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A Note on the Representation of Clifford Algebra

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Abstract: In this note we construct explicit complex and real matrix representations for the generators of real Clifford algebra $C\ell_{p,q}$. The representation is based on Pauli matrices and has an elegant structure similar to the fractal geometry. We find two classes of representation, the normal representation and exceptional one. The normal representation is a large class of representation which can only be expanded into $4m+1$ dimension, but the exceptional representation can be expanded as generators of the next period. In the cases $p+q=4m$, the representation is unique in equivalent sense. These results are helpful for both theoretical analysis and practical calculation. The generators of Clifford algebra are the faithful basis of $p+q$ dimensional Minkowski space-time or Riemann space, and Clifford algebra converts the complicated relations in geometry into simple and concise algebraic operations, so the Riemann geometry expressed in Clifford algebra will be much simple and clear.

Keywords: Clifford algebra, Pauli matrix, gamma matrix, matrix representation

AMS classes: 15A66, 15A30, 15B99

I. INTRODUCTION

Clifford algebra was firstly defined by W. K. Clifford in 1878[1], which combines length concept of Hamilton's quaternion(1843, [2]) and Grassmann's exterior algebra(1844, [3]). The introduce of Dirac's spinor equation [4] has greatly promoted the research on Clifford algebra. Further development of the theory of Clifford algebras is associated with a number of famous mathematicians and physicists: R. Lipschitz, T. Vahlen, E. Cartan, E. Witt, C. Chevalley, M. Riesz and others [5, 6, 7].

Due to its excellent properties, Clifford algebra has gradually become a unified language and efficient tool of modern science, and is widely used in different branches of mathematics, physics

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and engineering[8, 9, 10, 11, 12, 13, 14, 15]. Theoretically we have some equivalent definitions for Clifford algebras[16, 17]. For the present purpose, we use the original definition of Clifford, which is based on the generators of basis.

Definition 1 *Suppose V is n-dimensional vector space over field $\mathbb{R}$, and its basis $\{e_1, e_2, \cdots, e_n\}$ satisfies the following algebraic rules*

$$e_a e_b + e_b e_a = 2\eta_{ab} I, \quad \eta_{ab} = diag(I_p, -I_q), \quad n = p + q. \tag{1.1}$$

Then the basis

$$\mathbf{e}_k \in \{I, e_a, e_{ab} = e_a e_b, e_{abc} = e_a e_b e_c, \cdots, e_{12\cdots n} = e_1 e_2 \cdots e_n | 1 \le a < b < c \le n\} \tag{1.2}$$

together with relation (1.1) and number multiplication $C = \sum_k c_k \mathbf{e}_k$ $(\forall c_k \in \mathbb{R})$ form a 2^n-dimensional real unital associative algebra, which is called real universal **Clifford algebra** *$C\ell_{p,q} = \bigoplus_{k=0}^{n} \otimes^k V$, and $C = \sum_k c_k \mathbf{e}_k$ is called* **Clifford number**.

For $C\ell_{0,2}$, we have $C = tI + xe_1 + ye_2 + ze_{12}$ with

$$e_1^2 = e_2^2 = e_{12}^2 = -1, \; e_1 e_2 = -e_2 e_1 = e_{12}, \; e_2 e_{12} = -e_{12} e_2 = e_1, \; e_{12} e_1 = -e_1 e_{12} = e_2. \tag{1.3}$$

By (1.3) we find C is equivalent to a quaternion, that is we have isomorphic relation $C\ell_{0,2} \cong \mathbb{H}$.

Similarly, for $C\ell_{2,0}$ we have $C = tI + xe_1 + ye_2 + ze_{12}$ with

$$e_1^2 = e_2^2 = e_{12}^2 = 1, \; e_1 e_2 = -e_2 e_1 = e_{12}, \; e_2 e_{12} = -e_{12} e_2 = -e_1, \; e_{12} e_1 = -e_1 e_{12} = -e_2. \tag{1.4}$$

By (1.4), the basis is equivalent to

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

Thus (1.5) means $C\ell_{2,0} \cong \mathrm{Mat}(2, \mathbb{R})$.

For general cases, the matrix representation of Clifford algebra is an old problem with a long history. As early as in 1908, Cartan got the following periodicity of 8[16, 17].

Theorem 1 *For real universal Clifford algebra $C\ell_{p,q}$, we have the following isomorphism*

$$C\ell_{p,q} \cong \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{R}), & \text{if} \quad \mathrm{mod}\,(p-q, 8) = 0, 2 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}), & \text{if} \quad \mathrm{mod}\,(p-q, 8) = 1 \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if} \quad \mathrm{mod}\,(p-q, 8) = 3, 7 \\ \mathrm{Mat}(2^{\frac{n-2}{2}}, \mathbb{H}), & \text{if} \quad \mathrm{mod}\,(p-q, 8) = 4, 6 \\ \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}), & \text{if} \quad \mathrm{mod}\,(p-q, 8) = 5. \end{cases} \tag{1.6}$$

In contrast with the above representation for a whole Clifford algebra, we find the representation of the generators $(e_1, e_2 \cdots e_n)$ is more fundamental and important in the practical applications. For example, $C\ell_{0,2} \cong \mathbb{H}$ is miraculous in mathematics, but it is strange and incomprehensible in geometry and physics, because the basis $e_{12} \in \otimes^2 V$ has different geometrical dimensions from that of e_1 and e_2. How can e_{12} take the same place of e_1 and e_2? Besides, $C\ell_{2,0} \ncong C\ell_{0,2}$ is also abnormal in physics, because the different signs of metric are simply caused by different conventions.

For the generators in $1+3$ dimensional space-time, Pauli got the following result[17, 18, 19].

Theorem 2 *Consider two sets of* 4×4 *complex matrices* $\gamma^\mu, \beta^\mu, (\mu = 0,1,2,3)$. *The 2 sets satisfy the following* $C\ell_{1,3}$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\eta^{\mu\nu}, \quad (\mu,\nu = 0,1,2,3). \tag{1.7}$$

Then there exists a unique (*up to multiplication by a complex constant*) *complex matrix* T *such that*

$$\gamma^\mu = T^{-1}\beta^\mu T, \qquad \mu = 0,1,2,3. \tag{1.8}$$

This theorem is generalized to the cases of real and complex Clifford algebras of even and odd dimensions in [19, 20].

In this note we construct explicit complex and real matrix representations for the generators of Clifford algebra. The problem is aroused from the discussion on the specificity of the $1+3$ dimensional Minkowski space-time with Prof. Rafal Ablamowicz and Prof. Dmitry Shirokov. They have done a number of researches on general representation theory of Clifford algebra[16, 17, 19, 20, 21, 22, 23, 24]. Many isomorphic or equivalent relations between Clifford algebra and matrices were provided. However, the representation of generators provides some new insights into the specific properties of the Minkowski space-time and the dynamics of fields[25, 26, 27], and it discloses that the 1+3 dimensional space-time is really special.

II. THE CANONICAL MATRIX REPRESENTATION FOR GENERATORS OF CLIFFORD ALGEBRAS

Denote Minkowski metric by $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$, Pauli matrices σ^μ by

$$\sigma^\mu \equiv \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{2.1}$$

$$\sigma^0 = \widetilde{\sigma}^0 = I, \qquad \widetilde{\sigma}^k = -\sigma^k, \quad (k = 1,2,3). \tag{2.2}$$

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Define γ^μ by

$$\gamma^\mu = \begin{pmatrix} 0 & \widetilde{\vartheta}^\mu \\ \vartheta^\mu & 0 \end{pmatrix} \equiv \Gamma^\mu(m), \quad \vartheta_\mu = \mathrm{diag}(\overbrace{\sigma_\mu, \sigma_\mu, \cdots, \sigma_\mu}^{m}), \quad \widetilde{\vartheta}_\mu = \mathrm{diag}(\overbrace{\widetilde{\sigma}_\mu, \widetilde{\sigma}_\mu, \cdots, \widetilde{\sigma}_\mu}^{m}). \tag{2.3}$$

which forms the generator or grade-1 basis of Clifford algebra $C\ell_{1,3}$. To denote γ^μ by $\Gamma^\mu(m)$ is for the convenience of representation of high dimensional Clifford algebra. For any matrices C^μ satisfying $C\ell_{1,3}$ Clifford algebra, we have[25, 26]

Theorem 3 *Assuming the matrices C^μ satisfy anti-commutative relation*

$$C^\mu C^\nu + C^\nu C^\mu = 2\eta^{\mu\nu}, \tag{2.4}$$

then there is a natural number m and an invertible matrix K, such that $K^{-1}C^\mu K = \gamma^\mu$. This means in equivalent sense, we have unique representation (2.3) for generator of $C\ell_{1,3}$.

In this note, we derive complex representation of $C\ell(p,q)$ based on Thm.3, and then derived the real representations according to the complex representations.

Theorem 4 *Let*

$$\gamma^5 = i\mathrm{diag}(E, -E), \quad E \equiv \mathrm{diag}(I_{2k}, -I_{2l}), \quad k + l = n. \tag{2.5}$$

Other $\gamma^\mu, (\mu \le 3)$ are given by (2.3). Then the generators of Clifford algebra $C\ell_{1,4}$ are equivalent to $\forall \gamma^\mu$, $(\mu = 0, 1, 2, 3, 5)$.

Proof. Since we have gotten the unique generator γ^μ for $C\ell_{1,3}$, so we only need to derive γ^5 for $C\ell_{1,4}$. Assuming $4n \times 4n$ matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.6}$$

satisfies $\gamma^\mu X + X\gamma^\mu = 0$, $(\forall \mu = 0, 1, 2, 3)$. By $\gamma^0 X + X\gamma^0 = 0$ we get $D = -A$, $C = -B$. By $\gamma^k X + X\gamma^k = 0$ we get

$$\vartheta^k B + B\vartheta^k = 0, \qquad \vartheta^k A - A\vartheta^k = 0. \tag{2.7}$$

By the first equation we get $B = 0$, and then $X = \mathrm{diag}(A, -A)$. Assuming $A = (A_{ab})$, where $\forall A_{ab}$ are 2×2 matrices. Then by the second equation in (2.7) we get block matrix $A = (K_{ab}I_2) \equiv K \otimes I_2$, where K is a $n \times n$ matrix to be determined. In this paper, the direct product $\otimes$ of matrices is defined as Kronecker product.

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For $X^2 = I_{4n}$ we get $A^2 = I_{2n}$, and then $K^2 = I_n$. Therefore, there exists an invertible $n \times n$ matrix q such that $q^{-1}Kq = \mathrm{diag}(I_k, -I_l)$. Let $2n \times 2n$ block matrix $Q = q \otimes I_2$, we have

$$Q^{-1}AQ = \mathrm{diag}(I_{2k}, -I_{2l}) \equiv E, \qquad \vartheta^k Q = Q\vartheta^k. \tag{2.8}$$

Let $\gamma^5 = i\mathrm{diag}(E, -E)$, then all $\{\gamma^\mu, \mu = 0,1,2,3,5\}$ constitute the generators of $C\ell_{1,4}$. We prove the theorem.

Again assuming matrix X_1 satisfies $\gamma^\mu X_1 + X_1\gamma^\mu = 0$. By the above proof we learn that $X_1 = \mathrm{diag}(A_1, -A_1)$. Solving $X_1\gamma^5 + \gamma^5 X_1 = 0$, we get $X_1 = 0$ if $k \neq l$. In this cases we can not expand the derived γ^μ as matrix representation for $C\ell_{1,5}$. But in the case $k = l$, we find $X_1^2 = -I$ have solution, and A_1 has a structure of $i\gamma^1$. Then the construction of generators can proceed. In this case, we have the following theorem.

Theorem 5 *Suppose that $8n \times 8n$ matrices $A^\mu = diag(C^\mu, -C^\mu), \mu, \nu \in \{0,1,2,3\}$ satisfy*

$$A^\mu A^\nu + A^\nu A^\mu = 2\eta^{\mu\nu}, \qquad A^\mu \gamma^\nu_{2n} + \gamma^\nu_{2n} A^\mu = 0, \tag{2.9}$$

then there is an $8n \times 8n$ matrix K, such that

$$K^{-1}A^\mu K = diag(\gamma^\mu_n, -\gamma^\mu_n) \equiv \beta^\mu_{2n}, \qquad K\gamma^\mu_{2n} = \gamma^\mu_{2n}K. \tag{2.10}$$

In which γ^μ_n means n σ^μ in ϑ^μ. Then $\{\gamma^\mu_{2n}, \beta^\mu_{2n}\}$ constitute all generators of $C\ell_{2,6}$.

Proof. By $K\gamma^\mu_{2n} = \gamma^\mu_{2n}K$ we get $K = \mathrm{diag}(L, L)$ and $L = (L_{ab}I_2) \equiv \widetilde{L} \otimes I_2$, where $\widetilde{L} = (L_{ab})$ is a $2n \times 2n$ matrix to be determined. By (2.9) we have $C^\mu = (C^\mu_{ab}I_2) \equiv \widetilde{C}^\mu \otimes I_2$. Then $\widetilde{C}^\mu$ also satisfies $C\ell_{1,3}$ Clifford algebra. By Thm.3, there is a matric $\widetilde{L}$ such that $\widetilde{L}^{-1}\widetilde{C}^\mu\widetilde{L} = \gamma^\mu$. Then this K proves the theorem.

Since $(i\gamma^\mu)^2 = -(\gamma^\mu)^2$, instead of $C\ell_{p,q}$ we directly use $C\ell_{p+q}$ in some cases for complex representation. Similarly to the case $C\ell_4$, in equivalent sense we have unique matrix representation for $C\ell_8$. For $C\ell_9$, besides the generators constructed by the above Thm.5, we need another generator γ^9. By calculation similar to (2.8), we find $\gamma^9 = \mathrm{diag}(E, -E, -E, E)$ and $E = \mathrm{diag}(I_{2k}, -I_{2l}), k + l = n$. For $C\ell_{10}$, we also have two essentially different cases similar to $C\ell_6$. If $k \neq l$, γ^9 and the above generators cannot be expanded as generators of $C\ell_{10}$. We call this representation as **normal representation**. Clearly $k \neq l$ is a large class of representations which are not definitely equivalent. In the case of $k = l$, the above generators can be uniquely expanded as generators for $C\ell_{12}$. We call this representation as **exceptional representation**. The other generators are given by

$$\alpha^\mu_{4n} = \mathrm{diag}(\gamma^\mu_n, -\gamma^\mu_n, -\gamma^\mu_n, \gamma^\mu_n) \otimes I_4. \tag{2.11}$$

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In order to express the general representation of generators, we introduce some simple notations. I_m stands for $m\times m$ unit matrix. For any matrix $A=(A_{ab})$, denote block matrix

$$A\otimes I_m=(A_{ab}I_m),\qquad [A,B,C,\cdots]=\mathrm{diag}(A,B,C,\cdots). \tag{2.12}$$

Obviously, we have $I_2\otimes I_2=I_4$, $I_2\otimes I_2\otimes I_2=I_8$ and so on. In what follows, we use $\Gamma^\mu(m)$ defined in (2.3). For $\mu\in\{0,1,2,3\}$, $\Gamma^\mu(m)$ is $4m\times 4m$ matrix, which constitute the generator of $C\ell_{1,3}$. Similarly to the above proofs, we can check the following theorem by method of induction.

Theorem 6 1° *In equivalent sense, for $C\ell_{4m}$, the matrix representation of generators is uniquely given by*

$$\begin{gathered}\{\Gamma^\mu(n),\quad \left[\Gamma^\mu\left(\tfrac{n}{2^2}\right),\ -\Gamma^\mu\left(\tfrac{n}{2^2}\right)\right]\otimes I_2,\\ \left[\left[\Gamma^\mu\left(\tfrac{n}{2^4}\right),-\Gamma^\mu\left(\tfrac{n}{2^4}\right)\right],\ -\left[\Gamma^\mu\left(\tfrac{n}{2^4}\right),-\Gamma^\mu\left(\tfrac{n}{2^4}\right)\right]\right]\otimes I_{2^2},\\ \left[\Gamma^\mu\left(\tfrac{n}{2^6}\right),-\Gamma^\mu\left(\tfrac{n}{2^6}\right),-\Gamma^\mu\left(\tfrac{n}{2^6}\right),\Gamma^\mu\left(\tfrac{n}{2^6}\right),-\Gamma^\mu\left(\tfrac{n}{2^6}\right),\Gamma^\mu\left(\tfrac{n}{2^6}\right),\Gamma^\mu\left(\tfrac{n}{2^6}\right),-\Gamma^\mu\left(\tfrac{n}{2^6}\right)\right]\otimes I_{2^3},\cdots\}.\end{gathered} \tag{2.13}$$

In which $n=2^{m-1}N$, N is any given positive integer. All matrices are $2^{m+1}N\times 2^{m+1}N$ type.

2° *For $C\ell_{4m+1}$, besides (2.13) we have another real generator*

$$\gamma^{4m+1}=[E,-E,-E,E,-E,E,E,-E\cdots],\qquad E=[I_{2k},-I_{2l}]. \tag{2.14}$$

If and only if $k=l$, this representation can be uniquely expanded as generators of $C\ell 4m+4$.

3° *For any $C\ell_{p,q}$, $\{p,q|p+q\le 4m,\ \mod (p+q,4)\neq 1\}$, the combination of $p+q$ linear independent generators $\{\gamma^\mu,i\gamma^\nu\}$ taking from (2.13) constitutes the complete set of generators. In the case $\{p,q|p+q\le 4m,\ \mod (p+q,4)=1\}$, besides the combination of $\{\gamma^\mu,i\gamma^\nu\}$, we have another normal representation of generator taking the form (2.14) with $k\neq l$.*

4° *For $C\ell_m,(m<4)$, we have another 2×2 Pauli matrix representation for its generators $\{\sigma^1,\sigma^2,\sigma^3\}$.*

Then we get all complex matrix representations for generators of real $C\ell_{p,q}$ explicitly.

The real representation of $C\ell_{p,q}$ can be easily constructed from the above complex representation. In order to get the real representation, we should classify the generators derived above. Let $\mathbf{G}_c(n)$ stand for any one set of all complex generators of $C\ell_n$ given in Thm.6, exceptional representation or normal one, and set the coefficients before all σ^μ and $\widetilde{\sigma}^\mu$ as 1 or i. Denote $\mathbf{G}_{c+}$ stands for the set of complex generators of $C\ell_{n,0}$ and $\mathbf{G}_{c-}$ for the set of complex generators of $C\ell_{0,n}$. Then we have

$$\mathbf{G}_c=\mathbf{G}_{c+}\cup\mathbf{G}_{c-},\qquad \mathbf{G}_{c-}\cong i\mathbf{G}_{c+}. \tag{2.15}$$

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By the construction of generators, we have only two kinds of γ^μ matrices. One is the matrix with real nonzero elements, and the other is that with imaginary nonzero elements. This is because all nonzero elements of σ^2 are imaginary but all other $\sigma^\mu(\forall\mu\neq 2)$ are real. Again assume

$$\mathbf{G}_{c+}=\mathbf{G}_r\cup\mathbf{G}_i,\qquad \mathbf{G}_r=\{\gamma_r^\mu|\gamma_r^\mu \text{ is real}\},\qquad \mathbf{G}_i=\{\gamma_i^\mu|\gamma_i^\mu \text{ is imaginary}\}. \tag{2.16}$$

Denote $J_2=i\sigma^2$, we have $J_2^2=-I_2$. J_2 becomes the real matrix representation for imaginary unit i. Using the direct products of complex generators with (I_2,J_2), we can easily construct the real representation of all generators for $C\ell_{p,q}$ from $\mathbf{G}_{c+}$ as follows.

Theorem 7 1° *For $C\ell_{n,0}$, we have real matrix representation of generators as*

$$\mathbf{G}_{r+}=\{\gamma^\mu\otimes I_2 \;(\text{if } \gamma^\mu\in\mathbf{G}_r);\;\; i\gamma^\nu\otimes J_2\; (\text{if } \gamma^\nu\in\mathbf{G}_i)\}. \tag{2.17}$$

2° *For $C\ell_{0,n}$, we have real matrix representation of generators as*

$$\mathbf{G}_{r-}=\{\gamma^\mu\otimes J_2|\gamma^\mu\in\mathbf{G}_{r+}\}. \tag{2.18}$$

3° *For $C\ell_{p,q}$, we have real matrix representation of generators as*

$$\mathbf{G}_r=\left\{\Gamma_+^{\mu_a},\Gamma_-^{\nu_b}\;\middle|\;\begin{array}{l}\Gamma_+^{\mu_a}=\gamma^{\mu_a}\in\mathbf{G}_{r+},\;(a=1,2,\cdots,p)\\ \Gamma_-^{\nu_b}=\gamma^{\nu_b}\in\mathbf{G}_{r-},\;(b=1,2,\cdots,q)\end{array}\right\}. \tag{2.19}$$

Obviously we have $C_n^pC_n^q=(C_n^p)^2$ choices for the real generators of $C\ell_{p,q}$ from each complex representation.

Proof. By calculating rules of block matrix, it is easy to check the following relations

$$(\gamma^\mu\otimes I_2)(\gamma^\nu\otimes J_2)+(\gamma^\nu\otimes J_2)(\gamma^\mu\otimes I_2) = (\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)\otimes J_2, \tag{2.20}$$

$$(\gamma^\mu\otimes J_2)(\gamma^\nu\otimes J_2)+(\gamma^\nu\otimes J_2)(\gamma^\mu\otimes J_2) = -(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)\otimes I_2. \tag{2.21}$$

By these relations, Thm.7 becomes a direct result of Thm.6.

For example, we have 4×4 real matrix representation for generators of $C\ell_{0,3}$ as

$$i\{\sigma^1,\sigma^2,\sigma^3\}\cong\{\sigma^1\otimes J_2, i\sigma^2\otimes I_2,\sigma^3\otimes J_2\}\equiv\{\Sigma^1,\Sigma^2,\Sigma^3\}=$$
$$\left\{\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\right\}. \tag{2.22}$$

It is easy to check

$$\Sigma^k\Sigma^l+\Sigma^l\Sigma^k=-2\delta^{kl},\qquad \Sigma^k\Sigma^l-\Sigma^l\Sigma^k=2\epsilon^{klm}\Sigma_m. \tag{2.23}$$

III. DISCUSSION AND CONCLUSION

For different purpose, Clifford algebra has several different definitions, and 5 kinds are listed in references [16, 17, 28]. The following definition is the most commonly used in theoretical analysis.

Definition 2 *Suppose (V, Q) is an $n < \infty$ dimensional quadratic space over field $\mathbb{F}$, and A is a unital associative algebra. There is an injective mapping $J : V \to A$ such that*

i) $I \notin J(V)$;

ii) $(J(\mathbf{x}))^2 = Q(\mathbf{x})I, \quad \forall \mathbf{x} \in V$;

iii) $J(V)$ generates A.

Then the set A together with mapping J is called Clifford algebra $C\ell(V, Q)$ over $\mathbb{F}$.

The above definition includes the case of degenerate Clifford algebra $C\ell_{p,q,r}$. For example, if $Q(\mathbf{x}) = 0$, the Clifford algebra $C\ell_0$ becomes Grassmann algebra. In the non-degenerate case, if the standard orthogonal basis of V is introduced, we can derive Definition 1. The definition based on the quotient algebra of tensor algebra in V is introduced by Chevalley[16, 17], but it is too abstract for common readers. In the author's opinion, the most efficient and convenient definition of Clifford algebra should be as follows.

Definition 3 *Assume the element of an $n = p + q$ dimensional space-time $\mathbb{M}^{p,q}$ over $\mathbb{R}$ is described by*

$$d\mathbf{x} = \gamma_\mu dx^\mu = \gamma^\mu dx_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{3.1}$$

where γ_a is the local orthogonal frame and γ^a the coframe. The space-time is endowed with distance $ds = |d\mathbf{x}|$ and oriented volumes dV_k calculated by

$$d\mathbf{x}^2 = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\delta X^a \delta X^b, \tag{3.2}$$

$$dV_k = d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_k = \gamma_{\mu\nu\cdots\omega} dx_1^\mu dx_2^\nu \cdots dx_k^\omega, \quad (1 \le k \le n), \tag{3.3}$$

in which Minkowski metric $(\eta_{ab}) = \mathrm{diag}(I_p, -I_q)$, and Grassmann basis $\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k(\mathbb{M}^{p,q})$. Then the **Clifford-Grassmann number**

$$C = c_0 I + c_\mu \gamma^\mu + c_{\mu\nu}\gamma^{\mu\nu} + \cdots + c_{12\cdots n}\gamma^{12\cdots n}, \quad (\forall c_k \in \mathbb{R}) \tag{3.4}$$

together with multiplication rule of basis given in (3.2) and associativity define the 2^n-dimensional real universal **geometric algebra** *$C\ell_{p,q}$.*

In some sense, Definition 1 is for all scientists, Definition 2 is for mathematicians, and the definition of Chevalley is only for algebraists. However, the Definition 3 can be well understood by

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all common readers including high school students, which directly connects normal intelligence with the deepest wisdom of Nature[26, 27]. From the geometric and physical point of view, the definition of Clifford basis in Definition 1 is inappropriate, because in the case of non-orthogonal basis, $e_{12} = e_1 \odot e_2 + e_1 \wedge e_2 \in \Lambda^0 \cup \Lambda^2$ is a mixture with different dimensions, and the geometric meaning which represents is not clear. But the Grassmann basis in Definition 3 is not the case, where each term has a specific geometric meaning and has covariant form under coordinate transformation. The coefficients in (3.4) are all tensors with clear geometric and physical meanings.

To use the Definition 3, the transformation law of Grassmann basis under Clifford product is important. Now we discuss it briefly.

Theorem 8 *For γ^μ and $\gamma^{\theta_1\theta_2\cdots\theta_k} \in \Lambda^k$, we have*

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} = \left(g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} - g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + (-)^{k+1}g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}\right) + \gamma^{\mu\theta_1\cdots\theta_k}, \tag{3.5}$$

$$\gamma^{\theta_1\theta_2\cdots\theta_k}\gamma^\mu = \left((-)^{k+1}g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} + (-)^k g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}\right) + \gamma^{\theta_1\cdots\theta_k\mu}. \tag{3.6}$$

Proof. Clearly $\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} \in \Lambda^{k-1} \cup \Lambda^{k+1}$, so we have

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} = a_1 g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} + a_2 g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + a_k g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}} + A\gamma^{\mu\theta_1\cdots\theta_k}. \tag{3.7}$$

Permuting the indices θ_1 and θ_2, we find $a_2 = -a_1$. Let $\mu = \theta_1$, we get $a_1 = 1$. Check the monomial in exterior product, we get $A = 1$. Thus we prove (3.5). In like manner we prove (3.6).

In the case of multivectors $\gamma^{\mu_1\mu_2\cdots\mu_l}\gamma^{\theta_1\theta_2\cdots\theta_k}$, we can define **multi-inner product $\mathbf{A} \odot^k \mathbf{B}$** as follows[29]

$$\gamma^{\mu\nu} \odot \gamma^{\alpha\beta} = g^{\mu\beta}\gamma^{\nu\alpha} - g^{\mu\alpha}\gamma^{\nu\beta} + g^{\nu\alpha}\gamma^{\mu\beta} - g^{\nu\beta}\gamma^{\mu\alpha}, \tag{3.8}$$

$$\gamma^{\mu\nu} \odot^2 \gamma^{\alpha\beta} = g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}, \qquad \cdots \tag{3.9}$$

For example, we have

$$\gamma^{\mu\nu}\gamma^{\alpha\beta} = \gamma^{\mu\nu} \odot^2 \gamma^{\alpha\beta} + \gamma^{\mu\nu} \odot \gamma^{\alpha\beta} + \gamma^{\mu\nu\alpha\beta}. \tag{3.10}$$

The derivation of the paper is constructive, so it can be used for both theoretical analysis and practical calculation. From the results we find $C\ell_{1,3}$ has specificity and takes fundamental place in Clifford algebra theory. By the above representations of generators, we can easily get the relations between bases such as $\gamma^{abc} = i\epsilon^{abcd}\gamma_d\gamma^5$ in $C\ell_{1,3}$. The generators of Clifford algebra are the faithful bases of $p+q$ dimensional Minkowski space-time or pseudo-Riemann space as shown in (3.1)-(3.4), and Clifford algebra converts the complicated relations in geometry into simple and concise algebraic calculus[27], so the Riemann geometry expressed in Clifford algebra will be much simpler and clearer than current version.

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