A logical view of analogical reasoning

based on analogical proportions

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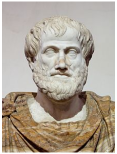
Tutorial UNILOG

Contents

Historical introduction

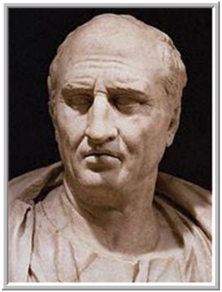
- Western world : Aristotle (384-322 BC)



- Eastern world : Mencius (A follower of Confucius : 372-289 BC)

- Idea of *analogical proportion*
- Use as a rhetorical argument
- Metaphoric use : "Messi is the Mozart of soccer"

Also a forgotten author



Philodemus

Epicurean philosopher Philodemus of Gadara (c. 110 - prob. c. 40 or 35 BC) whose library was buried in Herculanum eruption, and rediscovered in the XVIIIth century

De Lacy, P. H. and De Lacy, E. A. (1941). Philodemus : On Methods of Inference. A Study in Ancient Empiricism. American Philological Association, Philadelphia. With translation and commentary

Analogy

- analogy establishes a parallel between 2 situations on the basis of which, one concludes that what is true in the 1st situation may also be true in the 2nd
- Example
  situation 1 : $p(a), r(a,b), q(b)$
  situation 2 : $p(c), r(c,d)$

  $$\frac{\text{situation 1 : } p(a), r(a,b), q(b) \quad \text{situation 2 : } p(c), r(c,d)}{q(d)}$$
- cognitive psychology
  $\rightarrow$ *Structure Mapping Theory* (Deirdre Gentner)

Analogy - 2

- Analogical proportion "*a* is to *b* as *c* is to *d*"
  often denoted $a : b :: c : d$
- It establishes a *parallel*
  between the pair (a, b) and the pair (c, d)
- Case-based reasoning establishes a series of parallels
  between *known cases* $(< problem_i >, < solution_i >)$
  and a *new* $< problem_0 >$,
  for which one may think of
  a $< solution_0 >$ similar to $< solution_i >$
  as $< problem_0 >$ is similar to $< problem_i >$

Analogy - 3

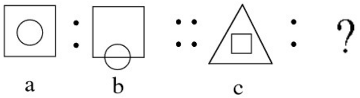
- For about 2300 years, there has been **no** attempt at formalizing analogical proportions
- analogy was regarded as *antagonistic* to logic,
  analogical reasoning, as a useful heuristics,
  in *full contrast* with deductive reasoning
- analogical reasoning *may provide wrong conclusions*
- (deductive) logical reasoning
  always provides valid conclusions
- but analogical reasoning is “creative”,
  may be useful when logical reasoning does not apply

Thomas Evans' *ANALOGY* program

- ANALOGY written in LISP, MIT, 1964

Main ideas

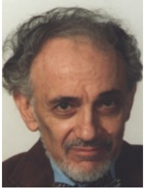
- Pb : "fig. A is to fig. B as fig. C is to fig. X ?"
  X belonging to a **given** *set* **S** *of candidate figures*
- Recognition and transformation of geometric figures



- Primitive input : description of the figures A, B, C, and in **S**
- Find an appropriate transformation rule from A to B to be compared with the transformations from C to each element of **S**
  solution X s. t. *transformation*(A $\rightarrow$ B) $\simeq$ *transformation*(C $\rightarrow$ X)

A forerunner

Sheldon Klein (1935 - 2005) - pages.cs.wisc.edu/ sklein/sklein.html



- B.A. (anthropology - 1956) Ph.D. (linguistics - 1963)
  Prof. of Computer Sciences and Linguistics University of Wisconsin
- *"Culture, mysticism & social structure and the calculation of behavior"*. Proc. Europ. Conf. in AI (ECAI'82), Orsay, 141-146, 1982
- A procedure for computing X such as A : B :: C : X, once A, B, C are encoded in a binary way feature by feature : $X = C \equiv (A \equiv B)$

(Non-logical) formalizations start to be proposed around 2000

Yves Lepage, 1997, 2001 ; François Yvon and Stroppa, 1995, 2005 ; Arnaud Delhay and Laurent Miclet, 2004

Proportions in mathematics

- relations between 2 ordered pairs (a, b) and (c, d)
- geometric proportion : equality of 2 *ratios*

  $a/b = c/d$

  arithmetic proportion : equality of 2 *differences* :

  $a - b = c - d$
- equivalent respectively

  to $ad = bc$ and to $a + d = b + c$
- enable us to "extrapolate" d

  as $d = c \times b/a$ ("*rule of three*"), or $d = c + (b - a)$
- *continuous* proportions where $b = c$ related to

  averaging : taking $b = c$ as the unknown yields

  the *geometric mean* $(ad)^{1/2}$

  and the *arithmetic mean* $(a + d)/2$

Analogical proportions postulates

1. $\forall a, b, R(a, b, a, b)$ (*reflexivity*) ;
2. $\forall a, b, c, d, R(a, b, c, d) \to R(c, d, a, b)$ (*symmetry*)
3. $\forall a, b, c, d, R(a, b, c, d) \to R(a, c, b, d)$ (*central permutation*)

$\forall a, b, c, d, R(a, b, c, d) \to R(d, b, c, a)$ (*external permutation*)

8 equivalent forms for an analogical proportion

R(a,b,c,d)	
R(c,d,a,b)	(by sym.)
R(c,a,d,b)	(by cent. permut.)
R(d,b,c,a)	(by sym.)
R(d,c,b,a)	(by cent. permut.)
R(b,a,d,c)	(by sym.)
R(b,d,a,c)	(by cent. permut.)
R(a,c,b,d)	(by sym.)

Boolean model

It is straightforward to get a basic Boolean model
- by reflexivity, $0101, 1010$ should belong to the relation
- and $0000, 1111$ as well since letting $a = b$
- central permutation then leads to add 0011 and 1100

$\Rightarrow$ we get the **minimal** model

$$\Omega_0 = \{0000, 1111, 0101, 1010, 0011, 1100\}$$

which is *stable under symmetry*

Other models - 1

Due to axioms, we should add to Ω_0 subsets of $\mathbb{B}^4$ stable w.r.t. symmetry and central permutation

1) 1 model with 6 elements : Ω_0 (the smallest one)

2) 1 model with 8 elements : $KI = \Omega_0 \cup S_2 =$
$\{0000, 1111, 0101, 1010, 0011, 1100, 0110, 1001\}$
first proposed by S. Klein (1982)
BUT "*a* is to *b* as *c* is *d*" $\rightarrow$ "*b* is to *a* as *c* is *d*"

3) 2 model with 10 elements :
$M_3 = \Omega_0 \cup S_3 =$
$\{0000, 1111, 0101, 1010, 0011, 1100, 1110, 1101, 1011, 0111\}$
$M_4 = \Omega_0 \cup S_4 =$
$\{0000, 1111, 0101, 1010, 0011, 1100, 0001, 0010, 0100, 1000\}$

Other models - 2

4) 2 models with 12 elements :
$M_5 = M_3 \cup S_2 = \{0000, 1111, 0101, 1010, 0011, 1100,$
$1110, 1101, 1011, 0111, 0110, 1001\}$,
$M_6 = M_4 \cup S_2 = \{0000, 1111, 0101, 1010, 0011, 1100,$
$0001, 0010, 0100, 1000, 0110, 1001\}$,
5) 1 model with 14 elements :
$M_7 = M_3 \cup S_4 = M_4 \cup S_3 = \Omega_0 \cup S_3 \cup S_4 =$
$\{0000, 1111, 0101, 1010, 0011, 1100,$
$1110, 1101, 1011, 0111, 0100, 1000, 0110, 1001\}$
6) 1 model with exactly 16 elements :
$\Omega = \Omega_0 \cup S_2 \cup S_3 \cup S_4 = \mathbb{B}$

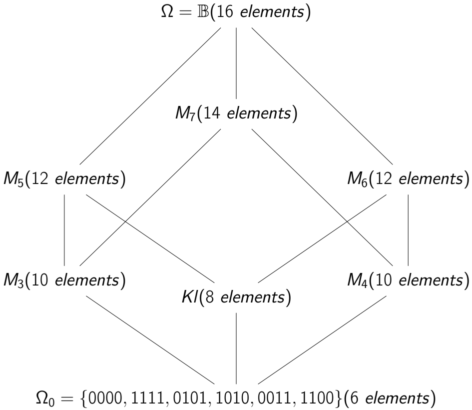


Figure – The lattice of Boolean models of analogy

Boolean analogical proportion "*a* is to *b* as *c* is to *d*"

a	b	c	d	$a : b :: c : d$	a	b	c	d	$a : b :: c : d$
0	0	0	0	**1**	1	0	0	0	0
0	0	0	1	0	1	0	0	1	0
0	0	1	0	0	1	0	1	0	**1**
0	0	1	1	**1**	1	0	1	1	0
0	1	0	0	0	1	1	0	0	**1**
0	1	0	1	**1**	1	1	0	1	0
0	1	1	0	0	1	1	1	0	0
0	1	1	1	0	1	1	1	1	**1**

$$a : b :: c : d = (a \wedge \neg b \equiv c \wedge \neg d) \wedge (\neg a \wedge b \equiv \neg c \wedge d)$$

"*a* differs from *b* as *c* differs from *d*, and vice-versa"

Analogical proportion truth table

Boolean patterns making analogical proportion true

a	b	c	d
0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

- *compatible* with $a - b = c - d$ **but** $a - b \in \{-1, 0, 1\}$
- analogical proportion is transitive :

$(a : b :: c : d) \wedge (c : d :: e : f) \Rightarrow a : b :: e : f$

Analogical proportions between vectors

- Items are represented by *vectors* of Boolean values $\vec{a} = (a_1, \ldots, a_n)$
- $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ iff $\forall i \in [1, n], \ a_i : b_i :: c_i : d_i$
- Pairing pairs (a, b) and (c, d)

	$\mathcal{A}_1$	...	$\mathcal{A}_{i-1}$	$\mathcal{A}_i$	...	$\mathcal{A}_{j-1}$	$\mathcal{A}_j$	...	$\mathcal{A}_{k-1}$	$\mathcal{A}_k$	...	$\mathcal{A}_{r-1}$	$\mathcal{A}_r$	...	$\mathcal{A}_{s-1}$	$\mathcal{A}_s$	...	$\mathcal{A}_n$
$\boldsymbol{a}$	1	...	1	0	...	0	1	...	1	0	...	0	1	...	1	0	...	0
$\boldsymbol{b}$	1	...	1	0	...	0	1	...	1	0	...	0	0	...	0	1	...	1
$\boldsymbol{c}$	1	...	1	0	...	0	0	...	0	1	...	1	1	...	1	0	...	0
$\boldsymbol{d}$	1	...	1	0	...	0	0	...	0	1	...	1	0	...	0	1	...	1

On attributes $\mathcal{A}_1$ to $\mathcal{A}_{r-1}$ $\vec{a}$ and $\vec{b}$ *agree* and $\vec{c}$ and $\vec{d}$ *agree* as well.
It contrasts with attributes $\mathcal{A}_r$ to $\mathcal{A}_n$, for which we can see that $\vec{a}$ *differs* from $\vec{b}$ as $\vec{c}$ *differs* from $\vec{d}$ (and vice-versa)

Example of analogical proportion

a calf *is to* **a cow** *as* **a foal** *is to* **a mare**

	mammal	young	equine	adult female	bovine	adult male
A : calf	1	1	0	0	1	0
B : cow	1	0	0	1	1	0
C : foal	1	1	1	0	0	0
D : mare	1	0	1	1	0	0

The columns are all binary analogical proportions.

$A \setminus B = \{ \text{ young } \} = C \setminus D$

$B \setminus A = \{ \text{ adult female } \} = D \setminus C$

Analogical proportion between *subsets*

Four subsets A, B, C and D are in AP ($A : B :: C : D$) when the *differences* between A and B are the same as between C and D.

$$A \backslash B = C \backslash D \quad \text{and} \quad B \backslash A = D \backslash C$$
$$\Leftrightarrow$$
$$A \cup D = B \cup C \quad \text{and} \quad A \cap D = B \cap C \,!$$

$A = \{a, b, c, h\}$, $B = \{a, b, d, e, h\}$, $C = \{f, c, h\}$ and $D = \{f, d, e, h\}$

$$A \backslash B = C \backslash D = \{c\} \quad \text{and} \quad B \backslash A = D \backslash C = \{d, e\}$$

	a	b	c	d	e	f	h
A	×	×	×				×
B	×	×		×	×		×
C			×			×	×
D				×	×	×	×

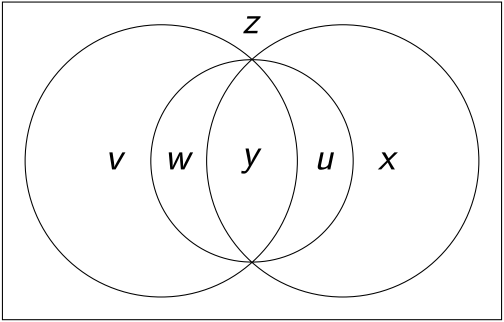
$$A \cup D = B \cup C = \{a, b, c, d, e, f, h\} \quad \text{and} \quad A \cap D = B \cap C = \{h\}$$

Analogical equation for lattice of subsets

Proposition (Y. Lepage)

In the Boolean lattice $(\wp(\Sigma), \cup, \cap, \Sigma \subseteq)$, *a 4-tuple* (A, B, C, D) *is in analogical proportion* $(A : B :: C : D)$ *iff*
there exists 6 subsets (u, v, w, x, y, z) *partitioning* $\wp(\Sigma)$ *such that*

$$A = u \cup w \cup y, B = v \cup w \cup y, C = u \cup x \cup y, D = v \cup x \cup y$$



Analogy is a matter of dissimilarity and similarity

Boolean setting : there are 4 comparison indicators

- 2 similarity indicators : a *positive* one $a \wedge b$ and a *negative* one $\neg a \wedge \neg b$
- 2 dissimilarity indicators : $\neg a \wedge b$ and $a \wedge \neg b$

$a : b :: c : d = (a \wedge \neg b \equiv c \wedge \neg d) \wedge (\neg a \wedge b \equiv \neg c \wedge d)$

"a differs from b as c differs from d, and vice-versa"

$a : b :: c : d = (a \wedge d \equiv b \wedge c) \wedge (\neg a \wedge \neg d \equiv \neg b \wedge \neg c)$

"what a and d have in common b and c have it also, positively and negatively "

Piaget's *logical proportion*,

but he never related it to analogy !

$LP_{Piaget}(\alpha, \beta, \gamma, \delta) = (\alpha \wedge \beta \equiv \gamma \wedge \delta) \wedge (\neg \alpha \wedge \neg \beta \equiv \neg \gamma \wedge \neg \delta)$

Analogical proportions : just compare 2 items !

- Starting with 2 *distinct* Boolean vectors $\boldsymbol{a}$ and $\boldsymbol{d}$ it is possible to find 2 other vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ s.t. $\boldsymbol{a} : \boldsymbol{b} :: \boldsymbol{c} : \boldsymbol{d}$ holds componentwise :
- $Agr(\boldsymbol{a}, \boldsymbol{d})$: the set of indices where $\boldsymbol{a}$ and $\boldsymbol{d}$ agree
  $Dis(\boldsymbol{a}, \boldsymbol{d})$: the set of indices where $\boldsymbol{a}$ and $\boldsymbol{d}$ differ
  $\Rrightarrow$ 2 new vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ s.t. :
  - $\forall i \in Agr(\boldsymbol{a}, \boldsymbol{d})$, $a_i = b_i = c_i = d_i$ (all 1, or all 0)
  - $\forall i \in Dis(\boldsymbol{a}, \boldsymbol{d}) (b_i = a_i$ and $c_i = d_i)$
    or $(b_i = \neg a_i$ and $c_i = \neg d_i)$

$\boldsymbol{a} = 0110, \boldsymbol{d} = 0011$: $Agr(\boldsymbol{a},\boldsymbol{d}) = \{1, 3\}$ $Dis(\boldsymbol{a},\boldsymbol{d}) = \{2, 4\}$

$\boldsymbol{b} = 0111$ and $\boldsymbol{c} = 0010$ make $\boldsymbol{a} : \boldsymbol{b} :: \boldsymbol{c} : \boldsymbol{d}$ true

if $Dif(\boldsymbol{a}, \boldsymbol{d})$ contains at least 2 indices, equation

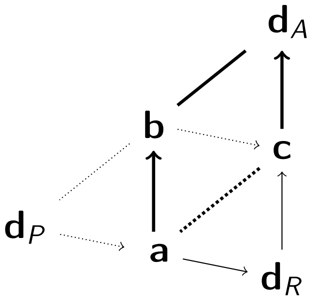
$a : x :: x' : d$ has solutions with $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{d}$ distinct

Two proportions associated with analogy

- *reverse analogy* : $\mathrm{Rev}(a, b, c, d) \triangleq$
  $((\neg a \wedge b) \equiv (c \wedge \neg d)) \wedge ((a \wedge \neg b) \equiv (\neg c \wedge d))$
  It reverses analogy into "b is to a as c is to d"
- *paralogy* : $\mathrm{Par}(a, b, c, d) \triangleq$
  $((a \wedge b) \equiv (c \wedge d)) \wedge ((\neg a \wedge \neg b) \equiv (\neg c \wedge \neg d))$
  what a and b have in common (positively or negatively), c and d have it also, and conversely
  $\mathrm{Rev}(b, a, c, d) \Leftrightarrow \mathrm{Ana}(a, b, c, d) \Leftrightarrow \mathrm{Par}(c, b, a, d))$
-

Ana				**Rev**				**Par**			
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	1	0	0	1	1	1	0	0	1
1	1	0	0	1	1	0	0	0	1	1	0
0	1	0	1	0	1	1	0	0	1	0	1
1	0	1	0	1	0	0	1	1	0	1	0

A geometric illustration

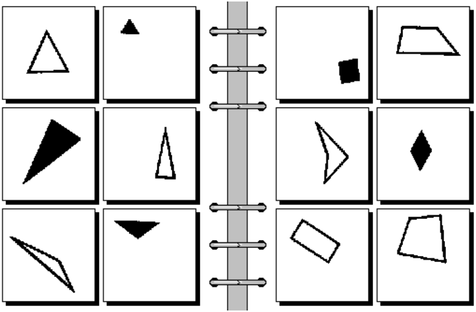


3 parallelograms

Inverse paralogy

- Switching the *positive* and *negative* similarity indicators for pair (c, d) in paralogy definition $\mathrm{Inv}(a, b, c, d) \triangleq$ $((a \wedge b) \equiv (\neg c \wedge \neg d)) \wedge ((\neg a \wedge \neg b) \equiv (c \wedge d))$
- "what a and b have in common, c and d do not have it and conversely" : a kind of "*orthogonality*"
- $A(a, b, c, d) \leftrightarrow I(a, \overline{b}, \overline{c}, d)$.
- Unique proportion *stable under any permutation* of 2 terms : $\mathrm{Inv}(a, b, c, d) \Leftrightarrow \mathrm{Inv}(b, a, c, d)$ $\Leftrightarrow \mathrm{Inv}(a, c, b, d) \Leftrightarrow \mathrm{Inv}(c, b, a, d)$
- *Bongard problems* easily expressed by Inv

Example of a Bongard problem



Transitivity

$T(a, b, c, d) \wedge T(c, d, e, f) \rightarrow T(a, b, e, f)$
A and P are transitive

R and I are **not** transitive,
but
$R(a, b, c, d) \wedge R(c, d, e, f) \rightarrow A(a, b, e, f)$
$I(a, b, c, d) \wedge I(c, d, e, f) \rightarrow P(a, b, e, f)$

Permutations

6 permutations exchanging the place of two elements

$p12, p13, p14, p23, p24, p34$

Proposition :

- **A** and **I** are the only logical proportions satisfying symmetry and being stable for permutation $p23$. The same result holds replacing $p23$ by $p14$.
- **P** and **I** are the only logical proportions satisfying symmetry and being stable for permutation $p12$. The same result holds replacing $p12$ by $p34$.
- **R** and **I** are the only logical proportions satisfying symmetry and being stable for permutation $p24$. The same result holds replacing $p13$ by $p24$.

Proposition :

***I* is the only logical proportion stable for *each* of the 6 permutations**

Proposition :

A is the unique proportion satisfying $T(a, b, a, b)$ and $p23$ (and thus also

Analogy,Reverse analogy,Paralogy,Inverse Paralogy

A				R				P				I			
0	0	0	0	0	0	0	0	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**
1	1	1	1	1	1	1	1	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**0**
0	0	1	1	0	0	1	1	0	1	1	0	0	1	1	0
1	1	0	0	1	1	0	0	1	0	0	1	1	0	0	1
0	**1**	**0**	**1**	**0**	**1**	**1**	**0**	0	1	0	1	0	1	0	1
1	**0**	**1**	**0**	**1**	**0**	**0**	**1**	1	0	1	0	1	0	1	0

Characteristic patterns

- 8 possible valuations for (a, b, c, d) never appear : they are of the form $x\ x\ x\ y$, $x\ x\ y\ x$, $x\ y\ x\ x$, or $y\ x\ x\ x$ with $x \neq y$ and $(x, y) \in \{0, 1\}^2$
- *characteristic pattern* : 2 lines of the table holds true as $(1 \equiv 1) \wedge (1 \equiv 1)$
- A **analogy** : **x y x y** same difference between a and b as between c and d
- R **reverse analogy** : **y x x y**
  differences between a and b and between c and d are in opposite directions
- P **paralogy** : **x x x x** what a and b have in common, c and d have it also
- I **inverse paralogy** : **x x y y**
  what a and b have in common, c and d do not have it, and conversely.
- the 4 other lines of the truth table are generated by the *characteristic patterns*
  of the *2 other proportions* that are *not opposed* to T

Patterns of the 4 homogeneous proportions : A summary

	Characteristic patterns	Missing patterns
Analogy	1010 and 0101	1001 and 0110
Reverse analogy	1001 and 0110	1010 and 0101
Paralogy	1111 and 0000	1100 and 0011
Inverse paralogy	1100 and 0011	1111 and 0000

Logical proportions

- Analogical proportion : a *comparison of comparisons*
  "a is to b as c is to d"
- A logical proportion $T(a, b, c, d)$ is the conjunction of 2 equivalences between indicators for (a, b) on one side and indicators for (c, d) on the other side
- Ex. : $((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((a \wedge b) \equiv (c \wedge d))$
  "a differs from b as c differs from d"
  and "a is similar to b as c is similar to d"

Mind it is not the analogical proportion !

What logical proportions have in common

- **120** *semantically distinct* proportions
- *All these proportions* share a remarkable property : they are true for exactly 6 patterns of values of *abcd* among $2^4 = 16$ possible values
previous example : true for
0000, 1111, 1010, 0101, 0001, and 0100

Logical proportions are quite rare among the $[{16 \atop 6}] =$ **8008** Boolean formulas involving 4 variables

Families of logical proportions

similarities : $s_1 = a \wedge b, s_2 = \neg a \wedge \neg b, s'_1 = c \wedge d, s'_2 = \neg c \wedge \neg d$
dissimilarities : $d_1 = a \wedge \neg b, d_2 = \neg a \wedge b, d'_1 = c \wedge \neg d, d'_2 = \neg c \wedge d$

- 4 homogeneous : 2 cond. $s_i = s'_k$ or 2 cond. $d_i = d'_k$
- 16 conditionals : $s_i = s'_k$ and $d_j = d'_l$
- 20 hybrids : $s_i = d'_k$ and $s_j = d'_l$
- 32 semi-hybrids : $s_i = s'_k$ or $d_j = d'_l$ and 1 hybrid cond.
- 48 degenerated : the same s_i (or s'_k, d_j, d'_l) in the 2 cond.

Conditional proportions

- of the form $s_i = s'_k$ and $d_j = d'_l$
- e.g., $((a \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge \overline{b}) \equiv (c \wedge \overline{d}))$
- a rule "if a then b" can be seen as a *three valued* entity called '**conditional object**', denoted $b|a$ (De Finetti).
  b|a is
  - *true* if $a \wedge b$ is true. The elements making it true are the examples of the rule "if a then b",
  - *false* if $a \wedge \overline{b}$ is true. The elements making it false are the counter-examples of the rule "if a then b",
  - *undefined* if $\overline{a}$ is true. The rule "if a then b" is then not applicable.
- so the above proportion may be denoted $b|a :: d|c$
  it expresses the semantical equivalence of the 2 rules "if a then b" and "if c then d" by stating that they have **the same examples**, i.e.
  $(a \wedge b) \equiv (c \wedge d))$
  and **the same counter-examples** $(a \wedge \overline{b}) \equiv (c \wedge \overline{d})$

4 noticeable hybrid proportions

$\equiv$ connectives link indicators of *different* kinds for (a, b) and for (c, d)

$$H_1(a,b,c,d) = (\neg a \wedge b \equiv \neg c \wedge \neg d) \wedge (a \wedge \neg b \equiv c \wedge d)$$

$$H_2(a,b,c,d) = (\neg a \wedge b \equiv c \wedge d) \wedge (a \wedge \neg b \equiv \neg c \wedge \neg d)$$

$$H_3(a,b,c,d) = (\neg a \wedge \neg b \equiv \neg c \wedge d) \wedge (a \wedge b \equiv c \wedge \neg d)$$

$$H_4(a,b,c,d) = (\neg a \wedge \neg b \equiv c \wedge \neg d) \wedge (a \wedge b \equiv \neg c \wedge d)$$

H₁				**H₂**				**H₃**				**H₄**			
1	1	1	0	1	1	1	0	1	1	1	0	1	1	0	1
0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0
1	1	0	1	1	1	0	1	1	0	1	1	1	0	1	1
0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0
1	0	1	1	0	1	1	1	0	1	1	1	0	1	1	1
0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0

- express that there is an intruder in $\{a, b, c, d\}$, which is not **a** (H_1), not **b** (H_2), not **c** (H_3), not **d** (H_4)
- solve puzzles of the type "Finding the odd one out"

The 4 heterogeneous proportions

- made of 3 of the pairs generated by the patterns $x\ x\ x\ y$, $x\ x\ y\ x$, $x\ y\ x\ x$, or $y\ x\ x\ x$

not a				not b				not c				not d			
0	0	0	1	0	0	0	1	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**0**
1	1	1	0	1	1	1	0	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**1**
0	0	1	0	0	0	1	0	0	1	1	1	0	1	1	1
1	1	0	1	1	1	0	1	1	0	0	0	1	0	0	0
0	**1**	**0**	**0**	**0**	**1**	**1**	**1**	0	1	0	0	0	1	0	0
1	**0**	**1**	**1**	**1**	**0**	**0**	**0**	1	0	1	1	1	0	1	1

- "among a, b, c, d, there is an **intruder** (which is true, or false, alone) which is not x" ($x = a, b, c$ or d)
  closely related with the idea of *spotting the odd one out*, or if we prefer of *picking the one that doesn't fit* among 4 items

Anomaly detection with heterogeneous proportions

Pick the odd one out in {bus, bicycle, car, truck}

	hasEngine	*canMove*	*canFly*	*canDrive*	*has4Wheels*
A : *bus*	1	1	0	1	1
B : *bicycle*	0	1	0	1	0
C : *car*	1	1	0	1	1
D : *truck*	1	1	0	1	1

For each $x \in a, b, c, d$ compute

$$N(x) = card(\{i \in [1, n] s.t. H_x(A_i, B_i, C_i, D_i) = 0\})$$

$$intruder = argmax_x N(x) \ (= \text{B})$$

Code independent logical proportions

- *code independent* property :

  $$T(a, b, c, d) \Leftrightarrow T(\neg a, \neg b, \neg c, \neg d)$$

- there only exist 8 logical proportions that satisfy it among the 120 ones

  they split into the 4 *homogeneous* proportions and the 4 *heterogeneous* logical proportions

Gradual properties

linearly ordered scale $\mathcal{L}$ may be an infinite chain $\mathcal{L} = [0, 1]$

a finite chain $\mathcal{L} = \{\alpha_0 = 0, \alpha_1, \cdots, \alpha_n = 1\}$ with $0 < \alpha_1 < \cdots < 1$

$\mathcal{L} = \{0, \alpha, 1\}$

- $A(a, b, c, d) = (a \wedge \neg b \equiv c \wedge \neg d) \wedge (\neg a \wedge b \equiv \neg c \wedge d)$
- central $\wedge$ equal to min ;
  $s \equiv t = \min(s \rightarrow_{Luka} t, t \rightarrow_{Luka} s) = 1 - |s - t|$;
  $s \wedge \neg t = \max(0, s - t) = 1 - (s \rightarrow_{Luka} t)$, i.e. $\wedge\neg$ is a bounded difference
- $A(a, b, c, d) = 1 - |\,(a - b) - (c - d)\,|$ if $a \geq b$ and $c \geq d$, or $a \leq b$ and $c \leq d$
  $A(a, b, c, d) = 1 - max(|a - b|, |c - d|)$ if $a \leq b$ and $c \geq d$, or $a \geq b$ and $c \leq d$
- fully true for 19 patterns in the 3-valued case : 9 following patterns
  $(1, 1, 1, 1)$; $(0, 0, 0, 0)$; $(\alpha, \alpha, \alpha, \alpha)$; $(1, 0, 1, 0)$; $(0, 1, 0, 1)$;
  $(1, \alpha, 1, \alpha)$; $(\alpha, 1, \alpha, 1)$; $(0, \alpha, 0, \alpha)$; $(\alpha, 0, \alpha, 0)$; $(1, 1, 0, 0)$;
  $(0, 0, 1, 1)$; $(1, 1, \alpha, \alpha)$; $(\alpha, \alpha, 1, 1)$; $(\alpha, \alpha, 0, 0)$; $(0, 0, \alpha, \alpha)$;
  $(1, \alpha, \alpha, 0)$; $(0, \alpha, \alpha, 1)$; $(\alpha, 1, 0, \alpha)$; $(\alpha, 0, 1, \alpha)$

Graded analogical proportion -1

- Attributes not necessarily Boolean :
  graded extensions of logical proportions of interest
- analogical proportion : 2 options that make sense

$$a : b ::_L c : d = \begin{cases} 1- \mid (a-b)-(c-d) \mid, \\ \quad \text{if } a \geq b \text{ and } c \geq d, \text{ or } a \leq b \text{ and } c \leq d \\ 1 - \max(\mid a-b \mid, \mid c-d \mid), \\ \quad \text{if } a \leq b \text{ and } c \geq d, \text{ or } a \geq b \text{ and } c \leq d \end{cases}$$

- Coincides with $a : b :: c : d$ on $\{0, 1\}$
- Equal to $\mathbf{1}$ if and only if $(a-b) = (c-d)$
- $a : b ::_L c : d = \mathbf{0}$ when the change inside one of (a, b) or (c, d) is *maximal*, while the other pair shows either no change, or an *opposite* change

Graded analogical proportion - 2

The second option :

- $a : b ::_C c : d =$
$$\min(1 - |\max(a, d) - \max(b, c)|, 1 - |\min(a, d) - \min(b, c)|)$$
- $a : b ::_C c : d = \mathbf{1}$
$\Leftrightarrow \min(a, d) = \min(b, c)$ and $\max(a, d) = \max(b, c)$
Only patterns (s, s, t, t), (s, t, s, t) (and (s, s, s, s)) enable the analogical proportion to be fully true ! !
- $a : b ::_L c : d = 1 \Rightarrow a : b ::_C c : d = 1$
- For instance, $0 : 0.5 ::_L 0.5 : 1 = 1$,
while $0 : 0.5 ::_C 0.5 : 1 = 0.5$

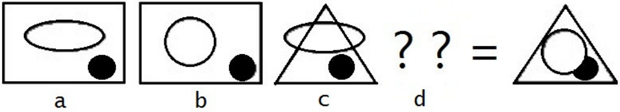
Analogical inference

- Equation $a : b :: c : x$ may not have a solution in $\mathbb{B}$
  neither $0 : 1 :: 1 : x$ nor $1 : 0 :: 0 : x$ have a solution
- when it exists (iff $(a \equiv b) \vee (a \equiv c)$ holds) it is unique
- $x = c \equiv (a \equiv b)$ (S. Klein 1982)
- Applies to Boolean vectors : look for $\vec{x} = (x_1, \cdots, x_n)$
  s.t. $\vec{a} : \vec{b} :: \vec{c} : \vec{x}$ holds :
  $\Rrightarrow$ n equations $a_i : b_i :: c_i : x_i$

analogical proportion solving process may be *creative*

$\vec{x} \neq \vec{a}$, $\vec{x} \neq \vec{b}$, $\vec{x} \neq \vec{c}$

Solving a puzzle



Example encoded with 5 Boolean predicates
hasRectangle(*hR*), *hasBlackDot*(*hBD*), *hasTriangle*(*hT*)
hasCircle(*hC*), *hasEllipse*(*hE*) (in that order)

	hR	*hBD*	*hT*	*hC*	*hE*
a	1	1	0	0	1
b	1	1	0	1	0
c	0	1	1	0	1
x	?	?	?	?	?
	0	1	1	1	0

The set counterpart

The analogical equation in D :

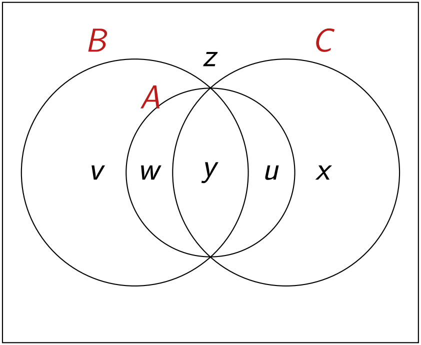
$$(A : B :: C : D)$$

has a solution iff

$$B \cap C \subseteq A \subseteq B \cup C$$

The solution is then unique and has the value

$$D = ((B \cup C) \backslash A) \cup (B \cap C)$$



$$D = v \cup x \cup y$$

General analogical inference

- $$\frac{\forall i \in \{1, ..., p\}, \quad a_i : b_i :: c_i : d_i \text{ holds}}{\forall j \in \{p+1, ..., n\}, \quad a_j : b_j :: c_j : d_j \text{ holds}}$$

  (Stroppa, Yvon, 2005)
- analogical reasoning amounts to finding completely informed triples $(\vec{a}, \vec{b}, \vec{c})$ suitable for inferring the missing value(s)
  of an incompletely informed item $(\vec{d})$
- if *several triples* leading to distinct conclusions
  a *voting* procedure may be used
- extends to gradual analogical proportions

Classification

- direct application of general inference principle
- one has to predict a class $cl(\vec{x})$ (viewed as a nominal attribute) for a new item $\vec{x}$
- successively applied to
  Boolean, nominal and numerical attributes
- analogical classifiers always give exact predictions when the classification process is governed by an affine Boolean function (which includes x-or functions) and *only in this case*

  does not prevent to get good results in other cases (as observed in practice)
- analogical proportions enforces a form of linearity

Logical reading of conformity

- $Even(a,b,c,d) =_{def} H_4(a,b,c,d) \vee Eq(a,b,c,d)$
  where $Eq(a,b,c,d) =_{def} (d=a) \wedge (d=b) \wedge (d=c)$
  $H_4(a,b,c,d) = 1$ if $(a,b,c,d) \in$
  $\{(1,1,0,1),(1,0,1,1),(0,1,1,1),(0,0,1,0),(0,1,0,0),(1,0,0,0)\}$
  $H_4(a,b,c,d) = 0$ otherwise.
  $H_4(a,b,c,d) = 1$ iff there is an *intruder* value $\neq d$ in $\{a,b,c,d\}$
- $Even(a,b,c,d)$ unchanged for any permutation of $\{a,b,c\}$
  $Even(a,b,c,d) = Even(\overline{a},\overline{b},\overline{c},\overline{d})$

Conformity of vector d with set $\mathcal{C}$

- for a feature i
  $Even(\mathcal{C}, d_i) = \Sigma_{(\vec{a},\vec{b},\vec{c})\in\mathcal{C}^3} Even(a_i, b_i, c_i, d_i)$

$Even(\mathcal{C}, d_i)$ high: there are few exceptions in $\mathcal{C}$, distinct from d_i

- wrt all features : $Even(\mathcal{C}, \vec{d}) =_{def} \Sigma_{i=1}^{n} Even(\mathcal{C}, d_i)$
- normalization : $Even^*(\mathcal{C}, \vec{d}) = \frac{|\mathcal{C}|}{\binom{|\mathcal{C}|}{3}} Even(\mathcal{C}, \vec{d})$

  where $Even^*(\mathcal{C}, \vec{d}) \in [0, n]$.

Classification algorithm : put the new item $\vec{d}$ in the class $\mathcal{C}$ that maximizes $Even^*(\mathcal{C}, \vec{d})$

Analogical prediction of preferences

$$\forall j \in [[1,n]], a_j^1 : b_j^1 :: c_j^1 : d_j^1 \text{ and } a_j^2 : b_j^2 :: c_j^2 : d_j^2$$

$$\vec{a^1} \preceq \vec{a^2} \,;\; \vec{b^1} \preceq \vec{b^2} \,;\; \vec{c^1} \preceq \vec{c^2}$$

— — — — — — — — — — — — — — —

$$\vec{d^1} \preceq \vec{d^2}$$

$$\forall j \in [[1,n]], a_j : b_j :: c_j : d_j$$

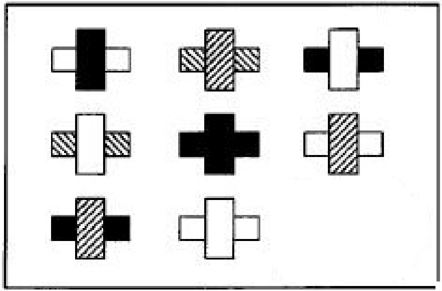
$$\vec{a} \preceq \vec{b},$$

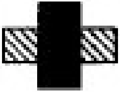
— — — — — — — — — — — —

$$\vec{c} \preceq \vec{d}$$

Raven tests : the solution is built, and not chosen

IQ tests : one is faced with a 3×3 matrix with 8 cells containing pictures ; one has to guess what is the right content of the empty 9th cell, *among 8 proposed solutions*





(a,b) :f(a,b) : :(c,d) :f(c,d)

$(pi[1,1], pi[1,2]) : pi[1,3] :: (pi[2,1], pi[2,2]) : pic[2,3]) ::$
$(pi[3,1], pic[3,2]) : pi[3,3])$

$(pi[1,1], pi[2,1]) : pi[3,1] :: (pi[1,2], pi[2,2]) : pi[3,2]) ::$
$(pi[1,3], pi[2,3]) : pi[3,3])$

	1	2	3
1	*WB*	*GG*	*BW*
2	*GW*	*BB*	*WG*
3	*BG*	*WW*	?i?ii

?i?ii = *GB*

- for the horizontal bars :

(W,G) : B : : (G, B) : W (horizontal analysis)
(W,G) : B : : (B,W) : ?i (horizontal analysis)
(W,G) : B : : (G, B) : W (vertical analysis)
(W,G) : B : : (B,W) : ?i (vertical analysis)

- for the vertical bars :

(B,G) : W : : (W, B) : G
(B,G) : W : : (G,W) : ?ii
(B,W) : G : : (G, B) : W
(B,W) : G : : (W,G) : ?ii

Predicting by analogy the expected value of a decision

- *generic scenario* : decision δ experienced in 2 situations sit_1, sit_2
  - in the presence or not of *special circumstances*,
  - leading to *good* or *bad* results

  depending on absence or presence of special circumstances

-

case	situation	special circumstances	decision	result
a	sit_1	*yes*	δ	*bad*
b	sit_1	*no*	δ	*good*
c	sit_2	*yes*	δ	*bad*
d	sit_2	*no*	$\boldsymbol{\delta}$	***good***

- **case-based** decision view : case d may be found *quite similar* to c

BUT a careful examination of cases a, b, c suggests another conclusion

- we may have in repository $\boldsymbol{R}$ a pair of cases (a', c') about sit_3 which may be a *counter-example* (or not) to what a, b, c suggest
  ■ *different triples* may lead to different predictions for the case d under consideration *majority vote* ? other methods ?

Modifying a decision by analogy

- a repertory of **recommended actions** in a *variety of circumstances*
  to take advantage of the creative capabilities of analogy
  for adapting a decision to the new situation :
  ☐ useful when decision has diverse options
- decisions : *Serve a tea with* or *without* **sugar**, with or *without* **milk**
  - in situation sit_1 with contraindication ($c\ i$), serve tea only
  - in situation sit_1 with no $c\ i$, serve tea with sugar
  - in situation sit_2 with $c\ i$ serve tea with milk

  What to do in situation sit_2 with no $c\ i$?

  Common sense suggests tea with sugar and milk
-

case	*situation*	*contraindication*	*decision*	*option*1	*option*2
a	sit_1	*yes*	δ	0	0
b	sit_1	*no*	δ	1	0
c	sit_2	*yes*	δ	0	1
d	sit_2	*no*	$\boldsymbol{\delta}$	**1**	**1**

Links and differences with case-based reasoning

- analogical proportion-based inference $\neq$ CBR :

takes advantage of **triples** for extrapolating conclusions while CBR exploits the similarity of the new case with stored cases considered one by one

- although "$< solution_1 >$ is to $< problem_1 >$ as $< solution_2 >$ is to $< problem_2 >$" may be regarded as an analogical proportion, the view presented here assumes that the vectors representing the 4 items in the analogical proportion "$\vec{a}$ is to $\vec{b}$ as $\vec{c}$ is to $\vec{d}$" are all defined on *the same* set of features

Analogical inequalities

- "a is to b **at least as much as** c is to d"
- $a : b \ll c : d =$
  $((a \wedge \neg b) \rightarrow (c \wedge \neg d)) \wedge ((\neg a \wedge b) \rightarrow (\neg c \wedge d))$
-
  - $a : b \ll a : b$
  - $a : b :: c : d \Rightarrow a : b \ll c : d$
  - $a : b :: c : d \Leftrightarrow ((a : b \ll c : d) \wedge (c : d \ll a : b))$
  - $(a : b \ll c : d) \Leftrightarrow (\neg a : \neg b \ll \neg c : \neg d)$
- $a : b \ll c : d$ holds true for the 6 patterns
  that makes analogical proportion true,
  plus the 4 patterns 0001, 0010, 1110, 1101
  $a : b \ll c : d$ true iff $(a : b :: c : d) \vee (a \equiv b)$ true
- When extended to the multiple-valued case, might

be of interest in *visual multiple-class categorization task*

for handling knowledge about semantic relationships

The analogical proportion-based inference view

As seen in analogy-based decision, we would rather suggest to exploit analogical proportions of the form

$$(<\mathit{problem}_1>,<\mathit{solution}_1>):(<\mathit{problem}_2>,<\mathit{solution}_2>)::(<\mathit{problem}_3>,<\mathit{solution}_3>):(<\mathit{problem}_0>,<\mathit{solution}_0>)$$

for extrapolating $< \mathit{solution}_0 >$ from 3 known cases $(\{(< \mathit{problem}_i >, < \mathit{solution}_i >) \mid i = 1, 3\})$ by solving $<\mathit{solution}_1>:<\mathit{solution}_2>::<\mathit{solution}_3>:<\mathit{solution}_0>$

where $< \mathit{solution}_0 >$ is unknown,

provided that $<\mathit{problem}_1>:<\mathit{problem}_2>::<\mathit{problem}_3>:<\mathit{problem}_0>$ holds

Analogical Proportion, Proportional Analogy, and Analogy

Analogical Proportion : four objets of the same kind

A foal is to a mare as a calf is to a cow.

fins are to scales as wings are to feathers.

Proportional Analogy : two couples of objects of the same kind

A foal is to equines as a calf is to bovines.

fins are to fishes as wings are to birds.

Analogy : Proportional Analogy shortened as a kind of metaphor

fins are the wings of fishes.

Metaphor

fins are like wings. life is a journey.

Formal Concept Analysis - Example of four concepts in WAP

	1	2	3	4
a			×	×
b	×		×	
c		×		×
d	×	×		

a	Foal
b	Mare
c	Calf
d	Cow

1	Female and adult
2	Bovine
3	Equine
4	Young

A Foal (Young Equine) is to a Mare (Female adult Equine)

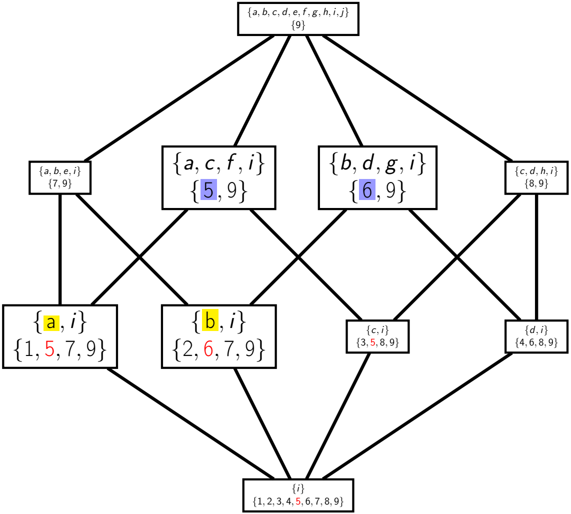
as

A Calf (Young Bovine) is to a Cow (Female adult Bovine)

Fins are to Fishes as Wings are to Birds.

i :	Part of an animal	9 :	Part of an animal
a :	Fins	5 :	Part of a Fish
b :	Wings	6 :	Part of a Bird
c :	Scales	7 :	Mobility part
d :	Feathers	8 :	Covering part
e :	Gills	1 :	Part of a Whale
f :	Beak	2 :	Part of a Bat
g :	Hooves	3 :	Part of a Snake
h :	Thick fur	4 :	Part of a Deinonychus

Fins are to Fishes as Wings are to Birds. *a* is to 5 as *b* is to 6.



	1	2	3	4	5	6	7	8	9
a	×				×		×		×
b		×				×	×		×
c			×		×			×	×
d				×		×		×	×
e					×				×
f						×			×
g							×		×
h								×	×
i	×	×	×	×	×	×	×	×	×

Concept Lattices

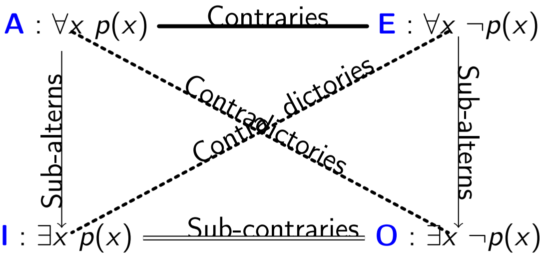
Weak Analogical Proportion between Concepts

$\{a, i\}$ $\{1, 5, 7, 9\}$	:	$\{b, i\}$ $\{2, 6, 7, 9\}$	::	$\{c, i\}$ $\{3, 5, 8, 9\}$	:	$\{d, i\}$ $\{4, 6, 8, 9\}$

Proportional Analogy : two objects, two attributes

a	is to	5	as	b	is to	6
a	$\Updownarrow$	5	$\Updownarrow\Updownarrow$	b	$\Updownarrow$	6
Fins	are to	Fishes	as	Wings	are to	Birds

The square of opposition



Aristotle

AffIrmo / NegO

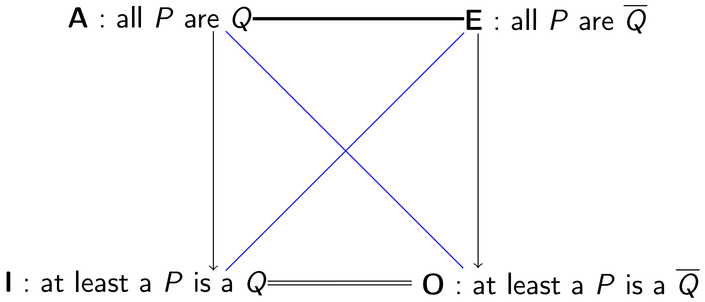
Another instance :

A : $\Box p$ **E** : $\Box\neg p$ **I** : $\Diamond p$ **O** : $\Diamond\neg p$

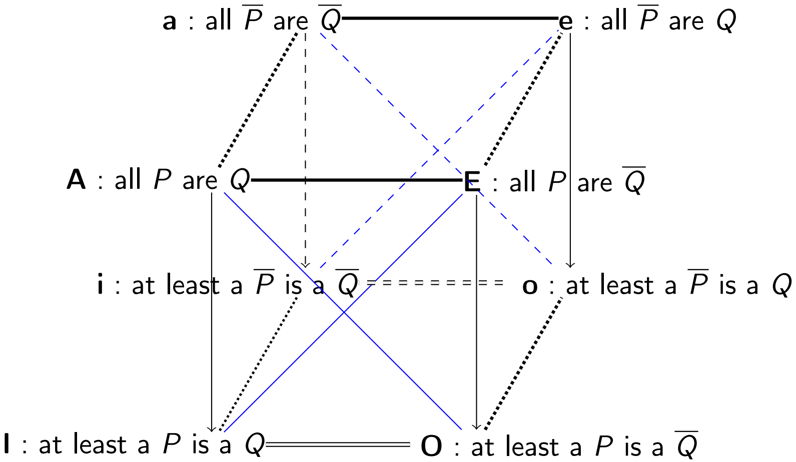
where $\Diamond p =_{def} \neg\Box\neg p$

(with $p \neq \bot, \top$)

From square to cube



Cube of opposition (after De Morgan)



6 squares ! 4 different structures ...

Piaget's group of logical transformations

logical formula $\phi = f(p, q, r, ...)$

- identity $I(\phi) = \phi$
- negation $N(\phi) = \neg\phi$
- reciprocation $R(\phi) = f(\neg p, \neg q, \neg r, ...)$
- correlation $C(\phi) = \neg f(\neg p, \neg q, \neg r, ...)$
- $N = RC$, $R = NC$, $C = NR$, et $I = NRC$

  Klein's group with 4 elements

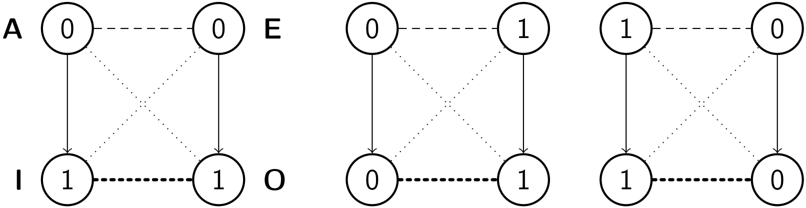
at work in the two *diagonal rectangles* **AaOo** and **EeIi**

Example : Propositional view of the analogical proportion

- Analogical proportion "a is to b as c is to d"
- $a : b :: c : d$
  $= ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$
- a **differs** from b as c **differs** from d, and conversely, b **differs** from a as d **differs** from c
- true for the following **6** patterns :
  $0 : 1 :: 0 : 1$ $\quad 1 : 0 :: 1 : 0$ $\quad 1 : 1 :: 0 : 0$
  $0 : 0 :: 1 : 1$ $\quad 1 : 1 :: 1 : 1$ $\quad 0 : 0 :: 0 : 0$
- both a matter of a *similarity* and *dissimilarity*

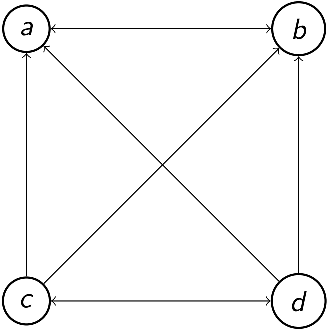
A valid square of oppositions makes an analogical proportion true !

- **A**, **E**, **I** ,**O** as the (Boolean-valued) vertices of a square of opposition **A** : **E** :: **I** : **O** form an analogical proportion when taken in this order since 0 : 0 :: 1 : 1, 0 : 1 :: 0 : 1 and 1 : 0 :: 1 : 0 are 3 of the 6 patterns that make an analogical proportion true
  3 valid squares :



- What about the 3 other patterns 1 : 1 :: 0 : 0, 1 : 1 :: 1 : 1 and 0 : 0 :: 0 : 0 ?

They make ... a square of agreement



The cube of opposition of comparison indicators

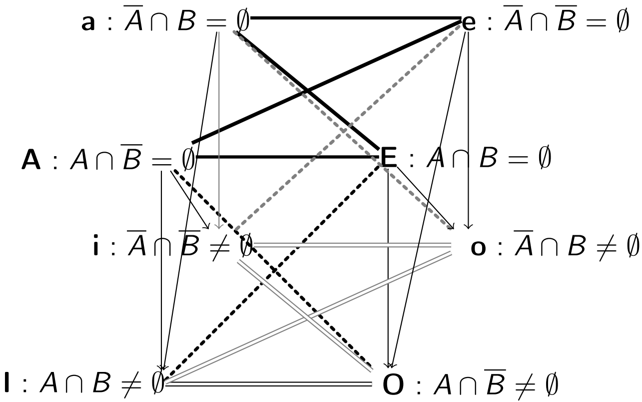


Figure – Cube of opposition of comparison indicators

Analogical proportion obtained from two pairs of mutually exclusive properties

$(q \wedge q' = \bot,\ r \wedge r' = \bot)$, and considering four items a, a', b, b' respectively described on the 4 properties (q, r, r', q') by $(1,1,0,0)$, $(1,0,1,0)$, $(0,1,0,1)$, $(0,0,1,1)$. For any vector component, $(a_i \wedge \neg a'_i \equiv b_i \wedge \neg b'_i) \wedge (\neg a_i \wedge a'_i \equiv \neg b_i \wedge b'_i)$ holds true, where $a = (a_1, a_2, a_3, a_4)$

a, a', b, b' make a kind of *square of opposition* (not the traditional one !) in the sense that a, a' satisfy q while b, b' satisfy q', and a, b satisfy r while a', b' satisfy r'. Diagonals ab' and $a'b$ link items that are *opposite* wrt properties q, q', r, r'.

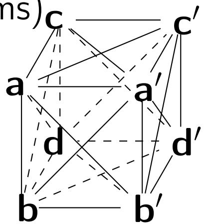
A new cube of opposition

Table 1	*s* (animal)	*p* (canid)	*q* (tame)	*r* (young)	*r'* (adult)	*q'* (wild)	*p'* (suidae)	*s'* (plant)
a *puppy*	1	1	1	1	0	0	0	0
a' *dog*	1	1	1	0	1	0	0	0
b *wolfcub*	1	1	0	1	0	1	0	0
b' *wolf*	1	1	0	0	1	1	0	0
c *piglet*	1	0	1	1	0	0	1	0
c' *pig*	1	0	1	0	1	0	1	0
d *yg.wd.boar*	1	0	0	1	0	1	1	0
d' *wildboar*	1	0	0	0	1	1	1	0

$a/a'/b/b'/c/c'/d/d'$

$= (a : a' :: b : b') \wedge (a' : b' :: c' : d') \wedge (a : a' :: c : c')$

The cube is associated to 128 syntactically distinct analogical proportions (including 32 degenerated ones with only 1 or 2 distinct items)



Conclusion

Present

- Analogy *formalized* in terms of *analogical proportion*
- It is both a matter of similarity and dissimilarity
- It belongs to the rich setting of logical proportions
- *Powerful* tool for different tasks : puzzle, IQ tests, creativity, ...
- *Competitive* results in classification and prediction
- *Shift of paradigm wrt similarity-based reasoing* : consider pairs of examples, can work with few data
- Provides a basis for interpolation and extrapolation

Future : This is just a beginning !

- *Theoretical issues* :
  - better understanding of why / how analogical classification works
  - (other) logical proportions : potential use ?
  - link / hybridization with other machine learning paradigms
  - joint use of analogical proportions and formal concept analysis
- Back to cognitive sciences

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