{C}$ be a category with initial objects. Let $G: \mathcal{C} \to \mathbf{1}$ be the constant functor. Find the left adjoints of G and their mediating morphism maps τ.
5. Let I: **AbGrp** $\Rightarrow$ **Grp** be the inclusion (forgetful) functor that forgets the commutativity of an abelian group. Show that the functor that takes a group C to the quotient group $C/[C, C]$, where $[C, C]$ is the derived (commutator) subgroup of C is a left adjoint of U. What is the unit?
6. Let I: **Tor** $\Rightarrow$ **AbGrp**, where **Tor** is the category of all torsion free groups (groups in which all elements have finite order) be the inclusion (forgetful) functor. Show that I has a right adjoint.
7. Let U: **Mod**$_R$ $\Rightarrow$ **AbGrp** be the forgetful functor that forgets the scalar multiplication. Show that the functor that maps an abelian group A to the tensor product $R \otimes \mathbb{Z}A$ is a left adjoint of U. What is the unit?
8. A group G is **Boolean** if every nonidentity element of G has order 2. Let U: **BoolGrp** $\Rightarrow$ **Grp** be the forgetful functor. Use the adjoint functor theorem to show that U has a left adjoint.
9. Let **Idem** be the category whose objects are ordered pairs (X, v), where $v: X \to X$ is an idempotent unary operation on X, that is, $v^2 = v$. The morphisms $f: (X, v) \to (Y, u)$ are the set functions f for which $f \circ v = u \circ f$. Let I: **Set** $\Rightarrow$ **Idem** be the functor for which

$$IX = (X, 1) \quad \text{and} \quad If = f$$

for $f: X \to Y$. Let F: **Idem** $\Rightarrow$ **Set** be the functor for which

$$F(X, v) = \mathrm{fix}(X, v) := \{x \in X \mid vx = x\}$$

and

$$Ff = f\,\big|_{\,\mathrm{fix}(X,v)}$$

for $f\colon (X, v) \to (Y, u)$. Verify that I and F are functors. Show that F is both a left and right adjoint for I.

10. Let $F\colon \mathcal{D} \Rightarrow \mathcal{C}$ and let $h\colon C \to C'$ and $k\colon C' \to C''$ be morphisms in $\mathcal{C}$. Suppose that

$$(S_C, u_C\colon C \to FS_C), (S_{C'}, u_{C'}\colon C' \to FS_{C'}), (S_{C''}, u_{C''}\colon C'' \to FS_{C''})$$

are universal pairs. Prove that

$$\tau_{u_{C''}} \circ k \circ h = \tau_{u_{C''}} \circ k \circ \tau_{u_{C'}} \circ h$$

11. (**Galois connections**) Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be preorders, which we can think of either as categories or as preordered sets.
   a) Show that an order-preserving function $F\colon P \to Q$ can be viewed as a functor from $\mathcal{P}$ to $\mathcal{Q}$ and conversely. Show that an order-reversing function $F\colon P \to Q$ can be viewed as a functor from $\mathcal{P}$ to $\mathcal{Q}^{\mathrm{op}}$ and conversely.
   b) A **Galois connection** from $\mathcal{P}$ to $\mathcal{Q}$ is a pair

$$(F\colon P \to Q, G\colon Q \to P)$$

   of antiparallel order-reversing maps for which

$$p \leq GFp \quad \text{and} \quad q \leq FGq \tag{126}$$

   for all $p \in P$ and $q \in Q$. Prove that for order-reversing maps F and G property (0) is equivalent to the statement that

$$p \leq Gq \quad \text{iff} \quad q \leq Fp$$

   c) If (F, G) is a Galois connection from $\mathcal{P}$ to $\mathcal{Q}$, show that $F\colon \mathcal{P} \Rightarrow \mathcal{Q}^{\mathrm{op}}$ and $G\colon \mathcal{Q}^{\mathrm{op}} \Rightarrow \mathcal{P}$ are adjoint functors.
   d) Show that if $F\colon \mathcal{P} \Rightarrow \mathcal{Q}^{\mathrm{op}}$ and $G\colon \mathcal{Q}^{\mathrm{op}} \Rightarrow \mathcal{P}$ are adjoint functors then (F, G) is a Galois connection from $\mathcal{P}$ to $\mathcal{Q}$.

12. (**The category of inverse adjunctions**) Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and let

$$F\colon \mathcal{C} \Rightarrow \mathcal{D}, G\colon \mathcal{D} \Rightarrow \mathcal{C}$$

and

$$F'\colon \mathcal{D} \Rightarrow \mathcal{E}, G'\colon \mathcal{E} \Rightarrow \mathcal{D}$$

be pairs of antiparallel functors. Let

$$\{\tau_{C,D}\colon \hom_{\mathcal{C}}(C, GD) \approx \hom_{\mathcal{D}}(FC, D)\}$$

and

$$\{\mu_{D,E}\colon \hom_{\mathcal{D}}(D, G'E) \dot{\approx} \hom_{\mathcal{E}}(F'D, E)\}$$

be inverse adjunctions (that is, inverses of adjunctions) from $\mathcal{C}$ to $\mathcal{D}$ and $\mathcal{D}$ to $\mathcal{E}$, respectively.

a) Show that we may compose the inverse adjunctions τ and μ to get an inverse adjunction

$$\{\lambda_{C,E}\colon \hom_{\mathcal{C}}(C, GG'E) \dot{\approx} \hom_{\mathcal{D}}(F', FC, E)\}$$

from $\mathcal{C}$ to $\mathcal{E}$ for the antiparallel functors $F'F\colon \mathcal{C} \Rightarrow \mathcal{E}$ and $GG'\colon \mathcal{E} \Rightarrow \mathcal{C}$.

b) Show that the unit $u_{\lambda,C}$ of λ is given by

$$u_{\lambda,C} = Gu_{\mu,FC} \circ u_{\tau,C}$$

c) Show that we can form a category **Adj** whose objects are categories and for which the morphisms in $\hom(\mathcal{C}, \mathcal{D})$ are the triples

$$(F\colon \mathcal{C} \Rightarrow \mathcal{D}, G\colon \mathcal{D} \Rightarrow \mathcal{C}, \{\lambda_{C,D}\})$$

where $\{\lambda_{C,D}\}$ is an inverse adjunction from $\mathcal{C}$ to $\mathcal{D}$.

13. Let V be a vector space over a field k. Let F_V : **Vect** $\Rightarrow$ **Vect** send W to $W \otimes V$, the tensor product over F and f: $W \to W'$ to

$$f \otimes 1_V\colon W \otimes V \to W' \otimes V$$

defined by

$$(f \otimes 1_V)(x \otimes v) \to fx \otimes v$$

Show that

$$(\mathcal{L}(V, W), e\colon \mathcal{L}(V, W) \otimes V \to W)$$

is a universal pair from F_V to W, where e is evaluation:

$$e(f \otimes v) = fv$$

for $f \in \mathcal{L}(V, W)$.

Answers to Selected Exercises

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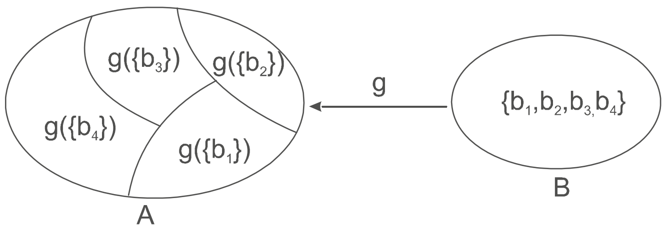
Chapter 1

3. Suppose that $f\colon FC \to FC'$ is an isomorphism. Since $F\colon \hom_{\mathcal{C}}(C, C') \leftrightarrow \hom_{\mathcal{D}}(FC, FC')$ is a bijection, there is a $g\colon C \to C'$ for which $Fg = f$. Moreover, the same argument applied to f^{-1} shows that $f^{-1} = Fh$ for some $h\colon C' \to C$. Thus,

$$F(h \circ g) = Fh \circ Fg = f^{-1} \circ f = 1_{FC} = F(1_C)$$

   and since $F\colon \hom_{\mathcal{C}}(C, C) \leftrightarrow \hom_{\mathcal{D}}(FC, FC)$, it follows that $h \circ g = 1_C$. Similarly, $g \circ h = 1_{C'}$ and so g is an isomophism.
5. The morphisms form a monoid under composition.
8. **Poset**$(\mathcal{P}, \leq)$ where $\mathcal{P}$ is nontrivial.
10. In $\mathbf{Mod}_R$ the zero module $\{0\}$ is a zero object. In **Rng**, where we postulate that a ring morphism must send 1 to 1, the trivial ring $\{0\}$ (in which $1 = 0$) is not initial but it is terminal.
13. Consider all subcategories of $\mathcal{C}$ with the same objects as that of $\mathbb{D}$ and that contain all of the morphisms of $\mathbb{D}$. The full subcategory is one such category. The intersection of all such categories is the smallest such category. $\mathcal{D}$ is the subcategory of $\mathcal{C}$ with objects the same as the objects of $\mathbb{D}$ and whose morphisms are the identity morphisms of objects in $\mathcal{D}$, the morphisms in $\mathbb{D}$ and the compositions of finite sequences of morphisms of $\mathbb{D}$.
15. We must show that $g = f^{-1}$. With reference to the figure below,



   where $B = \{b_1, b_2, b_3, b_4\}$, note that if $b_i \neq b_j$ in B, then

$$g(\{b_i\}) \cap g\big(\{b_j\}\big) = g\big(\{b_i\} \cap \{b_j\}\big) = g(\emptyset) = \emptyset$$

Also,

$$\bigcup g(\{b_i\}) = g\Big(\bigcup \{b_i\}\Big) = g(B) = A$$

Hence, the sets $g(\{b_i\})$ form a partition of A. Now we can define $f\colon A \to B$ to send the elements of $g(\{b_i\})$ to b_i for all $b_i \in B$. Clearly, $f^{-1} = g$.

17. To show that monic does not necessarily imply injective, intuitively speaking, if the images of all morphisms in a category $\mathcal{C}$ are confined to a "restricted" set, then f need only be well-behaved on this set in order to be left-cancellable. With this guidance, consider the category $\mathcal{C}$ whose objects are the subsets of $\mathbb{Z}$ and for which $\hom_{\mathcal{C}}(A, B)$ is the set of all *nonnegative* set functions from A to B, along with the identity function when $A = B$. In this case, the images of all nonidentity morphisms are contained in the natural numbers $\mathbb{N}$. Now, the absolute value function $\alpha\colon \mathbb{Z} \to \mathbb{N}$ is monic since $\alpha \circ f = f$ for all morphisms f in $\mathcal{C}$. On the other hand, α is clearly not injective.

    To show that injective (and therefore also monic) does not necessarily imply left-invertible, consider the inclusion map $\kappa\colon \mathbb{Z} \to \mathbb{Q}$ between rings, which is injective. However, κ is not left-invertible, since a left-inverse $\sigma\colon \mathbb{Q} \to \mathbb{Z}$ would satisfy $\sigma \circ \kappa = 1$, that is, $\sigma(k) = k$ for all integers k. This is not possible since, for example, it would imply that $\sigma(1/2) = \sigma(1)/\sigma(2) = 1/2$, which is not an integer.

    To see that epics are not necessarily surjective, if we can find a category in which each morphism leaving an object A is completely determined by its values on a proper subset S of A, then the inclusion map $i\colon S \to A$, which is not surjective, will be right-cancellable. To this end, the monoids $\mathbb{N}$ and $\mathbb{Z}$ are additive monoids. Moreover, the inclusion map $\kappa\colon \mathbb{N} \to \mathbb{Z}$ is not surjective. However, it is epic since if

$$g \circ \kappa = h \circ \kappa$$

for $g\colon \mathbb{Z} \to C$ and $h\colon \mathbb{Z} \to C$ then g and h agree on all nonnegative integers. This implies that $g = h$, since for $n > 0$, we have (where e is the identity in C)

$$\begin{aligned} g(-n) &= g(-n) * e \\ &= g(-n) * h(n-n) \\ &= g(-n) * [h(n) * h(-n)] \\ &= [g(-n) * h(n)] * h(-n) \\ &= [g(-n) * g(n)] * h(-n) \\ &= g(-n+n) * h(-n) \\ &= g(0) * h(-n) \\ &= h(-n) \end{aligned}$$

Finally, to see that surjective (and therefore epic) maps are not always right-invertible, let $C = \langle a \rangle$ be a cyclic group and let $H = \langle a^2 \rangle$. Consider the canonical projection map $\pi\colon C \to C/H = \{H, aH\}$. This map is surjective, but it is not right-invertible. In fact, any group morphism $\sigma\colon C/H \to C$ must send aH, which has order 2 to an element $\sigma(aH)$ of exponent 2 and so $\sigma(aH) = 1$. Since $\sigma(H) = 1$ as well, the only group morphism from C/H to C is the zero map.

19. Suppose that f is epic. Define α, β: $B \to B \otimes B$ by

$$\alpha(b) = 1 \otimes b \quad \text{and} \quad \beta(b) = b \otimes 1$$

Then

$$\alpha \circ f(a) = 1 \otimes f(a) = f(a)(1 \otimes 1) = f(a) \otimes 1 = \beta \circ f(a)$$

and so $\alpha \circ f = \beta \circ f$, whence $\alpha = \beta$, that is, $1 \otimes b = b \otimes 1$ for all $b \in B$. For the converse, suppose that $1 \otimes b = b \otimes 1$ for all $b \in B$. Let $\alpha \circ f = \beta \circ f$, where α, β: $B \to R$. The ring map $\alpha \circ f = \beta \circ f$: $A \to R$ makes R into an A-module via

$$a * r = (\alpha f(a)) \cdot r = (\beta f(a)) \cdot r$$

Now, define a map $\alpha \times \beta$: $B \times B \to R$ by

$$(\alpha \times \beta)(b, c) = (\alpha b)(\beta c)$$

which is A-bilinear since

$$(\alpha \times \beta)(ab, c) = \alpha(ab) \cdot \beta c = \alpha(f(a)) \cdot \alpha b \cdot \beta c = a*(\alpha \times \beta)(b, c)$$

and so there is a unique θ: $B \otimes B \to R$ for which

$$\theta(x \otimes y) = (\alpha \otimes \beta)(x, y)$$

In particular,

$$\theta(1 \otimes b) = (\alpha \otimes \beta)(1, b) = \beta b$$

and

$$\theta(b \otimes 1) = (\alpha \otimes \beta)(b, 1) = \alpha b$$

and so $\alpha = \beta$.

23. The figure below shows the factorization of f and g.

$$\begin{tikzcd}
A \arrow[rrr, "e"] \arrow[dr, "e_1"] \arrow[dd, "f"'] & & & B \arrow[dl, "e_2"'] \arrow[dd, "g"] \\
& I \arrow[dl, "m_1"] & J \arrow[l, dashed, "\exists\theta"'] \arrow[dr, "m_2"'] & \\
C \arrow[rrr, "m"'] & & & D
\end{tikzcd}$$

Since $(m \circ m_1) \circ e_1 = m_2 \circ (e_2 \circ e)$ and since each of these morphisms is in $\mathcal{E}$ or $\mathcal{M}$, there is an isomorphism $\theta\colon J \to I$ such that

$$e_1 = \theta \circ (e_2 \circ e)$$

Hence

$$f = m_1 \circ e_1 = m_1 \circ \theta \circ (e_2 \circ e) = (m_1 \circ \theta \circ e_2) \circ e$$

and we may take $f = h \circ e$.

Chapter 2

1. Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a contravariant functor. Then since $\mathcal{C}$ and its opposite category $\mathcal{C}^{\text{op}}$ have the same objects and the same morphisms (they even have the same hom-sets, but they are associated to different pairs of objects), F can also be thought of as mapping the objects and morphisms of $\mathcal{C}^{\text{op}}$ to the objects and morphisms of $\mathcal{D}$. Moreover, if $f\colon A \to B$ and $g\colon B \to C$ are morphisms in the opposite category $\mathcal{C}^{\text{op}}$, then

$$F\big(g \circ_{\text{op}} f\big) = F(f \circ g) = Fg \circ Ff = Ff \circ_{\text{op}} Fg$$

and so $F\colon \mathcal{C}^{\text{op}} \Rightarrow \mathcal{D}$ is a *covariant* functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$. Thus, a contravariant functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is a covariant functor $F\colon \mathcal{C}^{\text{op}} \Rightarrow \mathcal{D}$ and conversely.

3. For part a), to see that F is well defined, if $aG' = bG'$, then $b^{-1}a \in G'$ and so $\sigma(b^{-1}a) \in H'$ (since σ takes commutators to commutators). Hence, $(\sigma a)H' = (\sigma b)H'$. Also, $F1 = 1$ and if $\sigma\colon G \to H$ and $\tau\colon H \to K$, then

$$F(\tau\sigma)(aG') = (\tau\sigma a)K' = F\tau(\sigma a H') = F\tau(F\sigma a G')$$

and so $F(\tau\sigma) = F\tau F\sigma$. For part b), the canonical projection is natural.

5. First, we must show that a group homomorphism $f\colon G \to H$ maps $C(G)$ to $C(H)$. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in C(H)$ and so the subgroup of G that maps into $C(H)$ contains all commutators, and therefore contains $C(G)$. Clearly C preserves the identity and composition.

7. This map

$$\hom_{\mathcal{C}}(B, \cdot\,)\colon f \mapsto f^{\leftarrow}$$

for $f\colon X \to Y$ defines a natural transformation

$$\hom_{\mathcal{C}}(B, \cdot\,)^*\colon \hom_{\mathcal{C}}(A, \cdot\,) \stackrel{\cdot}{\to} \hom_{\mathbf{Set}}(\hom_{\mathcal{C}}(B, A), \hom_{\mathcal{C}}(B, \cdot\,))$$

by

$$[\hom_{\mathcal{C}}(B, \cdot\,)^*]_X\colon \hom_{\mathcal{C}}(A, X) \stackrel{\cdot}{\to} \hom_{\mathbf{Set}}(\hom_{\mathcal{C}}(B, A), \hom_{\mathcal{C}}(B, X))$$

where

$$[\hom_{\mathcal{C}}(B,\ \cdot\ )^*]_X(g_{A,X}) = g^{\leftarrow}$$

In this case, naturalness is

$$f^{\leftarrow\leftarrow} \circ [\hom_{\mathcal{C}}(B,\ \cdot\ )^*]_X = [\hom_{\mathcal{C}}(B,\ \cdot\ )^*]_Y \circ f^{\leftarrow}$$

for $f: X \to Y$ in $\mathcal{C}$. For any $g: A \to X$, this is

$$f^{\leftarrow\leftarrow}(g^{\leftarrow}) = (f \circ g)^{\leftarrow}$$

which is true, since

$$f^{\leftarrow\leftarrow}(g^{\leftarrow}) = f^{\leftarrow} \circ g^{\leftarrow} = (f \circ g)^{\leftarrow}$$

9. The identity map 1_C is sent to the map

$$\begin{pmatrix} 1_C \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix} = \begin{pmatrix} \rho_{C\times A,1} \\ \rho_{C\times A,2} \end{pmatrix} = 1_{C\times A}$$

If $f: C \to D$ and $g: D \to E$ then

$$\Pi_A(g \circ f) = \begin{pmatrix} g \circ f \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix}$$

that is,

$$\rho_{E\times A,1} \circ \Pi_A(g \circ f) = g \circ f \circ \rho_{C\times A,1}$$

and

$$\rho_{E\times A,2} \circ \Pi_A(g \circ f) = \rho_{C\times A,2}$$

But

$$\rho_{E\times A,1} \circ \Pi_A g \circ \Pi_A f = g \circ \rho_{D\times A,1} \circ \Pi_A f = g \circ f \circ \rho_{C\times A,1}$$

and

$$\rho_{E\times A,1} \circ \Pi_A g \circ \Pi_A f = g \circ \rho_{D\times A,1} \circ \Pi_A f = g \circ f \circ \rho_{C\times A,1}$$

11. For part 1), let $\lambda: F \dot{\approx} G$ be a natural isomorphism. Assume that G is faithful. To see that F is faithful, if $f, g \in \hom_{\mathcal{C}}(A, B)$ then

$$\begin{aligned}Ff = Fg &\Rightarrow \lambda_B \circ Ff \circ \lambda_A^{-1} = \lambda_B \circ Fg \circ \lambda_A^{-1}\\ &\Rightarrow Gf = Gg\\ &\Rightarrow f = g\end{aligned}$$

To see that F is full, let $h \in \hom_{\mathcal{C}}(FA, FB)$. Let

$$k = \lambda_B \circ h \circ \lambda_A^{-1} : GA \to GB$$

Since G is full, there is an f: $A \to B$ such that $Gf = k$ and so

$$h = \lambda_B^{-1} \circ k \circ \lambda_A \to \lambda_B^{-1} \circ Gf \circ \lambda_A = Ff$$

Hence F is full. For part 2), if $G \circ F$ is faithful then F is faithful, for if $f, g \in \hom_{\mathcal{C}}(A, B)$, we have

$$Ff = Fg \Rightarrow GFf = GFg \Rightarrow f = g$$

Also, if $G \circ F$ is full then G is full, for if $h \in \hom_{\mathcal{C}}(A, B)$ then there is an f: $A \to B$ such that $h = GFf = G(Ff)$.

13. A functor F: $M \Rightarrow$ **Grp** picks out a single group FM. A morphism $m \in M$ is sent to a group morphism Fm: $FM \to FM$ in End(FM). The definition of functor is equivalent to saying that $m \mapsto Fm$ is a group homomorphism.
15. For part a), we have Card $\circ$ $I(n) = \mathrm{Card}(n) = n$ and for f:$n \to m$, Card $\circ$ $I(f) = \mathrm{Card}(f)$ $= \theta_m \circ f \circ \theta_n = f$. For part b), let $F = I \circ \mathrm{Card}$. Note that $FS = I \circ \mathrm{Card}(S) = \mathrm{Card}(S)$, thought of as a set and for f: $S \to T$, $Ff = \theta_T \circ f \circ \theta_S^{-1}$. For any set S, let $\lambda_S = \theta_S$. Then

$$Ff \circ \theta_S = \theta_T \circ f \circ \theta_S^{-1} \circ \theta_S = \theta_T \circ f$$

which is the condition for F to be natural.
17. For part a), let f: $A \to B$ be a morphism in $\mathcal{C}$. Then there must exist morphisms Ff: $FA \to FB$ and Gf: $GA \to GB$ in $\mathcal{P}$, which implies that $FA \prec FB$ and $GA \prec GB$. Moreover, there are morphisms $\lambda(A)$: $FA \to GA$ and $\lambda(B)$: $FB \to GB$ in $\mathcal{P}$ if and only if $FA \prec GA$ and $FB \prec GB$, in which case transitivity implies that

$$Gf \circ \lambda(A) = \lambda(B) \circ Ff$$

Thus, there is a natural transformation from F to G if and only if $FA \prec GA$ for all objects A of $\mathcal{C}$.

For part b), by part a), there is a natural transformation λ from F to G if and only if $FA \prec GA$ for all objects A in $\mathcal{C}$, in which case there is exactly one natural transformation.
19. Since Obj(2) = {0, 1} and Mor(2) = $\{1_0, 1_1, 01\colon 0 \to 1\}$, a functor F from 2 to $\mathcal{D}$ sends 0 and 1 to a pair of objects $F(0)$ and $F(1)$ in $\mathcal{D}$ and the morphism 01 to a morphism $F(01)$: $F(0) \to F(1)$. Moreover, every arrow f: $A \to B$ in $\mathcal{D}$ gives rise to such a distinct functor F_f from 2 to $\mathcal{D}$. Hence, there is a bijection between the objects (functors) of $\mathcal{D}^2$ and the arrows of $\mathcal{D}$.

21. Define a functor F: Set$_*$ $\Rightarrow$ Set$_0$ as follows: F sends $A_* = A + \{*\}$ to A and if $f: A \to B$ is a partial function, then $Ff: A_* \to B_*$ sends all elements of $A_* \setminus \mathrm{dom}(f)$ to $*$. To see that F is a functor, note that $F1 = 1_*$. Also, let $f: A \to B$ and $g: B \to C$ be partial functions. We have

    a) if $x \in \mathrm{dom}(g \circ f)$ then $(g \circ f)_*(x) = x = g_*(f_*(x))$
    b) if $x \in \mathrm{dom}(f) \setminus \mathrm{dom}(g \circ f)$ then $(g \circ f)_*(x) = * = g_*(f_*(x))$
    c) if $x \notin \mathrm{dom}(f)$ then $(g \circ f) * (x) = * = g_*(f_*(x))$

    and so $(g \circ f)_* = g_* \circ f_*$. Now, any pointed function $\sigma: A_* \to B_*$ is the image under F of the partial function $\sigma: \sigma^{-1}(B) \to B$ obtained by restricting σ to $\sigma^{-1}(B)$. Finally, if f, g: $A \to B$ are partial functions for which $f_* = g_*$, then clearly $f = g$. hence, F is an isomorphism.

25. This follows from the contravariant version of Yoneda's lemma. Alternatively, the contravariant functors $\hom_{\mathcal{C}}(\,\cdot\,, A)$ and $\hom_{\mathcal{D}}(F\,\cdot\,, B)$ from $\mathcal{C}$ to Set are covariant functors from $\mathcal{C}^{\mathrm{op}}$ to Set. Hence, for any $f: Y \to X$ in $\mathcal{C}$, the condition of naturalness is

$$\lambda_{A,B}(Y) \circ \hom_{\mathcal{C}}(f, A) = \hom_{\mathcal{D}}(Ff, B) \circ \lambda_{A,B}(X)$$

But $\hom_{\mathcal{C}}(f, A) = f^{\to}$ and $\hom_{\mathcal{D}}(Ff, B) = (Ff)^{\to}$, and so

$$\lambda_{A,B}(Y) \circ f^{\to} = (Ff)^{\to} \circ \lambda_{A,B}(X)$$

Taking $X = A$ and applying to 1_A gives

$$\lambda_{A,B}(Y) \circ f^{\to} 1_A = (Ff)^{\to} \circ \lambda_{A,B}(A) 1_A$$

for any $f: Y \to A$. But $f^{\to} 1_A = 1_A f = f$ and so we get

$$\lambda_{A,B}(Y) f = \lambda_{A,B}(A) 1_A \circ Ff$$

Finally, suppose that

$$\lambda_{A,B}(X) f = (Ff)^{\to} g = g \circ Ff$$

for all $f: X \to A$, where $g \in \hom_{\mathcal{D}}(FA, B)$. Then if $f: Y \to X$ in $\mathcal{C}$, we have for any $h: X \to A$,

$$\begin{aligned}
(\lambda_{A,B}(Y) \circ f^{\to}) h &= \lambda_{A,B}(Y)(h \circ f) \\
&= g \circ F(h \circ f) \\
&= g \circ Fh \circ Ff \\
&= (Ff)^{\to} \circ (g \circ Fh) \\
&= (Ff)^{\to} \circ \lambda_{A,B}(X) h
\end{aligned}$$

and so $\lambda_{A,B}(Y) \circ f^{\to} = (Ff)^{\to} \circ \lambda_{A,B}(X)$, showing that $\lambda_{A,B}$ is natural.

Chapter 3

1. The pair

$$(F(X), j\colon X \to F(X))$$

where $F(X)$ is the field generated by the elements of X, that is, the field of all rational functions in the variables X over F, and j is inclusion, is universal for X and U.

5. Let $U\colon \mathrm{Alg}(F) \Rightarrow \mathrm{Set}$ be the forgetful functor, where $\mathrm{Alg}(F)$ is the category of F-algebras. The pair $(F[x], j\colon \{x\} \to UF[x])$ is universal from $\{x\}$ to U. For given any pair $(A, g\colon \{x\} \to U A)$, we define $\tau\colon F[x] \to A$ by $\tau(p(x)) = p(g(x))$.

7. A couniversal pair $(V, h\colon S \to V)$ for a set S has the property that for every set function $f\colon W \to S$, there is a unique linear map $\tau\colon W \to V$ for which $h \circ \tau = f$. But $\tau 0 = 0$ and so $h0 = f0$ for all f. Hence, if $|S| \geq 2$ then there is no couniversal pair. If $S = \{s\}$, then take $V = \langle v \rangle$ and h to be the constant function. If $S = \emptyset$, take $V = \{0\}$ and h to be the empty function.

9. To see that a) and b) are equivalent, note that

$$\tau_{C,E}^{-1}(g \circ h) = Gg \circ \tau_{C,D}^{-1}(h)$$

for all $h\colon U \to D$ and $g\colon D \to D'$ is equivalent to

$$\tau_{C,D'}\left(Gg \circ \tau_{C,D}^{-1}(h)\right) = g \circ h$$

for all $h\colon U \to D$ and $g\colon D \to D'$. But $\alpha = \tau_{C,D}^{-1}(h)$ runs through all morphisms from C to GD as h runs through all morphisms from U to D and so this is equivalent to

$$\tau_{C,D'}(Gg \circ \alpha) = g \circ \tau_{C,D}(\alpha)$$

for all $\alpha\colon C \to GD$, which is in turn equivalent to

$$\tau_{C,D'} \circ (Gg)^{\leftarrow} = g^{\leftarrow} \circ \tau_{C,D}$$

Part c) comes directly from b) by setting $h = 1_U$. Conversely, if 3) holds then

$$\tau_{C,D'}^{-1}(g \circ h) = G(g \circ h) \circ \tau_{C,D}^{-1}(1_U) = Gg \circ Gh \circ \tau_{C,D}^{-1}(1_U) = Gg \circ \tau_{C,D'}^{-1}(h)$$

which is 2).

11. By a theorem, the family of bijections

$$\{\tau_{C,D}\colon \hom_{\mathcal{C}}(C, FD) \leftrightarrow \hom_{\mathcal{D}}(S, D)\}_{D \in \mathcal{D}}$$

is a natural isomorphism and so $\hom_{\mathcal{C}}(C, F\,\cdot\,)$ is representable if and only if the pair

$$\mathcal{U} = (S, u\colon C \to FS) \quad \text{where} \quad u = \tau_{C,S}^{-1}(1_S)$$

is universal.

Chapter 4

3. Let $f\colon A \to B$. A cone is any morphism $g\colon X \to A$. If $t\colon T \to A$ is terminal, then the mediating morphism θ for the identity map $1\colon A \to A$ satisfies $t \circ \theta = 1$ and so t is right-invertible. Also, t is monic and so an isomorphism. Conversely, an isomorphism $t\colon T \approx A$ is terminal.
5. The cones over $\{f\colon A \to B\}$ are the same as the cones over the diagram $\{f\colon A \to B,$ $f\colon A \to B\}$ and so a terminal cone is an equalizer of $\{f, f\}$, that is, an isomorphism.
7. A morphism in **Field** is either the zero map or an embedding. Let $\mathcal{P} = (P, \rho_1\colon P \to A, \rho_2\colon P \to B)$ be a product. Since $\hom(A, B) \neq \emptyset$, projections are surjective and so isomorphisms. Thus, $P = A \times B$ is isomorphic to A and to B. The product construction implies that for any $(X, f\colon X \to A, g\colon X \to B)$ there is a unique $\theta\colon X \to P$ such that

$$\rho_1 \circ \theta = f \quad \text{and} \quad \rho_2 \circ \theta = g$$

   Now, if $f = 0$ then $\theta = 0$ since ρ_1 is an isomorphism. Hence, $g = 0$. Thus, the cone $(B, 0, 1_B)$ has no mediating morphism.
9. Lexicographic order does not work because the second projection is not monotone. However, product order works just fine.
13. Consider the category with 4 objects and 9 morphisms, as shown in the commutative diagram below What is the dual of this result?

$$\begin{tikzcd} A & B \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] & P \arrow[l, "h"'] \arrow[r, "k"] & C \end{tikzcd}$$

   Since the only cone over $\{B, C\}$ is (P, h, k), this must be the product of B and C. However, h is not epic since $f \circ h = g \circ h$ but $f \neq g$.
15. For part a), let $f, g\colon M \to N$ be R-maps. Let $A = \operatorname{im}(f - g)$ and let $\pi\colon N \to N/A$ be the canonical projection map. Then for $m \in M$,

$$\pi \circ (f - g)(m) = A$$

   and so $\pi \circ f(m) = \pi \circ g(m)$, whence $\pi \circ f = \pi \circ g$. Now, if $h\colon N \to X$ satisfies $h \circ f = h \circ g$, it follows that $h \circ (f - g) = 0$ and so $A = \operatorname{im}(f - g) \leq \ker(h)$. Hence, there is a unique map $\theta\colon N/A \to X$ for which $\theta \circ \pi = h$, as desired.

   For part b), let $f, g\colon R \to S$ be ring homomorphisms. Let $A = \operatorname{im}(f - g)$ and let $\pi\colon S \to S/I$ where $I = \langle A \rangle$ is the ideal generated by A and π is the canonical projection map. Then for $r \in R$,

$$\pi \circ (f - g)(r) = I$$

   and so $\pi \circ f(r) = \pi \circ g(r)$, whence $\pi \circ f = \pi \circ g$. Now, if $h\colon S \to X$ satisfies $h \circ f = h \circ g$, it follows that $h \circ (f - g) = 0$ and so $A = \operatorname{im}(f - g) \leq \ker(h)$. Hence, there is a unique map $\theta\colon S/I \to X$ for which $\theta \circ \pi = h$, as desired.

17. Since u_x: $\{x\} \to S_x$, we have

$$G\kappa_x \circ u_x\colon \{x\} \to GC$$

and so $u|_{\{x\}} = Gk_x \circ u_x$. Now, let f: $X \to GE$ for $E \in \mathcal{D}$. Then $f\,|_{\{x\}}$: $\{x\} \to GE$ and so there is a unique τ_x: $S_x \to E$ for which

$$G\tau_x \circ u_x = f\big|_{\{x\}}$$

The definition of coproduct implies that there is a unique θ: $C \to E$ for which

$$\theta \circ \kappa_x = \tau_x$$

for all $x \in X$ and so there is a unique θ for which

$$G(\theta \circ \kappa_x) \circ u_x = f\big|_{\{x\}}$$

(this follows from $\theta \circ \kappa_x = \theta' \circ \kappa_x$ for all x implies $\theta = \theta'$). This is equivalent to

$$(G\theta \circ G\kappa_x \circ u_x)(x) = f(x)$$

that is,

$$(G\theta \circ u)(x) = f(x)$$

and so $G\theta \circ u = f$, as desired.

19. Consider the category shown below.

$$\begin{tikzcd}
\{1,2,3\} \arrow[r, "2"] \arrow[d, "1"'] \arrow[dr, dashed, "1" near start] & \{2\} \arrow[d, shift right, "1"'] \\
\{1\} \arrow[r, "1"'] \arrow[ur, dashed, "2" near start] & \{1,2\} \arrow[u, shift right, "2"']
\end{tikzcd}$$

21. The essence of the proof is that there is no real difference between the two diagrams in the figure below

$$\begin{tikzcd}
& X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
A \arrow[dr, "f"] & & B \arrow[dl, "g"'] \\
& T &
\end{tikzcd}$$

where f and g are the unique morphisms associated to the terminal object T. This follows from the fact that $f \circ \alpha$ and $g \circ \beta$ must be the unique morphism from X to T, which makes the right-hand diagram commute. Thus, a limit in the cones over the left-hand diagram, that is, a product, is a limit in the cones over the right-hand diagram, which is a pullback for this diagram.

23. Consider the figure below

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\mu"'] \arrow[drr, dashed, "\exists!\theta"] \arrow[drrr, "\beta"] \arrow[ddr, "\alpha"'] & & & \\
& A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & B_1 \arrow[r, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[d, "h_3"] \\
& A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, "g_2"'] & C_2
\end{tikzcd}$$

where $g_2 f_2 \alpha = h_3 \beta$. Since the right-hand square is a pullback and since $(f_2\alpha, \beta)$ is a cone, there is a unique θ: $X \to B_1$ for which

$$g_1\theta = \beta \quad \text{and} \quad h_2\theta = f_2\alpha \tag{*}$$

Since (α, θ) is a cone for the square on the left, there is a unique μ: $X \to A_1$ for which

$$h_1\mu = \alpha \quad \text{and} \quad f_1\mu = \theta \tag{**}$$

Now, μ is a mediating arrow for the entire rectangle, for we have

$$h_1\mu = \alpha \quad \text{and} \quad g_1 f_1\mu = g_1\theta = \beta$$

As to uniqueness, if

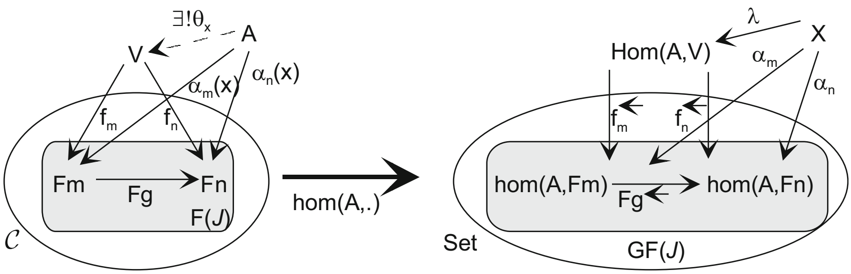
$$h_1\mu_0 = \alpha \quad \text{and} \quad g_1 f_1\mu_0 = \beta$$

then we claim that μ_0 satisfies $(**)$, in which case $\mu_0 = \mu$. To see this, we have already that $h_1\mu_0 = \alpha$. Also, we claim that $f_1\mu_0 = \theta$, in which case, $g_1 f_1\mu_0 = g_1\theta = \beta$ and we are done. To show that $f_1\mu_0 = \theta$, we show that $f_1\mu_0$ satisfies $(*)$. First, $g_1 f_1\mu_0 = \beta$ by assumption. Second,

$$h_2 f_1 \mu_0 = f_2 h_1 \mu_0 = f_2 \alpha$$

which is the second condition in $(*)$.

25. As shown in the figure below,



let $\mathcal{X} = \big(X, \{\alpha_n \colon X \to F_n\}\big)$ be a cone over $G \circ F$. Then

$$(Fg)^{\leftarrow} \circ \alpha_m = \alpha_n$$

that is, for any $x \in X$,

$$(Fg)[\alpha_m(x)] = \alpha_n(x)$$

This shows that

$$\mathcal{A}_x = (A, \{\alpha_n(x) \colon A \to Fn\})$$

is a cone over F. Hence, there is a unique $\theta_x \colon A \to V$ for which

$$f_n \circ \theta_x = \alpha_n(x)$$

Let $\lambda \colon X \to \hom(A, V)$ be defined by $\lambda(x) = \theta_x$. Then

$$\big(f_n^{\leftarrow} \circ \lambda\big)(x) = f_n^{\leftarrow}(\theta_x) = f_n \circ \theta_x = \alpha_n(x)$$

and so λ is a mediating morphism. As to uniqueness, if

$$\big(f_n^{\leftarrow} \circ \mu\big)(x) = \alpha_n(x)$$

then

$$f_n \circ \mu(x) = \alpha_n(x)$$

which implies that $\mu(x) = \theta_x = \lambda(x)$.

As to the dual, first we note that for $A \in \mathcal{C}^{\text{op}}$, the hom functor

$$\hom_{\mathcal{C}^{\text{op}}}(A, \ \cdot\ ): \mathcal{C}^{\text{op}} \Rightarrow \textbf{Set}$$

preserves limits, that is,

$$\hom_{\mathcal{C}}(\ \cdot\ , A): \mathcal{C}^{\text{op}} \Rightarrow \textbf{Set}$$

preserves limits. But a limit in $\mathcal{C}^{\text{op}}$ is a colimit in $\mathcal{C}$ and so the dual is that $\hom_{\mathcal{C}}(\ \cdot\ , A)$ sends colimits in $\mathcal{C}$ to limits in **Set**.

27. Suppose that $\mathcal{K} = \left(V, \{f_n \mid n \in \mathcal{J}\}\right)$ is a cone of F, whose image $G\mathcal{K} = \left(GV, \{Gf_n \mid n \in \mathcal{J}\}\right)$ is a limit of $F \circ G$. If $\mathcal{X} = \left(X, \{g_n \mid n \in \mathcal{J}\}\right)$ is a cone over F, then its image $G\mathcal{X} = \left(GX, \ \{Gg_n \mid n \in \mathcal{J}\}\right)$ is a cone over $F \circ G$ and so there is a unique map $\theta: GX \to GV$ for which $Gf_n \circ \theta = Gg_n$. Since G is fully faithful, there is a unique $\lambda: X \to V$ for which $\theta = G\lambda$ and so $G(f_n \circ \lambda) = Gg_n$, whence $f_n \circ \lambda = g_n$. Thus, λ is a unique cone morphism from $\mathcal{K}$ to $\mathcal{X}$.

30. The elements of the quotient S/N have the form $s + N$, where $s \in S$ has finite support. If $\rho_i(s) \neq 0$ and $\rho_j(s) \neq 0$ for $i < j$, then since $[\rho_i(s)]_{i,j} \in N$, we have

$$sN = \left(s - [\rho_i(s)]_{i,j}\right) + N$$

where the latter has ith coordinate equal to 0 and support that is properly contained in the support of $s + N$. It follows that any $x \in S/N$ has the form $s + N$, where $|\text{supp}(s)| = 1$, that is, $x = [a_i]_i + N$ for $a_i \in M_i$.

Now let us examine the elements of N. The generators of N are

$$[a_i]_{i,j} = [a_i]_i - \left[f_{i,j}(a_i)\right]_j$$

where $a_i \in M_i$ and $i < j$. If $k > j$, then we can write

$$\begin{aligned}[a_i]_{i,j} &= [a_i]_i - \left[f_{i,j}(a_i)\right]_j \\ &= [a_i]_i - \left[f_{i,k}(a_i)\right]_k + \left(\left[-f_{i,j}(a_i)\right]_j - \left[-f_{i,k}(a_i)\right]_k\right)\end{aligned}$$

If $b_j = -\ f_{i,j}(a_i)$, then

$$f_{j,k}(b_j) = -f_{j,k}\, f_{i,j}(a_i) = -f_{i,k}(a_i)$$

and so

$$[a_i]_{i,j} = [a_i]_i - \left[f_{i,k}(a_i)\right]_k + \left(\left[b_j\right]_j - \left[f_{j,k}(b_j)\right]_k\right)$$

which is the sum of two generators, each of which has last term of index k. Thus, since any $x \in N$ is a finite sum of generators, and since the index set I is directed, there is an index k for which x is the finite sum of generators whose last terms all have index k. Moreover, since we can add generators that have the same pair of indices, we may assume that the first indices in this sum are distinct. Thus,

$$\begin{aligned} x &= \left([a_{i_1}]_{i_1} - \left[f_{i_1,k}(a_{i_1})\right]_k + \cdots + \left([a_{i_n}]_{i_n} - \left[f_{i_n,k}(a_{i_n})\right]_k\right)\right. \\ &= \left([a_{i_1}]_{i_1} + \cdots + [a_{i_n}]_{i_n} - \left(\left[f_{i_1,k}(a_{i_1})\right]_k + \cdots + \left[f_{i_n,k}(a_{i_n})\right]_k\right)\right. \end{aligned}$$

where the i_j are distinct and less than k.

Now, if $|\text{supp}(x)| = 1$, that is, if x has exactly one nonzero coordinate, then x must have the form $x = \left[a_{i_j}\right]_{i_j}$ for some j, and $f_{i_j,k}\left(a_{i_j}\right) = 0$. In simpler notation, the elements of N that have support of size 1 are of the form $x = [a_i]_i$ for $a_i \in M_i$ and for which $f(a_{i,k}) = 0$ for some $k > i$.

The pair $\left(\text{dirlim}(\mathcal{M}), \{\pi \circ \kappa_i \mid i \in I\}\right)$ is an initial object in the category of cones under the diagram $\mathcal{M}$, that is, it is the colimit in the categorical sense. For if $(X, \{g_i\})$ is a cone under $\mathcal{M}$, then from the definition of direct sum, there is a unique mediating arrow $\theta\colon S \to X$ for which

$$\theta \circ \kappa_i = g_i$$

Now, if $x = [a_i]_{i,j}$ is a generator of N, then

$$\begin{aligned} \theta(x) &= \theta\left([a_i]_{i,j}\right) \\ &= \theta([a_i]_i) - \theta\left(\left[f_{i,j}(a_i)\right]_j\right) \\ &= \theta\kappa_i(a_i) - \theta\kappa_j\left(f_{i,j}(a_i)\right) \\ &= g_i(a_i) - g_j\left(f_{i,j}(a_i)\right) \end{aligned}$$

and since $(X, \{g_i\})$ is a cone under $\mathcal{M}$, we have

$$\theta(x) = g_i(a_i) - g_j\left(f_{i,j}(a_i)\right) = g_i(a_i) - g_i(a_i) = 0$$

Thus, $x \in \ker(\theta)$, from which we get $N \le \ker(\theta)$. it follows that θ induces a map $\overline{\theta}\colon S/N \to X$ defined by

$$\overline{\theta}([a_i]_i + N) = \theta([a_i]_i) = \theta\kappa_i(a_i) = g_i(a_i)$$

Thus,

$$\left(\overline{\theta} \circ \pi_N \circ \kappa_i\right)(a_i) = g_i(a_i)$$

and so $\overline{\theta} \circ (\pi_N \circ \kappa_i) = g_i$. Moreover, if $\mu \circ (\pi_N \circ \kappa_i) = g_i$, then

$$\mu([a_i]_i + N) = \overline{\theta}([a_i]_i + N)$$

and so $\mu = \overline{\theta}$, which shows that $\overline{\theta}$ is unique.

Chapter 5

1. In terms of adjunctions, we have

$$\tau_{C,D}\colon \hom_{\mathcal{C}}(C, GD) \dot{\leftrightarrow} \hom_{\mathcal{D}}(FC, D)$$

and

$$\sigma_{E,C}\colon \hom_{\mathcal{E}}(E, G'C) \dot{\leftrightarrow} \hom_{\mathcal{C}}(F'E, C)$$

The composition

$$\lambda_{E,D} = \tau_{F'E,D} \circ \sigma_{GD,E}$$

maps

$$\hom_{\mathcal{E}}(E, G'GD) \dot{\leftrightarrow} \hom_{\mathcal{C}}(F'E, GD) \dot{\leftrightarrow} \hom_{\mathcal{D}}(FF'E, D)$$

and the map is natural since

$$\lambda_{E,\cdot} = \tau_{F'E,\cdot} \circ \sigma_{G\cdot,E} = \tau_{F'E,\cdot} \circ (\sigma_{\cdot,E} * G)$$

and

$$\lambda_{\cdot,D} = \tau_{F'\cdot,D} \circ \sigma_{GD,\cdot} = (\tau_{\cdot,D} * F') \circ \sigma_{GD,\cdot}$$

3. $F(X) = F[X]$, the ring of polynomials in X. $f\colon X \to Y$ can be extended to polynomials.
5. Let $u_C = \pi_{[C,C]}$ be projection modulo the derived subgroup $[C, C]$. Then if A is an abelian group and $f\colon C \to A$ is a group morphism, it is easy to see that $[C, C] \le \ker(f)$ and so there is a unique $\tau\colon C/[C, C] \to A$ for which

$$\tau \circ \pi_{[C,C]} = f$$

This shows that $(C/[C, C], \pi_{[C,C]}\colon C \to C/[C, C])$ is a universal pair from C to U.
7. Let $u_A\colon a \mapsto 1 \otimes a$. Then if M is an R-module and $f\colon A \to UM$ is a morphism of abelian groups, let $\tau\colon R \times A \to M$ be defined by

$$\tau(r, a) = rf(a)$$

Since this map is $\mathbb{Z}$-bilinear, there is a unique R-map $\tau'\colon R \otimes A \to M$ for which $\tau'(r \otimes a) = rf(a)$. Moreover,

$$\tau' \circ u_A\colon a \to \tau'(1 \otimes a) = f(a)$$

9. Note first that

$$\mathrm{fix}(X, v) := \{x \in X \,|\, vx = x\} = vX$$

To see that F is a left adjoint of I, we show that there is a universal pair

$$\big(F(X, v), u_{(X,v)} \colon (X, v) \to IF(X, v)\big)$$

that is,

$$\big(vX, u_{(X,v)} \colon (X, v) \to (vX, 1)\big)$$

for every $(X, v) \in$ **Idem**. This amounts to showing that for every $f\colon (X, v) \to (Y, 1)$, that is, for every set function $f\colon X \to Y$ for which $f \circ v = f$, there is a unique $\tau\colon vX \to Y$ for which

$$\tau \circ u_{(X,v)} = f$$

If $u_{(X,v)}(x) = vX$, then this becomes

$$\tau \circ v = f$$

and so this uniquely defines τ as equal to $f|_{vX}$.

To see that F is a right adjoint of I, we must show that there is a couniversal pair

$$((X, 1), v_X \colon FIX \to X)$$

that is,

$$((X, 1), v_X \colon X \to X)$$

for every set X. Let $v_X = 1_X$. Given $f\colon vY \to X$, let $\tau\colon (Y, v) \to (X, 1)$ be defined by setting $\tau(y) = f(vy)$. Then $\tau(y) = \tau(vy)$ and so τ is a morphism. Also, $\tau(vy) = f(vy)$ and so $1 \circ F\tau = f$.

11. a) If $F\colon P \to Q$ is isotone then $p \le p'$ implies that $Fp \le Fp'$. Put in categorical terms, if $f\colon p \to p'$ then $Ff\colon Fp \to Fp'$. b) If

$$p \le GFp \quad \text{and} \quad q \le FGq$$

and if $p \le Gq$ then applying F gives

$$q \le FGq \le Fp$$

Also, if $q \le Fp$ then applying G gives

$$p \le GFp \le Gq$$

Conversely, if $p \leq Gq$ iff $q \leq Fp$ then since $Fp \leq Fp$, we have for $q = Fp$, $p \leq GF\,p$. The other is similar. c) Since $p \leq GFp$ there is a unique morphism u_p: $p \to GFp$. If $q \in Q$ and f: $p \to Gq$ in $\mathcal{P}$ then $p \leq Gq$ which implies that $q \leq Fp$ in $\mathcal{Q}$, whence there is a unique τ: $q \to Fp$ in $\mathcal{Q}$ and so τ: $Fp \to q$ in $\mathcal{Q}^{\mathrm{op}}$. Moreover, $G\tau$: $GFp \to Gq$ in $\mathcal{P}$ and so

$$p \leq GFp \leq Gq$$

which is the statement that $G\tau \circ u_p = f$. Thus, $(Fp, u_p{:}\ p \to GFp)$ is universal from p to G. d) If

$$u = \{u_p\}: I_{\mathcal{P}} \stackrel{\cdot}{\to} GF$$

is the unit of the adjoint pair, then u_p: $p \to GFp$ and so $p \leq GFp$. Similarly, if

$$v = \{v_q\}: FG \stackrel{\cdot}{\to} I_{\mathcal{Q}}$$

is the counit then v_q: $q \to FGq$ and so $q \leq FGq$. Thus, (F, G) is a Galois connection. (The fact that F and G are functors implies that they are order-reversing between $\mathcal{P}$ and $\mathcal{Q}$.)

13. For any f: $X \otimes V \to W$, we seek a unique τ_f: $X \to \mathcal{L}(V, W)$ with the property that

$$e \circ (\tau_f \otimes 1_V) = f$$

that is,

$$\tau_f(x)(v) = f(x \otimes v)$$

But this uniquely defines τ_f as $\tau_f\,(x) = f\,(x \otimes \cdot\,)$. Note that

$$\begin{aligned} \tau_f(ax + by)(v) &= f((ax + by) \otimes v) \\ &= f(ax \otimes v + by \otimes v) \\ &= [a\tau_f(x) + b\tau_f(y)](v) \end{aligned}$$

and so τ_f is linear.

Index of Symbols

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$\Rightarrow$ Functor
$\leftrightarrow$ Bijection
$\stackrel{\cdot}{\to}$ Natural transformation
$\stackrel{\cdot}{\leftrightarrow}$ Natural bijection
$\approx$ Isomorphism
$\stackrel{\cdot}{\approx}$ Natural isomorphism
$\dashv$ Left adjoint
$\vdash$ Right adjoint
1_A Identity morphism
$(A \to \mathcal{C})$ Comma category of arrows leaving A
$(\mathcal{C} \to A)$ Comma category of arrows entering A
$(A \to G)$ Comma category of arrows leaving A entering G
$(G \to A)$ Comma category of arrows entering A leaving G
$\mathcal{A}(v, w)$ Set of arcs between v and w in a digraph
$\mathcal{B} \times \mathcal{C}$ Product category
$\mathcal{C}^{\mathrm{op}}$ Opposite category
$\mathcal{C}, \mathcal{D}, \mathcal{E}$ Categories
$\mathcal{C}^{\to}$ Category of arrows
$\mathbf{Cone}_{\mathcal{C}}(F)$ *or* $\mathbf{Cone}_{\mathcal{C}}(\mathbb{D})$ Category of cones
$\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, etc. Diagrams
$\mathbb{D}(F\colon \mathcal{J} \Rightarrow \mathcal{C})$ Diagram in $\mathcal{C}$ with functor F and index category $\mathcal{J}$
$\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$ Category of diagrams
$\mathcal{D}^{\mathcal{C}}$ Functor category
$f^{\leftarrow}$ Follow by f
$f^{\to}$ Preceed by f
$\hom_{\mathcal{C}}(A, B)$ Hom-set
$\hom_{\mathcal{C}}(A, -)$ Hom-set category
$\hom_{\mathcal{C}}(A, \cdot\,)$ Hom-set functor
$\mathcal{K}, \mathcal{L}$ Cones and cocones
$\mathbf{Mor}(\mathcal{C})$ Morphisms of $\mathcal{C}$
$\mathbf{Obj}(\mathcal{C})$ Objects of $\mathcal{C}$
$\mathcal{V}(D)$ Vertex class of a digraph

References

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