WORD PROBLEMS AND LINEAR FUNCTIONS 4/1

SHOPPING FOR PANTS You want to buy a pair of pants. You are thinking of buying them at Premium Pants at the regular price. That is because Fashion Passion sells the same pair of pants for one third more. But Fashion Passion is having a sale where everything is one third off. Where should you buy your pants ?

HAIR You have two daughters, Ruta and Lina. Ruta is born without any hair and her hair grows at a rate of one inch per year. Lina is born with hair that is one inch long and it grows at a rate of two inches per year. They both have hair that is twelve inches long.

a) Let t be time in years, with t=0 when Ruta is born. Define a function R which gives the length of Ruta's hair (in inches) at time t.

b) How old is Ruta ?

c) Let s be time in years, with s=0 when Lina is born. Define a function L which gives the length of Lina's hair (in inches) at time s.

d) How old is Lina ?

e) Now define a function H which gives the length of Lina's hair (in inches) at time t, where t=0 when *Ruta* is born. Graph the functions R and H on the same graph.

CALCULATING TAX You are a single person who is filing taxes with the 1040 EZ form. Suppose you earned no interest during the year, but had total wages W (in dollars). Of this money, 5,550 is not taxable, whereas the rest is taxed at a rate of 15%.

a) Define a function T (in dollars) which determines the amount of tax for the year.

b) How much tax must you pay if you earned 10,000 dollars ?

c) If you did earn 10,000 dollars, how much more must you earn for your taxes to double ?

d) How much must you have earned if you need not pay any tax at all ?

SHOPPING FOR PANTS II Lo and behold, you find a coupon in the newspaper. Premium Pants offers five dollars worth of socks when you purchase any pair of pants. Does this affect your decision ?

THE OLD CAR AND THE NEW CAR Your family has two cars, an old car and a new car. Both cars take off together on a trip. They both start off with full gas tanks. They both maintain a speed of 60 miles per hour. The old car has a 15 gallon gas tank and gets 12 miles to the gallon. The new car has a 10 gallon gas tank and gets 30 miles to the gallon.

a) For the old car, define a function y (in gallons) which gives the amount of gas in the tank at time t (in hours).

b) When is the gas tank of the old car empty ?

c) For the new car, define a function y (in gallons) which gives the amount of gas in the tank at time t (in hours).

d) When is the gas tank of the new car empty ?

e) When do both cars have the same amount of gas in their tanks ?

RIDDLE Six years ago I was three times younger than my father is now. In four years, my father will be twice as old as me. How old am I ?

HOMEWORK DUE 4/9: 7.1 #36, #52. 7.2 #16, #18, #20, #42, #47, #50. 3.2 #8. 3.4 #16.

TWO LINEAR EQUATIONS IN TWO UNKNOWNS 4/3

ONE WISE MAN AND TWO WISE GUYS

Liu Yuan-te is marching onward to Shu and seeks the help of a wise man. At the top of the highest mountain he finds three old men sitting under a tree. They all start talking at once. The first says, "This tree is twice my age, but fifty years ago it was three times my age." The second says, "This tree is twice my age, but when I was half my age, then it was three times my age." The third says, "This tree is twice my age, but when it was half its age, it was fifty years younger than I am now."
Which is the wise man ?

GENERAL PROBLEM Let x and y be variables. Let a, b, c, d, e, f be constants. Given the equations $ax + by = e$ and $cx + dy = f$, solve for x and y.

a) When is there exactly one solution and what is it ?

b) When are there infinitely many solutions and what are they ?

c) When is there no solution ?

COUNTING TRAINS The towns of Attica, Bakersville, and Concord are connected by trains. The trains only run during the day; they leave in the morning and arrive in the evening. Every morning half of the trains in Attica leave for Bakersville and the other half leave for Concord. Every morning all of the trains in Bakersville leave for Concord. Every morning all of the trains in Concord leave for Attica. There are five trains in all. At night, when none of the trains are running, how many are there in each of the respective towns ?

DEMOCRATIC CONVENTION At the Democratic convention in June, every delegate has been pledged to one of three candidates: Bill Clinton, Jerry Brown, and Paul Tsongas. As none of the three candidates has a convincing majority, the party leadership decides to free all of the delegates so that they can vote for whoever they like. Because of this, Bill Clinton enlists the support of 20% of Jerry Brown's old delegates and 20% of Paul Tsongas' old delegates. Jerry Brown enlists the support of 10% of Bill Clinton's old delegates and 20% of Paul Tsongas' old delegates. Paul Tsongas enlists the support of 10% of Bill Clinton's old delegates and 10% of Jerry Brown's old delegates. After this a new vote is taken, with the results that Bill Clinton wins 50%, Jerry Brown wins 30%, and Paul Tsongas wins 20% of the total vote.

a) What percentage of the delegates did each of the three candidates have going into the convention ?

b) Who gained delegates and who lost delegates as a result of the party leadership's decision ?

THREE LINEAR EQUATIONS IN THREE VARIABLES 4/6

GAUSSIAN ELIMINATION relies on the following three *elementary operations:*

1. Multiply both sides of an equation by a nonzero constant.
2. Interchange the order in which two equations of a system are listed.
3. To one equation add a multiple of another equation in the system.

KEEPING IN TOUCH A daughter, a mother, and a grandmother write letters to each other regularly because they live in different cities. During the course of a year each writes as many letters as she receives (but not necessarily from the same people). The daughter writes three times more often to the mother than to the grandmother. The mother writes twice as often to the grandmother than to the daughter. The grandmother writes two letters to the daughter for every three letters she writes to the mother.

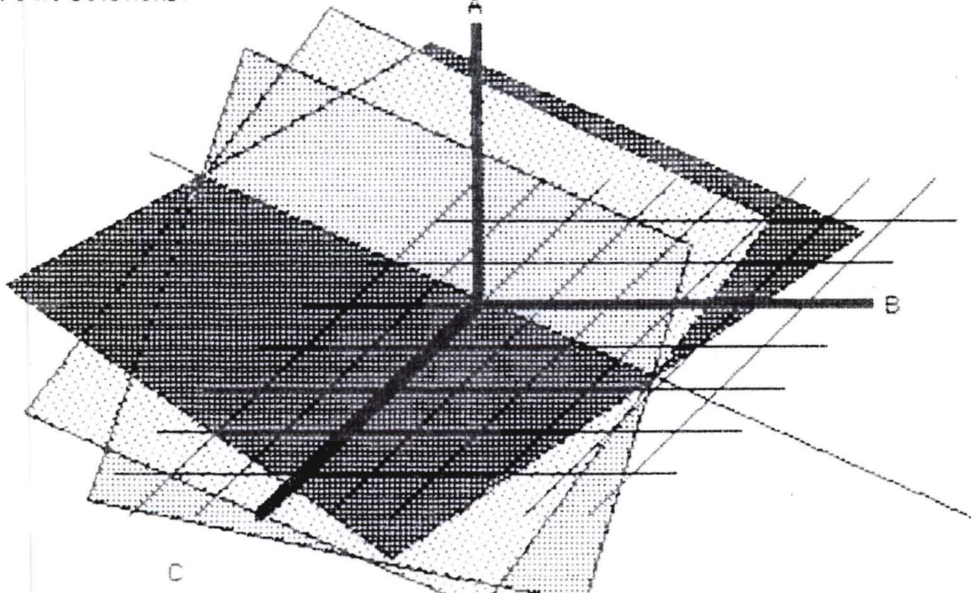
a) Define functions D, M, and G to give the number of letters the daughter, mother, and grandmother write during the year, respectively.

b) Solve for D, M, and G. How many solutions are there ? What are they ?

c) Suppose that during the year the three write ninety letters in all. How many solutions are there ? What are they ?

If we have two linear equations with two variables, then we may picture these equations with a two dimensional graph.

If we have three linear equations with three variables, then we must use a three dimensional graph. Here each linear equation represents a plane. If the three planes intersect in a point, then we have a unique solution. If they intersect in a line, as in the picture below, then we have infintely many solutions with one independent variable. If they intersect in a plane, then we have infintely many solutions with two independent variables. If there are no points common to all three planes, then there are no solutions.



CENTRAL PLANNING After a Soviet coup the leaders of the coal industry, the steel industry, and the train system get together to decide on the number of workers they need to keep all of their industries going. "Let us agree", they say, "that first we each need a certain number of workers just to take care of our own needs. The coal industry needs 2 million workers just to keep the mines open and operating. The steel industry needs 2 million workers just to maintain the mills and factories and keep them running. The train system needs 1 million workers just to keep the trains and the tracks in good condition."

Then the three turn to discuss what they need from each other. Regardless of how many workers they were to have or what exactly they were working on or how productive they were to be, they agreed on the following guidelines:

"For every four people working in the coal industry there must be one person in the steel industry (to produce mining equipment)."

"For every four people working in the steel industry there must be one person in the train system (to deliver raw materials) and two people in the coal industry (to supply coal for our furnaces)."

"For every four people working in the train system there must be four people in the coal industry (to supply coal for our engines) and three people in the steel industry (to produce tracks and trains).

a) Define a function C that gives the number of workers that must be working in the coal industry. Define a function S that gives the number of workers that must be working in the steel industry. Define a function T that gives the number of workers that must be working in the train system. (Recommendation: use units of "millions of workers")

b) How many workers must there be working in each of the industries ?

QUADRATIC EQUATIONS:
FINDING MAXIMUMS AND MININUMS 4/8

QUADRATIC EQUATIONS are those which can be written in the form below, where x is a variable and a, b, c are constants with $a \neq 0$.

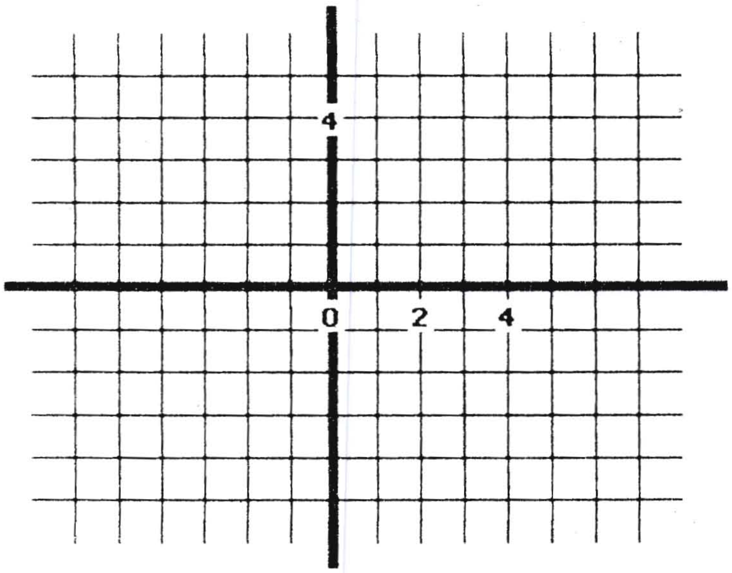
$$ax^2 + bx + c = 0$$

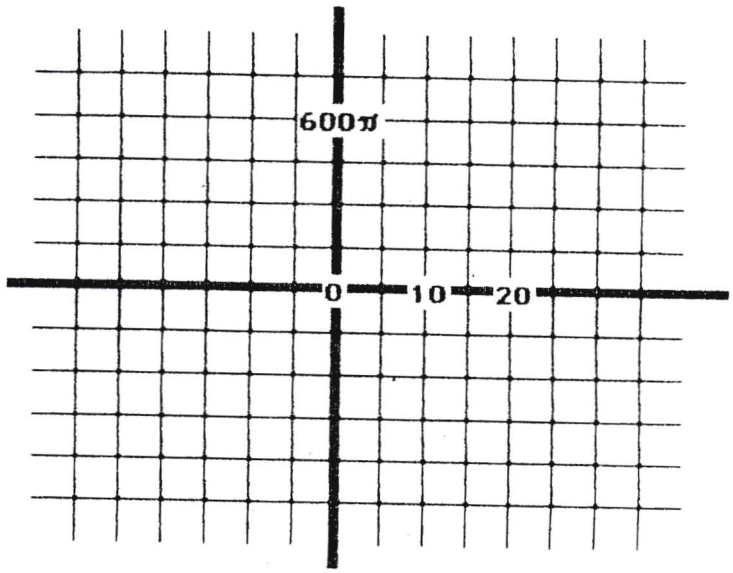
RECTANGLES A rectangle has a perimeter of 8 feet.

a) Define a function A that gives the area of the rectangle in terms of a single variable.

b) Graph the function A.

c) Find the dimensions (length and width) for which the rectangle has the greatest area.





HAZARDOUS WASTE You are designing a suitcase that is to look just like a container for hazardous waste. It is to be cylindrical in shape and bright yellow. The sum of the length, the width, and the height must be 60 inches so that it can be checked on board an airplane. You decide to maximize its surface area so that it would be highly visible.

a) Define a function S that gives the surface area of the container in terms of a single variable.

b) Graph the function S.

c) Find the dimensions (radius and height) which yield the greatest surface area.

GOING TO KANSAS CITY A new interstate highway is being built to connect Wichita, Kansas with the Northeast. In the outskirts of Wichita there are three farming towns more or less along the way of the proposed highway. (One is 2 miles east and 1 mile north, the other is 3 miles east and 2 miles north, and the last is 5 miles east and 3 miles north). Each wants the highway to pass by as close as possible. The builders insist that in the outskirts of Wichita the highway be straight as a line, but they are willing to discuss the exact direction in which the highway will go. What direction for the highway will best accomodate all three towns ?

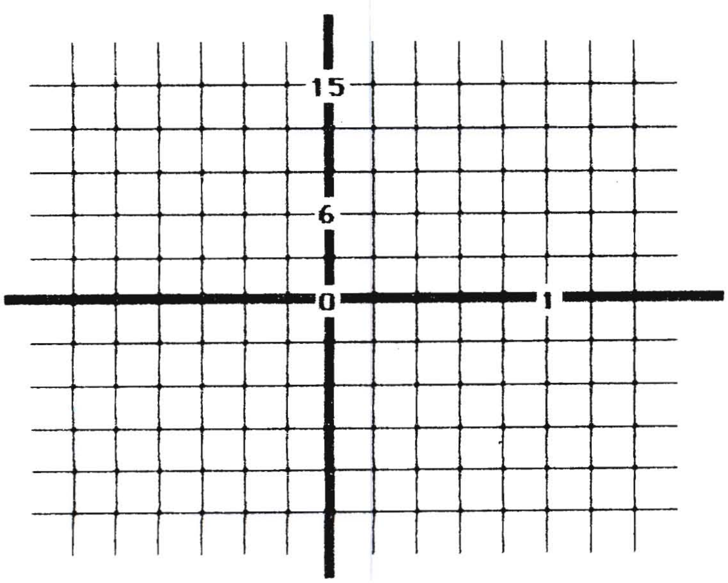
We attack this type of problem by using "the method of least squares". We consider a line $y=mx$, extending from Wichita, and we will be looking for the most suitable slope m. To do this we calculate the vertical distance from each town to the line. Let T be the sum of the squares of the vertical distances.

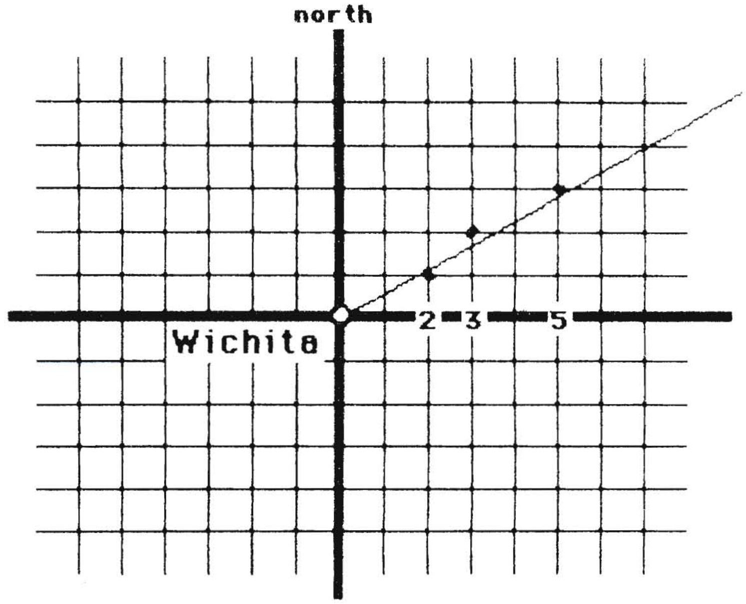
a) Note that T is a function of m. For what m is T a minimum ?

Having solved for m, note that the line $y=mx$ passes very close to all three towns. The function T gives a measure of how far off a line is from the towns. Note that T is a sum of squares: each square represents a positive contribution from a town. If a line is too far off from any single town, that line is severly penalized because the vertical distance to that town is squared. This technique is usually used to find m,b such that a line $y=mx+b$ will be most representative of a collection of data. In this case the function T is a function of two variables, m and b, and in this class we do not have the techniques for determining when T is a minimum.

$$T = (2m-1)^2 + (3m-2)^2 + (5m-3)^2$$

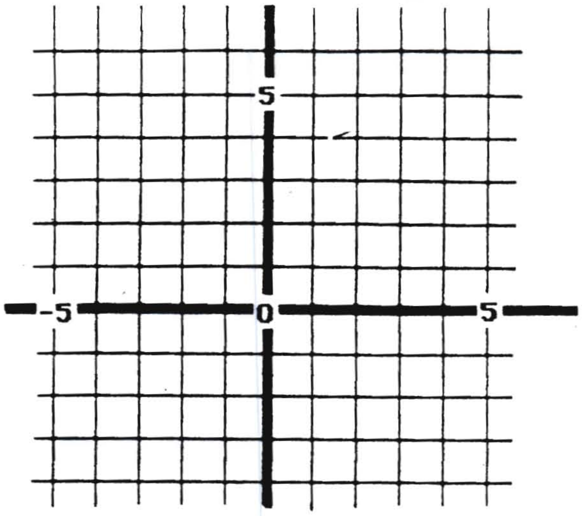
$$T = 38m^2 - 46m + 14$$





Line with best fit: $y = \frac{11}{19}x$

$y = 2x^2$ and $y = \frac{1}{2}x^2$



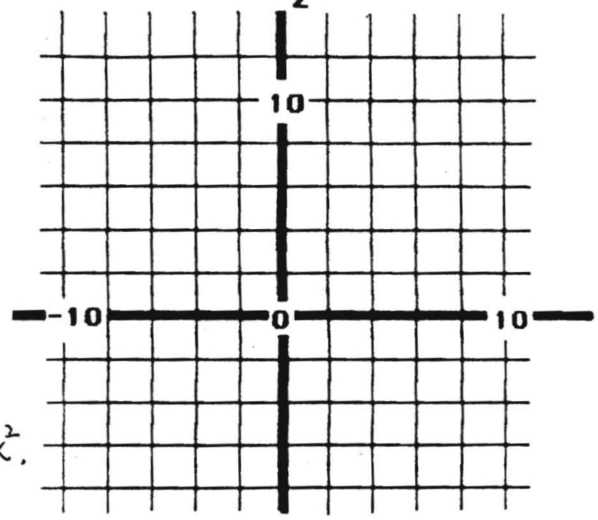
to make the graph steeper, multiply the function by a, where $a > 1$.

Example: $y = 2x^2$

to make the graph shallower, multiply the function by a, where $1 > a > 0$.

Example: $y = \frac{1}{2}x^2$

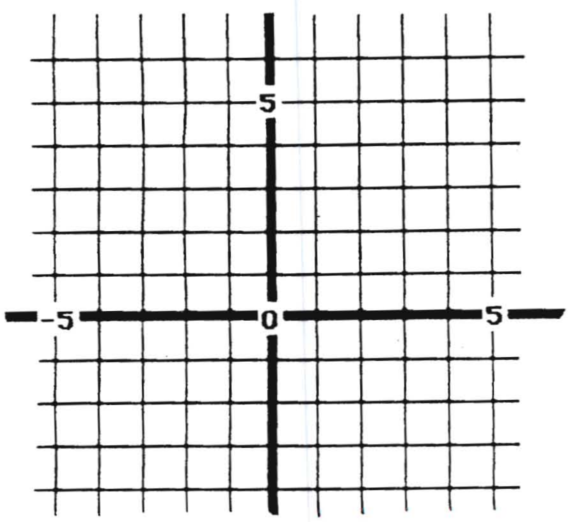
$y = \frac{1}{2}x^2$



What is the difference in shape between a steep parabola and a shallow parabola ?
Suppose $y = ax^2$,
If we replace x with $\frac{1}{a}x$ and y with $\frac{1}{a}y$, then this is equivalent to changing the scales in the x and y directions in equal proportions - this is the same as moving closer to or farther away from the graph.
But this gives us $\frac{1}{a}y = \left(\frac{x}{a}\right)^2 a$ or $y = x^2$.
Consequently, the graph of $y = ax^2$ just looks like the graph of $y = x^2$ does from close up or far away.

This is a special property of parabolas.

$y = x^2 - 2x + 4$



Given a quadratic function $y = ax^2 + bx + c$ we can by completing the square write it in the form:

$$y = a^2\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4}\right)$$

so that it's graph can be gotten from the graph $y = x^2$ by moving closer or farther away from the graph, moving the graph to the right or to the left, and moving it up or down.
THEREFORE THE GRAPH OF EVERY QUADRATIC FUNCTION IS A PARABOLA !

But an easier way to graph is to find where $y = ax^2 + bx + c$ and the line $y = c$ intersect: the axis of symmetry will lie in

HOMEWORK DUE: 4/17   3.2 #47, #52   3.5 #22 3.6 #14
2.5 #52, #58, #62   Chapter 3 review exercise #14, #36, #66

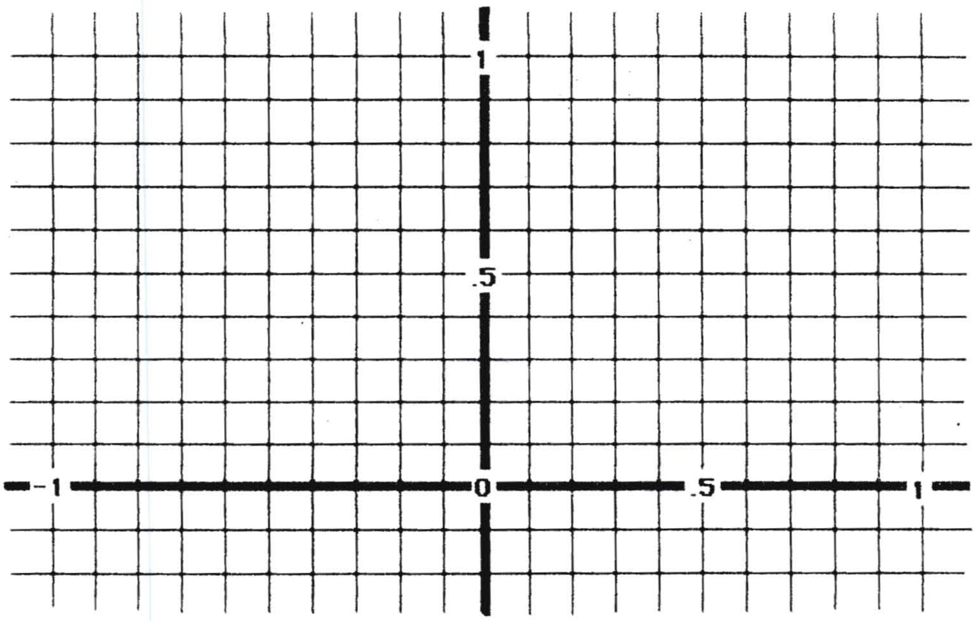
RECOMMENDED PROBLEMS:

2.2 #15, #25, #33, #35, #39,
2.4 #17, #19, #21, #27, #31
2.5 #1, #3, #29, #47, #49
7.1 #33, #37, #39, #55, #57
pg.444 #9, #17, #41, #45, #47

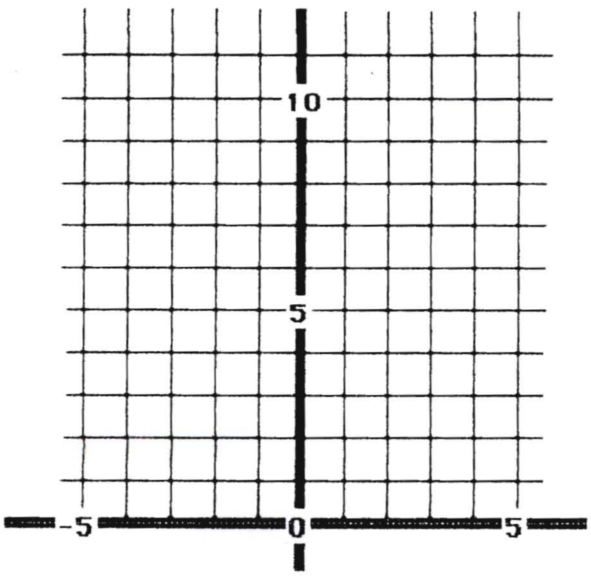
3.2 #11, #19, #25, #31, #41
3.3 #3, #7, #13, #15, #23
3.4 #3, #7, #11, #15, #17
3.5 #9, #11, #13, #35, #41
3.6 #7, #9, #11, #13, #17

THE GRAPH OF A QUADRATIC FUNCTION IS A PARABOLA 4/10

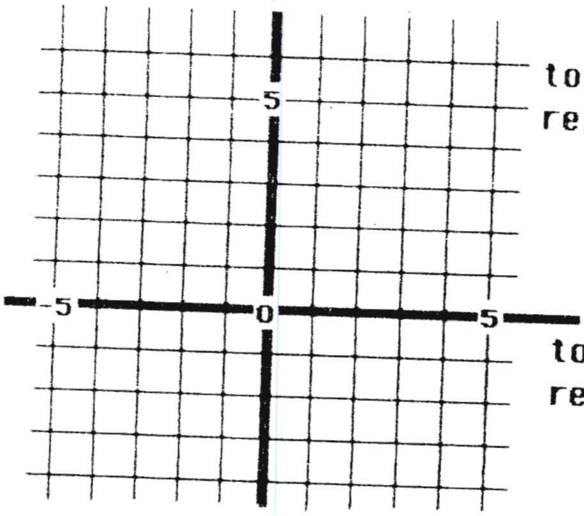
$y = x^2$



$y = x^2$



$y = x^2+3$ and $y = x^2-3$



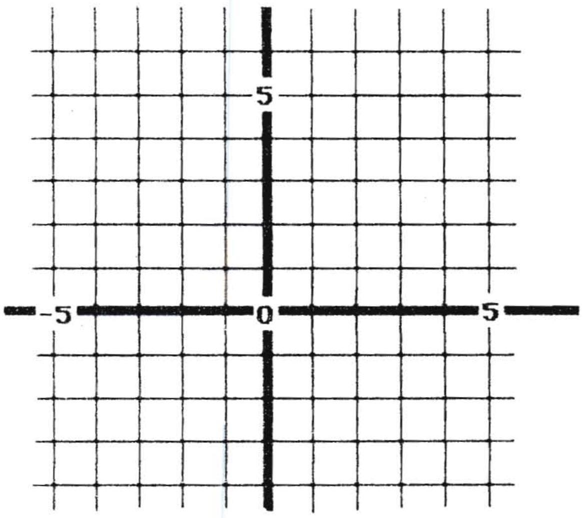
to move the graph up k squares, replace y with (y−k)

Example: $y-3 = x^2$ so that $y = x^2+3$

to move the graph down k squares, replace y with (y+k)

Example: $y+3 = x^2$ so that $y = x^2-3$

$y = x^2 +4x +4$ and $y = x^2-4x +4$



to move the graph k squares to the right, replace x with (x−k)

Example: $y = (x-2)^2$ which is $y = x^2-4x +4$

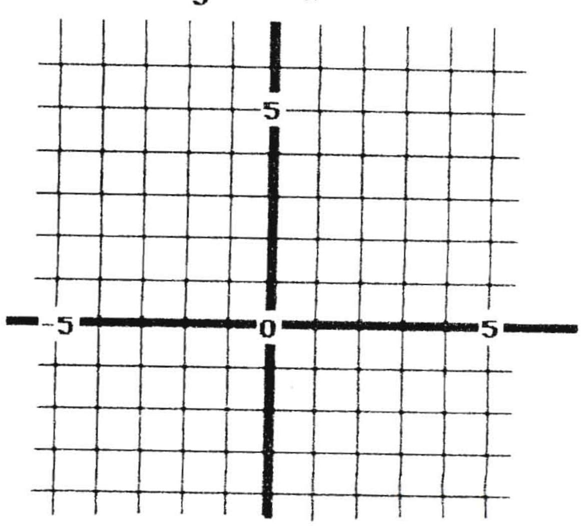
to move the graph k squares to the left, replace x with (x+k)

Example: $y = (x+2)^2$ which is $y = x^2 +4x +4$

to get a parabola facing down, multiply y by −1.

Example: $(-y) = x^2$ which is $y = -x^2$

$y = -x^2$



READING: 3.5 Wednesday, 4/15 3.6 Friday, 4/17 The midterm will cover Sections 2.1, 2.2, 2.4, 2.5, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 7.1, 7.2.

THE GRAPH OF A QUADRATIC FUNCTION IS A PARABOLA 4/13

Consider the quadratic function $y = ax^2+bx+c$

$$y = ax^2+bx+c$$

Multiply both sides by a

Complete the square by adding $\frac{b^2}{4}$ to both sides. Having thus rewritten the function, let $K_1 = \frac{b^2}{4}-ac$ $K_2 = \frac{b}{2}$ $K_3 = a$

$$(ay + (\tfrac{b^2}{4}-ac)) = (ax + \tfrac{b}{2})^2$$

Then we see that the graph of the function is gotten by taking $y = x^2$ and manipulating its graph in the following three ways:

I) Replacing y with $y + K_2$ so that the graph moves up or down

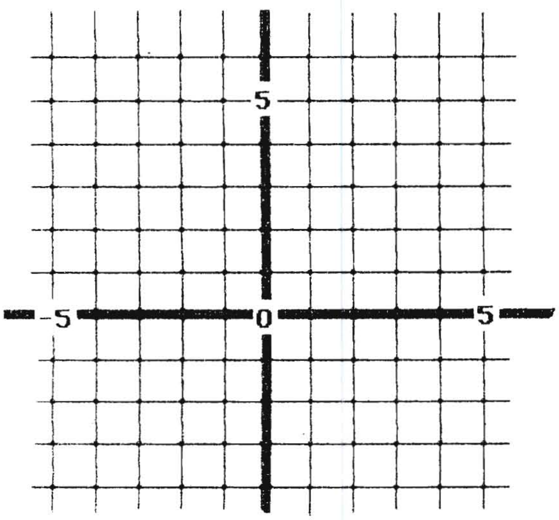
II) Replacing x with $x + K_1$ so that the graph movesto the right or to the left.

III) Replacing both x with K_3x and y with K_3y so that the graph appears closer or farther away.

$$(K_3y + K_1) = (K_3x + K_2)^2$$

Question: What are the x-intercepts of the function $y = ax^2+bx+c$?

$y = x^2 - 2x + 4$

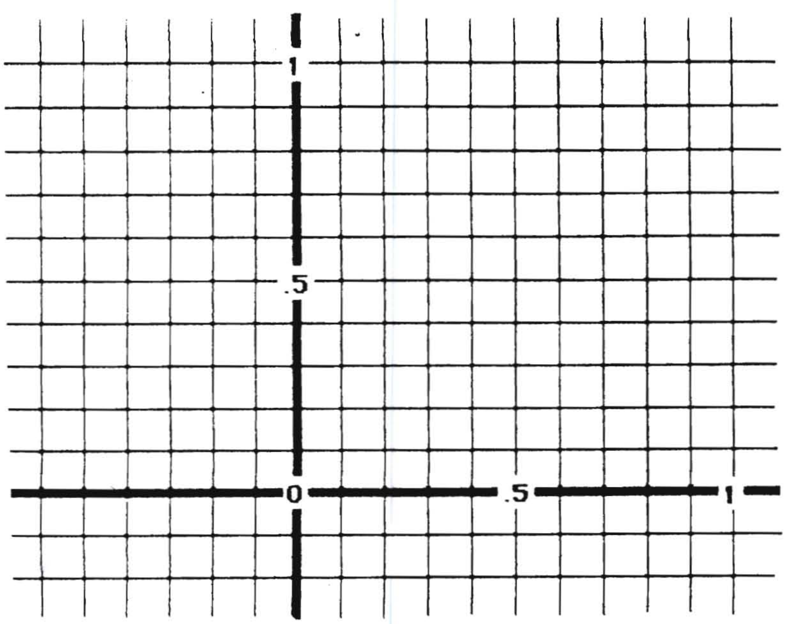


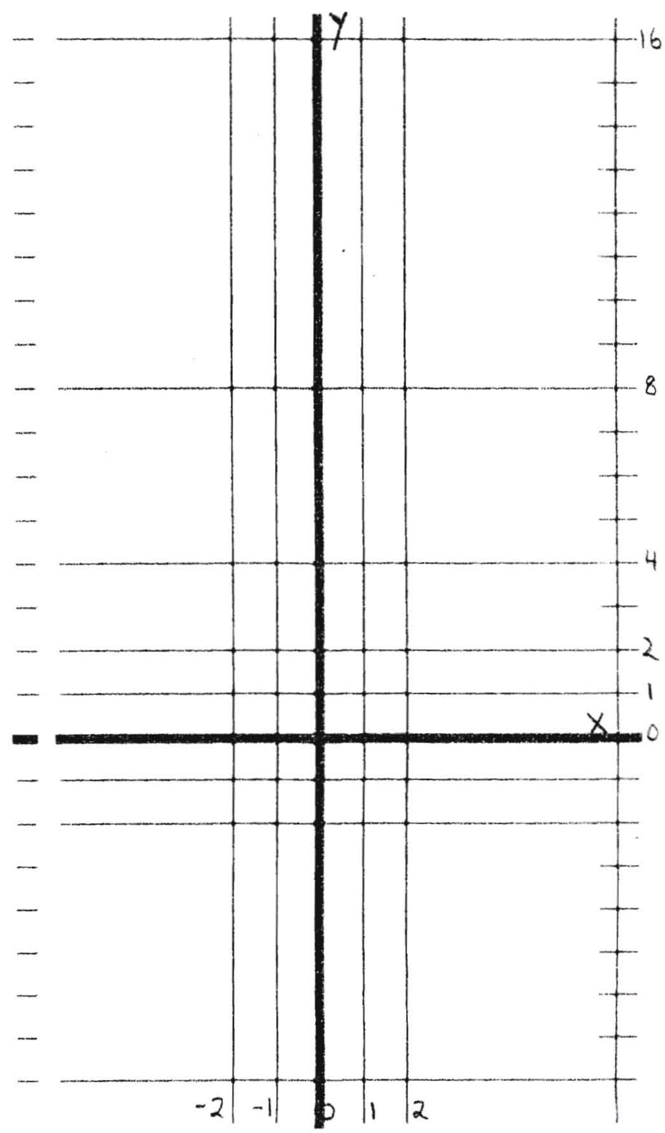
The argument above demonstrates that the graph of a quadratic function is always a parabola!

This fact together with the fact that $y = x^2$ has an axis of symmetry make for a quick technique of graphing quadratic functions.
Find where the function $y = ax^2+bx+c$ intersects with the horizontal line $y = c$. The axis of symmetry must be exactly between the points of intersection,

The Family of Functions $y = x^N$

		$\lim_{x\to-\infty}$	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2		$\lim_{x\to+\infty}$
x^4											
x^3											
x^2											
x^1	x										
$x^{\frac{1}{2}}$	$\sqrt{x}$										
x^0	1										
$x^{-\frac{1}{2}}$	$\frac{1}{\sqrt{x}}$										
x^{-1}	$\frac{1}{x}$										
x^{-2}											
x^{-3}											





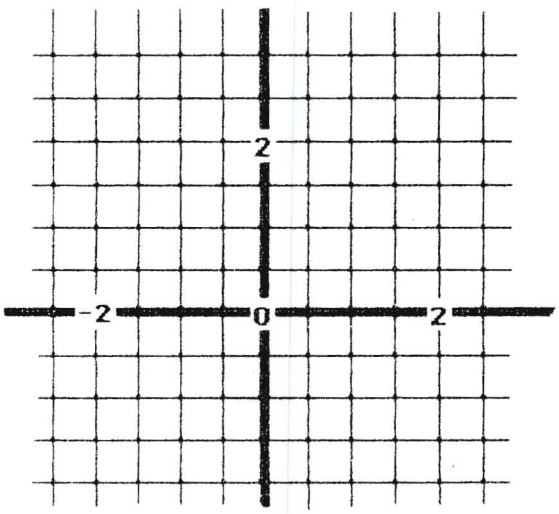
COMPARING THE SIZE OF POLYNOMIALS 4/15

Be sure to memorize the shapes of the basic graphs!

When we are interested in the <u>difference</u> in size between polynomials, we subtract one from the other.
Example: when is the volume of a cube bigger than the length of its side ?

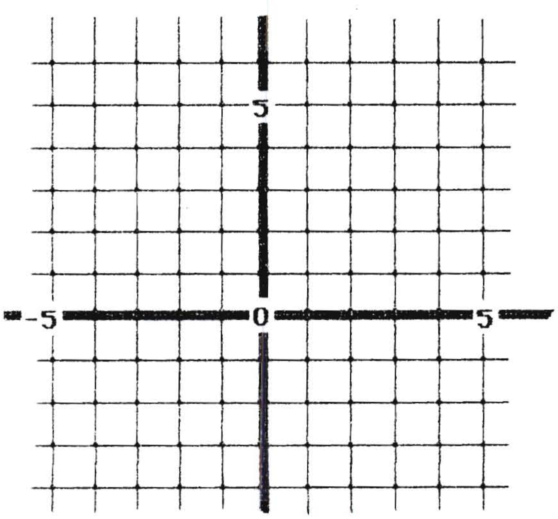
Know how to find x-intercepts.
Where is the function positive, where is it negative ?

$y = x^3 - x$



When we are interested in the <u>relative difference</u> in the size of two polynomials, we look at their ratio.
In this example it would be $\frac{x^3}{x}$ or $\frac{x}{x^3}$, as we choose.

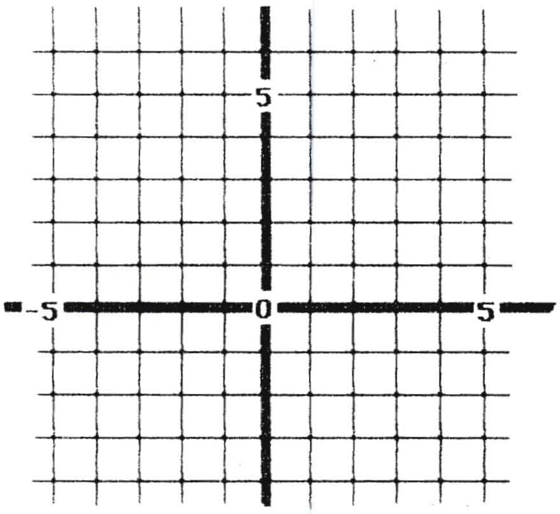
RECTANGLE OF FIXED AREA. If a rectangle has area 1 and length x, then the ratio $\frac{1}{x}$ gives the width.



NOTE: do not do homework problem 2.5 #62. It has a typographical error

Note do not do homework problem 2.5 #62. It has a typographical error.

COMPARING PLEASURE: You find the apartment of your dreams. The rent is $1,000 per month. You want to live there forever but all the same they make you pay first and last months' rent before you even move in. How do you compare, at any given time, the amount of dollars you have spent on rent with the amount of value you have received ?



COMPARING VOLUME Given width w, one box has dimensions w, w + 100, w + 100, and another box has dimensions w, w + 10, 2w. When w is very large, how do the volumes of the boxes compare ?

HOMEWORK DUE 4/23: 1.3 #98, 2.3 #58, 2.5 #18, 3.4 #27, #32, #34, 3.6 #20, pg.202 #72, 10.1 #2, #27.

POLYNOMIALS AND THEIR BUILDING BLOCKS 4/17

THE MOST SIGNIFICANT MATH PROBLEM OF ALL: There are N people at a party. You brought a rectangular cake. You cut the cake across its length, say x times, and across its width, say y times, so that each piece has the same shape. What are the ways you can cut the cake so that each person gets a piece and there are none left over ?

integers	ways of factoring	prime decomposition
1		
2		
3		
4		
5		
6		
7		
12		
24		
360		

The numbers ... -3, -2, -1, 0, 1, 2, 3, ... are called the <u>integers</u>.
The building blocks of the integers are the <u>prime numbers</u>.
The first several prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23...
The Fundamental Theorem of Arithmetic says that each integer can be expressed uniquely as a product of prime numbers and ± 1.

SECRET CODES: You can use this fact to write secret codes! Factor the number 5,467,500,000 and find the answer to a riddle for Easter: "Dig through ice and find silver. Dig through silver and find gold."

POLYNOMIALS ALSO FACTOR INTO BUILDING BLOCKS: If you allow for imaginary numbers, then the Fundamental Theorem of Algebra says that each polynomial $p(x)$ can be expressed uniquely as a product of linear factors $(x+C_i)$ and a constant K.

$$p(x) = K(x+C_1)(x+C_2) \ldots (x+C_n)$$

for example, $x^4 - 1 = (x+1)(x-1)(x+i)(x-i)$ where $i = \sqrt{-1}$ is the imaginary number. If we don't allow for the imaginary number, then some of the "chunks" of the polynomial don't break down:

$$x^4 - 1 = (x+1)(x-1)(x^2+1)$$

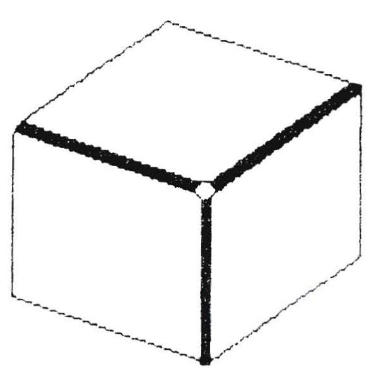
NOTE: As we code logical statements into polynomials,
multiplication means "and"
addition means "either... or..."

	<- g								b ->	
					1					
$(g + b)$				1		1				
$(g + b)(g + b) = g^2 + 2gb + b^2$			1		2		1			
$(g + b)(g + b)(g + b) = g^3 + 3g^2b + 3gb^2 + b^3$		1		3		3		1		
$(g + b)(g + b)(g + b)(g + b) = g^4 + 4g^3b + 6g^2b^2 + 4gb^3 + b^4$	1		4		6		4		1	

GIRLS AND BOYS: A family has three children. What is the chance that there are two boys and a girl ?

BLUE EYES AND BROWN EYES: Each person has a pair of genes that determines eye color. The gene for brown eyes, "B", wins out over the gene for blue eyes, "b". Therefore the pairs "BB" and "Bb" make for brown eyes, and only the pair "bb" makes for blue eyes. When a child is born, it gets one gene from each parent.
Suppose that the father has "Bb" and the mother has "Bb" as well. What is the chance that their child has blue eyes ?

BIG BOX: A wooden box is in the shape of a cube. The box is made of material that is 1 inch thick.
Suppose that the inside of the box has volume x^3. Then the outside of the box has volume $(x + 2)^3$.
A) Expand the latter polynomial and give a geometrical explanation for each term.
B) What happens to the difference between the volumes of the inside and the outside of the box as x gets large ?



METHODS OF PROOF 4/20

BOX OFFICE When the box office opens there are N people in line waiting to buy tickets. It takes one minute to help each person. Consider the time spent by the N people waiting in line and purchasing tickets once the box office has opened. If we add it all up, how much would it be ?

$$1 + 2 + 3 + 4 + \dots + N = ?$$

LEX LUTHOR, the archvillain of Superman, was mischievous and ingenious as a child. His math teacher once punished him by having him stay after class and add up all the numbers from one to one hundred. To her suprise, after five minutes he presented her with the answer. How did he do it ?

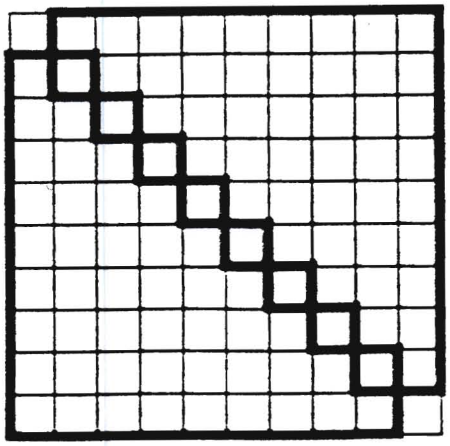
$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \dots + i + \dots + 100 = \frac{(100+1)(100)}{2}$$

BIJECTION Bijection is the most illuminating method of proof. It involves matching the structure behind the problem with an identical structure that is more conducive to our intuition. It is an act of translating the original problem into a language where it makes more intuitive sense.

For our problem, we may consider the side of the staircase given in the diagram. The area of each rectangle corresponds to a term in the sum and the total area is $1 + 2 + \dots + N$.

If we take two such staircases and lay them opposite one another, then we get an "N+1 by N+1" square with N+1 "1 by 1" squares missing. Therefore:

2×(area of staircase) = (area of "N+1 by N+1") − (N+1)×(area of "1 by 1")



INDUCTION Induction is a powerful technique to use when we suspect that a formula is true for all integers $N > 0$, but we lack proof.

"Suppose that for each integer $N > 0$ we have a statement P_N for which the following two conditions hold:

1. P_1 is true.
2. For each integer $k > 0$, if P_k is true, then P_{k+1} is true.

Then all the statements are true; that is, P_N is true for all integers $N > 0$."
see pg.557

In our case the statement P_N states that "$1 + 2 + \ldots + N = \frac{N(N+1)}{2}$", and we are trying to show that the statement P_N is true for all $N > 0$. If we demonstrate that the two above mentioned conditions hold, then what we are trying to show will be true by induction.

We first show that P_1 is true. This statement states that $1 = 1$.
We see that this statement is indeed true.

Now we are allowed to suppose that P_k is true, that is,

$$1 + 2 + \ldots + k \quad =$$

Using the assumption that P_k is true, we want to show that P_{k+1} is true. Let us add $k+1$ to each side of the equation above. This gives us:

$$1 + 2 + \ldots + k + (k+1) \quad =$$
$$1 + 2 + \quad \ldots \quad + (k+1) \quad =$$

We see that if we assume that P_k is true, then it follows that P_{k+1} is true. Therefore the second condition holds. This proves by induction that P_N is true for all integers $N > 0$.

The upshot is that we know that P_N is true because we can prove it from P_{N-1}, which we can prove from P_{N-2}, which we can prove from P_{N-3}, and we pass along the burden of proof in this manner until we get to P_1, when we can stop because <u>we know that P_1 is true!</u>

MATH 4C
FIRST MIDTERM

Name: ______________________________ **April 22, 1992**

Andrius Kulikauskas

Do all eight problems. They are all worth the same number of points. Show all of your work. No calculators. No notes. Relax. Good luck!

1) You have four dollars worth of change. It consists of nickels and quarters. You have three times as many nickels as you have quarters. How many nickels do you have, and how many quarters ?

2) Select nonzero numbers for A, B, C so that the two linear equations in x and y have no solutions:

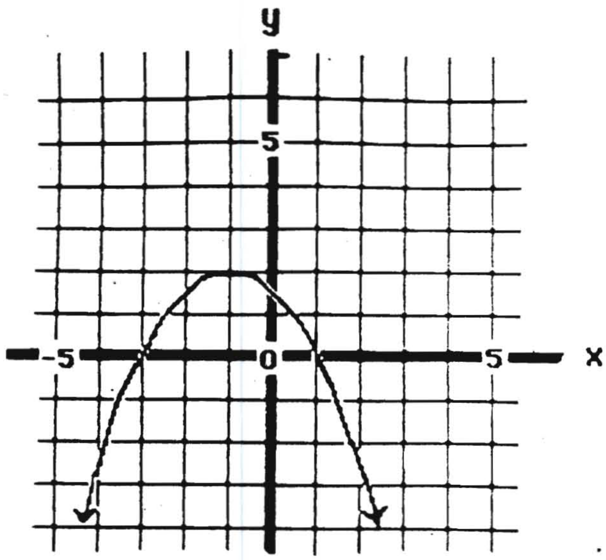
$$2x + 3y = 5$$
$$Ax + By = C$$

3) A one ounce letter costs 29 cents to mail. A two ounce letter costs 52 cents to mail. Assume that the price of mailing a letter (in cents) is a linear function of its weight (in ounces). Write down this function.

4) A baseball is thrown up into the air. Its height is given by $144t - 16t^2$ where time t is in seconds. At what time does it achieve its maximum height ?

5) You are given the equation $x^2 + 2px + q = 0$ where p and q are constants. Solve for x in terms of p and q by completing the square. No other method will be counted correct.

6) Graph $y = 4x^3 - x$.



7) The graph in the picture is a parabola. It includes the points $(-3, 0)$, $(-2, \frac{3}{2})$, $(-1, 2)$, $(0, \frac{3}{2})$, $(1, 0)$. Write down the quadratic function that is described by this graph.

8) Graph $y = \dfrac{-x-3}{x-4}$

HOMEWORK DUE 4/30: 4.4 #51, #60, #76, #78, #80
4.5 #4, #43 (change the two earthquakes to Joshua Tree Quake 6.1 and Oakland Quake 7.2) Chapter 4 review #82, #99, #124 (replace the words "relative growth rate" with "annual percentage increase" so that, for example, the population of China increases by a factor of 1.014 every year.)

READ: Monday: 4.1 and 4.3, Wednesday: 4.4.

EXPONENTIAL GROWTH 4/24

CHECKERS AND CHESS A sultan from India once asked that there be invented a game of strategy by which he might test his skill at making war. Many games were created, but he was most pleased by the games of checkers and chess, and sought to reward their creators.

The creator of checkers said, "Each day find a square on the checker board and stack a thousand coins. Do this until all the squares have been filled. Then I will be suitably rewarded."

The creator of chess said, "On the first square of the chess board place but one coin. Tommorow, place as many coins on the second square as there are on the first square. And the next day, place as many coins on the third square as there are on the other two. In this way, each day find an empty square and place as many coins on it as there are already on the board. Do this until all the squares have been filled. Then I will be suitably reward."

How were they each rewarded ?

MANHATTAN The Dutch came to New York around the year 1640. Legend has it that they purchased the island of Manhattan from the Indians for 20 dollars. We suppose that the Indians deposited that money in a fund earning ten percent a year.

A) How much was the money worth in 1650 ?
How much was the money worth in 1803 ?
How much was the money worth in 1992 ?
How much was the money worth in 1492 ?

B) When it grows ten percent a year, how long does it take for money to double ?

C) How much do you think Manhattan is worth ? If it was worth 20 dollars to begin with, by what percentage did its worth increase each year ?

ADAM AND EVE The population of the world is 5.5 billion. This number has been increasing by 2 percent every year.

A) When was there only one person in the world ?
When were there two people in the world ?

B) How many times has the population of the world ever doubled ?

C) It is thought that the total number of humans that ever lived is 100 billion. When will that many people be living in the world at one time ?

D) What is wrong with this model ?

THE PROPERTIES OF LOGARITHMS 4/27

$x = \log_{10} P$ is equivalent to $P = 10^x$

$\log_{10} P + \log_{10} Q = \log_{10} PQ$

What lies halfway between the largest things and the smallest things in the universe ?

If we stacked a trillion dollar bills, one on top of the other, how far would they reach ? What if we placed them end to end ?

$\log_{10} P - \log_{10} Q = \log_{10} \frac{P}{Q}$

Which is larger, the Earth with respect to the solar system, or the sun with respect to the Milky Way ?

If the Earth was the size of a human, what would be the diameter of its orbit ?

In relative terms, which is longer, a strand of hair or a DNA molecule ?

At least how old must the universe be ?

$\log_{10} P^n = n \log_{10} P$

How many particles could fit into the universe as we know it ?

If electrons and protons were thought to have the same density, how much more must a proton weight ?

How many spiders could cover a human being ? How many humans could cover a skyscraper ?

Is the Earth smooth ?

Suppose you start with one bacteria. Every hour they split in two. When would they fill the known universe ?

They say that on one square mile there live as many insects as there are people in the entire world. If people lived this densely, how many of them could the Earth support ?

$\log_{10} 1 = 0$

$\log_{10} 10 = 1$

$\log_{10} \frac{1}{Q} = -\log_{10} Q$ $\qquad 10^{\log_{10} P} = P$

$\log_{10} P$ is only defined when $P > 0$.

THE SCALE OF THE UNIVERSE

x				$\log_{10}x$
10^{26}	distance to farthest known quasars			26
10^{25}				25
10^{24}	distance to thousand closest galaxies			24
10^{23}				23
10^{22}				22
10^{21}	the Milky Way galaxy's diameter			21
10^{20}				20
10^{19}				19
10^{18}	distance to hundred closest stars			18
10^{17}		1 light-year		17
10^{16}				16
10^{15}				15
10^{14}	our solar system's diameter			14
10^{13}				13
10^{12}	diameter of Jupiter's orbit			12
10^{11}	diameter of Earth's orbit	supergiant star's diameter		11
10^{10}		giant star's diameter		10
10^{9}	diameter of the moon's orbit	sun's diameter		9
10^{8}	Jupiter's diameter			8
10^{7}	Earth's diameter			7
10^{6}	moon's diameter	length of California		6
10^{5}	asteroid's diameter			5
10^{4}	height of Mount Everest			4
10^{3}	height of the clouds above	1 mile		3
10^{2}	height of skyscraper			2
10^{1}	height of tree			1
10^{0}	height of human	1 meter	radio waves	0
10^{-1}	size of kitten			-1
10^{-2}	size of spider	1 inch		-2
10^{-3}	size of ameba			-3
10^{-4}	thickness of hair or paper	length of a DNA molecule		-4
10^{-5}	size of human cell			-5
10^{-6}	size of bacteria	size of cell organelles	visible light	-6
10^{-7}	size of virus			-7
10^{-8}	thickness of cell wall	size of protein	ultraviolet rays	-8
10^{-9}	size of amino acid	width of a DNA molecule		-9
10^{-10}	size of atom		X-rays	-10
10^{-11}				-11
10^{-12}				-12
10^{-13}	diameter of atom's nucleus		gamma rays	-13
10^{-14}	diameter of proton			-14
10^{-15}	diameter of electron (classical)			-15
10^{-16}				-16
10^{-17}				-17
10^{-18}	present experimental limits in the search for quarks			-18

Be sure to have read 4.1, 4.3, 4.4.
READ: Friday: 10.2, Monday: 2.3 and A.3.

NUMERATION SYSTEMS 4/29

COMPUTER BRAIN: A computer has a two word vocabulary: "yes" and "no".
A) How can it write numbers ?
B) How many numbers can it write if they have m or fewer digits ?
C) How can it add numbers ?

SETS AND SUBSETS: Suppose that a certain family consists of n people. Mathematically, we say that the family is a set S with n elements. If members of the family go to a movie, we say that those who went to the movie form a subset M of the set S.
A) What are the possible subsets of S and how many does it have in all ?
B) Suppose that k people went to the movie. In other words, the subset M has size k. Each way of choosing k people out of n people gives us a different subset. How many different ways are there of choosing k people from n people ? Equivalently, how many subsets does S have of size k ?

An example: The subsets of the set {1, 2, 3, 4, 5} of size 3:

{3, 4, 5}
{2, 4, 5}
{2, 3, 5}
{2, 3, 4}
{1, 4, 5}
{1, 3, 5}
{1, 3, 4}
{1, 2, 5}
{1, 2, 4}
{1, 2, 3}

CREATURE WITH TEN FINGERS: Most humans use a system of numeration that is based on ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

A) What does the string of symbols 2536 mean ?

B) What are humans doing when they multiply together 2536 and 1300 ?

NEXT MIDTERM: Monday, May 18th. Will cover sections #2.3, #2.7, #4.1, #4.2, #4.3, #4.4, #4.5, #5.1, #5.2, #10.1, #10.2, #A.3.
HOMEWORK DUE: 5/7 1.8 #56, 4.1 #22, #34 4.2 #22, #36 4.3 #39 10.1 #8, #10, #42, 10.2 #53
READ: Monday 2.3 and A.3, Wednesday: 2.7

IF X IS LARGE ENOUGH, THEN $2^X > X^n$. 5/1

A THEOREM ON THE SIZE OF POLYNOMIALS:
Consider a polynomial $p(x)$ where

$$p(x) = a_{n+1}x^{n+1} + a_nx^n + \dots \dots + a_3x^3 + a_2x^2 + a_1x^1 + a_0$$

and a_{n+1} is positive, that is, $a_{n+1} > 0$. Then there always exists X large enough such that $p(X) > X^n$ and such that furthermore $p(x) > x^n$ for all $x \geq X$.

How can we prove that the theorem above is true ?

RACING: Suppose that x high school students are swimming in a race. We assume that there are never any ties. How many possible outcomes are there for the race - that is, the students can finish in how many different orders ?

x! means the product $x(x-1)(x-2)(x-3)\dots \dots(5)(4)(3)(2)(1)$

Suppose that you are choosing k students out of x. For example, suppose that the fastest k students are to be on the swim team. You don't care what order they appear in, but you need to know which k students they will be. Equivalently, you have a set of size x and you are choosing a subset of size k. How many ways are there of doing this ?
One solution is to hold a race and to only take note of who finishes in each of the first k places. The number of possible outcomes is given by $x(x-1)\dots$ $\dots(x-(k-2))(x-(k-1))$. There is a correspondence between these outcomes and the outcomes that we get by first choosing k students that are to be on the team, and then holding a race between them.
Therefore:

[The number of ways of choosing k out of x][$(k)(k-1)\dots(3)(2)(1)$] = [$x(x-1))\dots \dots(x-(k-2))(x-(k-1))$]

[The number of ways of choosing k out of x][k!] = $\dfrac{x!}{(x-k)!}$

[The number of ways of choosing k out of x] = $\dfrac{x!}{k!\,(x-k)!}$

Fix k. (This means that in what follows we don't let the value of k change. For example, in the expression x^k we let x change, but not k).
Then **[**the number of ways of choosing k out of x**]** is a polynomial in x whose term of highest power is

By a previous argument we have shown that there exists X large enough such that at this value the polynomial is greater than X^{k-1}, and furthermore, we have that for all $x \geq X$:

[the number of ways of choosing k out of x**]** $>$ x^{k-1}

A THEOREM ON THE SIZE OF 2^x: Consider x^n where n is a positive integer. There always exists X large enough such that $2^X > X^n$ and such that $2^x > x^n$ for all $x \geq X$.

PROOF: Fix n. Let $k = n+1$.

Recall that if a set has x elements, then 2^x is the total number of subsets that it has, and **[**the number of ways of choosing k out of x**]** is the number of subsets that it has of size k. Note that the former is strictly bigger than the latter.

Therefore

2^x $>$ **[**the number of ways of choosing k out of x**]**

The above statement is true for all x. Moreover, there exists X such that for all $x \geq X$

[the number of ways of choosing k out of x**]** $>$ x^{k-1}

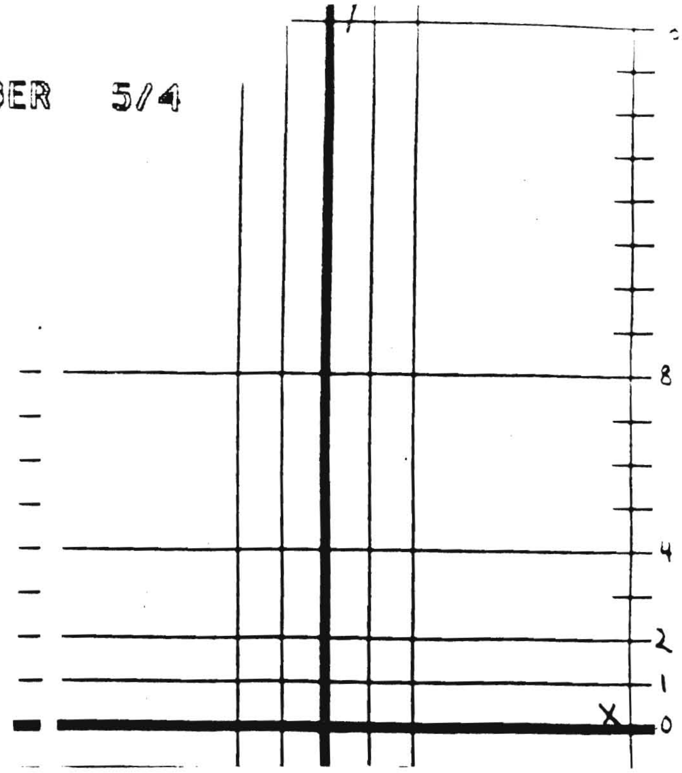
Finally, we note that $k = n+1$ so that $x^{k-1} = x^n$. We therefore conclude that for all $x \geq X$

$$2^x > x^n$$

What does $y = 2^x$ mean when
x is a positive integer ?
Note that we have the property
$(2^a)(2^b) = 2^{a+b}$

How do we define it when x is zero ?
How do we define it when x is a negative integer ?

In order to be self-consistent, we figure out these definitions by refering to the property that
$(2^a)(2^b) = 2^{a+b}$



How do we define $y = 2^x$ when x is a fraction ? We refer to the same property. For example, if $x = 1/2$, then we write $(2^{1/2})(2^{1/2}) = 2^1$. It must be the case that $2^{1/2} = \sqrt{2}$. In general, if $x = m/n$, then we multiply $2^{m/n}$ n times to get:

$$(2^{m/n})(2^{m/n}) \ldots \ldots(2^{m/n})(2^{m/n})(2^{m/n})$$
$$= 2^{m/n + m/n + \ldots \ldots + m/n + m/n + m/n}$$
$$= 2^m.$$

Therefore $2^{m/n} = \sqrt[n]{2^m}$

But does $\sqrt{2}$ exist ? Is there such a number ?
It depends on what we mean by number.
Rational numbers are those which can be written as fractions (with an integer in the numerator and an integer in the denominator).
The Greeks thought that all numbers are rational - they thought of numbers as ratios. They were shocked to prove that **$\sqrt{2}$ is not rational**. How did they prove this ?

In the spirit of Kant, we ponder the
SOURCES OF TRUTH

If a statement is SELF-CONSISTENT, which is to say that it does not contradict itself, then it is *possibly true*. "Let x = 3" : it is possibly true that x = 3.

If a statement is a CONSEQUENCE OF ANOTHER TRUTH, then it is *actually true*. This is to say that it's truth is real in as much as it is a consequence of another truth which is real. For example, if we have accepted, even for the moment, that it is true that x=3 , then we attribute reality to the consequence that 2x=6.

If a statement's NEGATION CONTRADICTS ITSELF, then it is *necessarily true*. A statement must be either true or false. If it's negation is false, then it must be true - the nature of this relation is one of necessity. If "$\sqrt{2}$ is a rational number" contradicts itself, then it is false, and it must be the case that "$\sqrt{2}$ is not a rational number" is true.

Mathematical definitions are a matter of self-consistency. Any definition of a number which is self-consistent is a legitimate object of study. For example, there is no problem in allowing for $\sqrt{-1}$.
Homework problems are a matter of consequences. If we are given that $x^2+3x+7=0$, then what is x? In solving the problem, we assume that the situation is real.
Mathematical proofs are a matter of necessity. Proof by contradiction is the strategy around which mathematics is built. In the proof that "$\sqrt{2}$ is not a rational number" we start by assuming that the opposite is true and arriving at a contradiction.

We note that although the sources of truth are different, we think of the nature of the resulting truth as being the same. Therefore the sources of truth are but forms by which truth manifests itself.
Kant claimed that our minds apply these forms in organizing everything we think. Conservation laws (which state, for example, that mass is conserved) merely apply self-consistency to concepts like mass.
We apply the relation between a statement and its consequences whenever we use the law of cause and effect (by which every effect has a cause).
Action at a distance (the way that the force of gravity keeps track of all masses at all times) is an application of the picture in which truth manages to keep track of the truth and falsehood of all statements, so that all true statements share the same truth.

$\sqrt{2}$ IS NOT A RATIONAL NUMBER.

We will show that if we suppose that $\sqrt{2}$ is rational, then we will arrive at a contradiction. As mathematics does not allow for contradictions, it must be the case that $\sqrt{2}$ is not rational.

Suppose that $\sqrt{2}$ is rational. Then it can be written as a fraction of integers m/n where m and n have no common divisors: in particular, it can not be the case that 2 divides both m and n.

But if $m/n = \sqrt{2}$, then $m^2/n^2 = 2$.

So $m^2 = 2n^2$. This means that 2 divides m^2. If 2 divides $(m)(m)$, then it must divide one of the m. This means that m must be an even number!

So we can write $m=2k$, where k is an integer. Then $m^2/n^2 = 4k^2/n^2 = 2$. So $2k^2 = n^2$. This means that 2 divides n^2. By the argument above, n must be an even number!

Therefore m and n are both even numbers. But then they have a common divisor 2. This contradicts what we assumed! So $\sqrt{2}$ is not rational!

READ: Friday: 2.7, Monday: 5.1, Wednesday: 5.2.

RATIONAL NUMBERS AND DECIMAL EXPANSIONS 5/6

$Y = 2^X$ WHEN X IS RATIONAL

How do we define $\sqrt{2}$?

$1^2 = 1$	<	2	<	$2^2 = 4$
$(1.4)^2 = 1.96$	<	2	<	$(1.5)^2 = 2.25$
$(1.41)^2 = 1.9881$	<	2	<	$(1.42)^2 = 2.0164$
$(1.414)^2 = 1.999396$	<	2	<	$(1.415)^2 = 2.002225$

We use this procedure in general to get a decimal expansion for $2^{1/n}$. To get $2^{m/n}$ we take $2^{1/n}$ to the mth power. This gives $(2^{1/n})^m = 2^{m/n}$.

$2^X > 1$ IF X IS RATIONAL AND $X > 0$. If x is rational, and $x > 0$, then we can write $x = m/n$ where m and n are both positive integers. Then $2^{1/n} > 1$, so if we take both sides to the mth power we see that $2^{m/n} > 1$.

$Y = 2^X$ IS AN INCREASING FUNCTION ON THE DOMAIN $\{x \mid x \text{ is rational}\}$ Suppose that a and b are rational, and $b > a$. Then $b = a + r$ where r is positive and rational. So $2^b = 2^{a+r} = 2^a 2^r > 2^a$.

DECIMAL EXPANSIONS We can write every rational number as a decimal expansion. We do this by using long division !

divide 1 into 2	1/2	=	.50000000000000000000000000...
divide 1 into 3	1/3	=	.33333333333333333333333333...
divide 2 into 3	2/3	=	.66666666666666666666666666...
divide 1 into 4	1/4	=	.25000000000000000000000000...
divide 2 into 4	2/4	=	.50000000000000000000000000...
divide 3 into 4	3/4	=	.75000000000000000000000000...
divide 1 into 7	1/7	=	.142857142857142857142857142 8...
divide 2 into 7	2/7	=	.285714285714285714285714285 7...
divide 3 into 7	3/7	=	.428571428571428571428571428 5...
divide 4 into 7	4/7	=	.571428571428571428571428571 4...
divide 5 into 7	5/7	=	.714285714285714285714285714 2...
divide 6 into 7	6/7	=	.857142857142857142857142857 1...

When we divide two integers by long division, the process may end, in which case the decimal expansion ends in zeros. Or the process may never end, but since the number of possible remainders is finite (in fact, it's no bigger than the denominator), we soon get a remainder that we had before. As soon as this happens we are assured of getting a repeating pattern. This proves the following theorem: *Every rational number is a decimal expansion that ends in a repeating pattern*

The converse is also true: *Every decimal expansion that ends in a repeating pattern is a rational number.* We consider an example. But first note that

$$1 \qquad = \qquad (1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \dots)$$

$$\frac{1}{1-x} \qquad = \qquad 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \dots$$

Given a number like 75.92121212121212121212121.... we say that 75.9 is the prefix. We say that the pattern consists of repeating the integer 21, that it has period of length 2, and that it starts after the first digit to the right of the decimal point.

$$\begin{aligned}
& 75.92121212121212121212121\dots \\
= \; & 75.9 + .021 + .00021 + .0000021 + .000000021 + \dots \\
= \; & 75.9 + (.021)(1 + .01 + .0001 + .000001 + \dots) \\
= \; & 75.9 + (.021)(1 + (1/100) + (1/100)^2 + (1/100)^3 + \dots) \\
= \; & \\
= \; & \frac{759}{10} + \frac{21}{1000}\left[\frac{1}{1-\frac{1}{100}}\right]
\end{aligned}$$

In general, let the prefix be a rational number A, and the pattern consist of repeating the integer J, and the length of its period be the integer s, and let it start after the t-th digit. Then the decimal expansion equals:

$$\begin{aligned}
& A + (J)(10^{-t-s}) + (J)(10^{-t-2s}) + (J)(10^{-t-3s}) + (J)(10^{-t-4s}) + \dots \\
= \; & A + [(J)(10^{-t})][1 + 10^{-s} + 10^{-2s} + 10^{-3s} + 10^{-4s} + \dots] \\
= \; & A + [(J)(10^{-t})][1 + (10^{-s}) + (10^{-s})^2 + (10^{-s})^3 + (10^{-s})^4 + \dots] \\
= \; & A + (J)(10^{-t})\left[\frac{1}{1-10^{-s}}\right]
\end{aligned}$$

which is a rational number.

THE DECIMAL EXPANSION OF $\sqrt{2}$ DOES NOT HAVE A REPEATING PATTERN Otherwise $\sqrt{2}$ would be a rational number.

For our purposes WE WILL THINK OF REAL NUMBERS AS BEING DECIMAL EXPANSIONS. There is only one problem with this: sometimes the same real number x has two different decimal expansions.

$$1.000000000\dots \quad = \quad 1/3 + 2/3 \quad = \quad .9999999999\dots$$

Note that there are no rational numbers in between the two decimal expansions. Mathematicians therefore identify the real number x with the set of rational numbers less than d, where d is a decimal expansion of x. It is the case that whatever decimal expansion we choose for x, we get the same set of rational numbers.

2.7#46, 4.1#34 (another chance at it!), 5.1#42, #45,
5.2#8, #10, #28, #30, 5.3 #38, 6.5 #42

READ: Monday: 5.1, Wednesday: 5.2, 5.5 Friday: 5.3

THE SIZE OF AN INFINITE SET 5/8

TWO SETS HAVE THE SAME SIZE if there is a one-to-one function f(x) whose domain is one set, and whose range is the other set.
Domain: set of input values
Range: set of output values

1 -> A
2 -> B
3 -> C

FOUR EQUIVALENT DEFINITIONS OF ONE-TO-ONE FUNCTION

1) To every input x, f(x) assigns a different output.
2) If f(a) = f(b), then a = b.
3) The graph of f(x) passes the horizontal line test.
4) The function f(x) has an inverse function $f^{-1}(x)$. If the function f(x) takes input x and gives output y = f(x), then the inverse function $f^{-1}(x)$ does the opposite: it takes input y = f(x) and gives output x.

The set of positive integers is the same size as the set of positive even integers. Use the function f(x) = 2x.
1 -> 2
2 -> 4
3 -> 6
4 -> 8 etc.

The set of positive integers is the same size as the set of all integers. Use the function f(x) which is x/2 if x is even, and is -(x-1)/2 if x is odd.
1 -> 0
2 -> 1
3 -> -1
4 -> 2
5 -> -2
6 -> 3
7 -> -3

The set of positive integers is the same size as the set of all fractions of positive integers.

1/1	2/1	3/1	4/1	5/1	6/1 ....
1/2	2/2	3/2	4/2	5/2	6/2 ....
1/3	2/3	3/3	4/3	5/3	6/3 ....
1/4	2/4	3/4	4/4	5/4	6/4 ....

A set is *countably infinite* if it is the same size as the set of positive integers.

2^x IS A ONE-TO-ONE FUNCTION. We still have not rigorously defined what 2^x means when x is a real number. But we expect that if $a < b$, then $2^a < 2^b$. Therefore each input will be assigned a different output.

Consequently, $y = 2^x$ has an inverse function. We call it $x = \log_2 y$, or reversing notation (so that x is the variable), we write it: $y = \log_2 x$. What are the domain and range of $y = 2^x$? It follows that *the set of all real numbers x* and *the set of all positive real numbers x* have the same size!

In the same way, $\{x \mid 0 < x < 1\}$ and $\{x \mid 1 < x < \infty\}$ have the same size: Look at the function $y = 1/x$. It is one-to-one.
When its domain is $0 < x < 1$, its range is $1 < x < \infty$.

IN BETWEEN ANY TWO RATIONAL NUMBERS THERE IS A REAL NUMBER
Take the average of the two rational numbers.

IN BETWEEN ANY TWO REAL NUMBERS THERE IS A RATIONAL NUMBER
Suppose that the first place where the decimal expansions differ is at the nth digit. Let x be the larger of the two numbers. Then consider the rational number which consists of the first s digits of x. For example:

.23587324965293457982375968435987293893561854108З...
.235873
.23587123572356981248239568001245803345340349887O...

DECIMAL EXPANSIONS ARE MORE NUMEROUS THAN RATIONAL **NUMBERS** Suppose not. Then we could make a list of all the decimal expansions d where .00000... ≤ d ≤ .99999...

1) .**2**9284763729102867498721349837277783990108793847235879...
2) .2**3**5674820834679734569286379587039875982703857928379 87...
3) .71**2**3491725972943570945671212948182759287435019205724 8...
4) .098**7**9823203945798723652875472039465987938475787987877...
5) .0000**0**0345555598900344448987783945883475878222978934 85...
6) .99899**8**7234573847584789293877686500008702340485480209 8...
7) .097898**7**98234578484775839457932875827394579873485783 79...
8) .4445435**3**45398700000000000000000000000000000000000000...
9) etc.

Whatever the list may be, the following procedure yields a decimal expansion which is not on the list !

Procedure: Construct a new decimal expansion D in the following way. Make sure that its first digit is different from the first digit of the first decimal expansion on the list. Make sure that its second digit is different from the second digit of the second decimal expansion on the list. In general, make sure that its nth digit is different from the nth digit of the nth decimal expansion on the list.

For example, if the list is given as above, then we construct a new decimal expansion D in the following way:

The first digit is not 2
The second digit is not 3
The third digit is not 2
The fourth digit is not 7
The fifth digit is not 0
The sixth digit is not 8
The seventh digit is not 7
The eighth digit is not 3 etc.

Now D is a decimal expansion, and .000... ≤ D ≤ .999... , so it must be somewhere on the list! Say it is in the nth place on the list. But by the way we constructed it, D is different from the nth decimal expansion on the list (just look at the nth digit !).
Therefore D is not on the list. We have arrived at a contradiction.

Therefore the set of decimal expansions is not countably infinite. There is no procedure for listing all decimal expansions.

Footnote: The Axiom of Choice allows us to make the choice "the first digit is not 2" without specifying how to do it. Some mathematicians rely on this Axiom all the time, while others avoid it whenever they can. We can avoid this Axiom by specifying a rule. For example: If the digit was 0, change it to 1. If the digit was 1, 2, 3, 4, 5, 6, 7, 6, 8, or 9, then change it to 0.

A RIGHT TRIANGLE IS HALF OF A RECTANGLE 5/11

A rectangle is completely determined by its width and its length.
The shape of a rectangle - by the ratio of its width to its length.

A right triangle is completely determined by any two sides.
The shape of a right triangle - by the ratio of any two sides.

The shape of a right triangle is also determined by the size of any angle (except for the right angle). Therefore we can define a function so that the input is an angle, and the output is a ratio of two sides. Such a function will be one-to-one: different angles will yield different ratios.

THE HEIGHT OF A TREE I We see a tree in the distance and we want to know its height. We walk closer until the following occurs: when we extend our arm at full length, the tree is the size of our thumb. From this position we walk towards the tree and count our paces until we get to the tree. We count exactly 100 paces.

A) How tall is the tree ?

B) When the tree looked the size of our thumb, how far away were we from its top ?

THE HEIGHT OF A TREE II We stand at the base of a tree and on it we mark with chalk the height of our eye - say it is 5 feet above the ground. We then walk 80 feet away from the tree. We have an instrument for measuring angles. We calibrate our instrument so that it reads 0° when we look at the chalk marking. We find that it reads 45° when we look at the top of the tree.

A) How tall is the tree ?

B) There is a bird at the top of the tree. How far away is the bird ?

FLYING A KITE We get our kite up into the air. We reel out all 100 yards of twine. We sit down and watch our kite. It is flying at an angle of 65° above the ground.

A) How high up is the kite ?

B) The sun is directly overhead. How far away is the kite's shadow ?

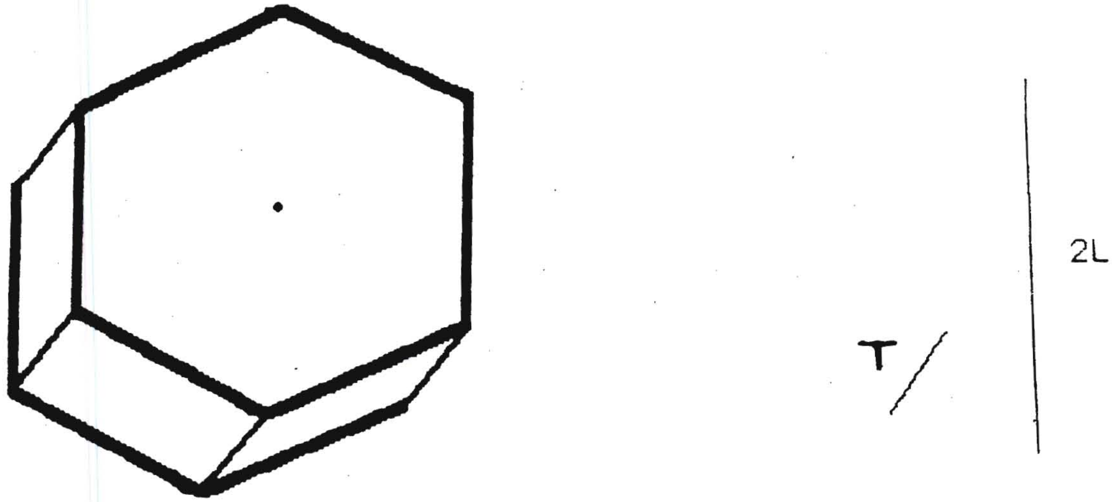
ROWING A river is 1/2 mile across. A rower is making her way across. In still waters she can row 8 miles per hour. She is rowing very hard in the direction perpendicular to the river bank. She does not see that the current is strong and that she is drifting 60° off course.

A) How fast is the river flowing and how far down the bank will she find herself ?

B) How fast is the boat moving and what is the total distance that she will have travelled ?

C) In what direction should she row if she wants to get across the river as fast as possible ?

VOLUME OF HONEYCOMB Honeybees build a chamber which has a face the shape of a hexagon. The thickness of the chamber is T, and the greatest distance across the hexagon is 2L. How much honey can the chamber hold ?

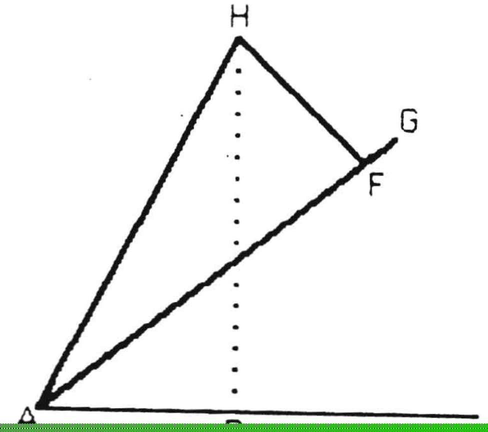


LOVERS' TRIANGLE In a crowded ballroom there is one person per square meter. Monsieur Z admires Mademoiselle X. She returns his glance. They stand exactly **a** meters apart. Then he turns his head **C** degrees to the left. At a distance of **b** meters stands Monsieur Y, who is aware of all and moves to cut him off from her.

How many people are trapped inside this lovers' triangle ?

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

STUDYING ON THE BEACH You are on the beach and the sun is directly overhead. You have your notebook cracked open $\alpha = 45°$. You crack it open another $\beta = 30°$. The notebook is L inches across. What is the length of the shadow ?



[illegible]

[illegible]

HOMEWORK DUE 5/21 5.4 #46, 5.5 #24, #32 6.1 #30 6.3 #40, #44, #48 6.5 #38, #44 6.6#58

READ: Friday: 5.3, Wednesday: 6.1, Friday: 6.2, Wednesday: 6.3

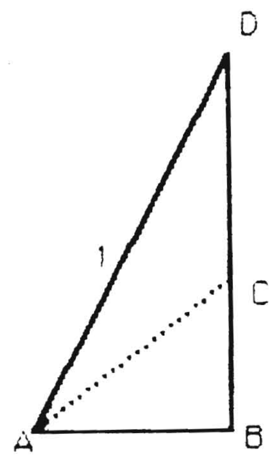
A TRIANGLE WITH AN OBTUSE ANGLE IS THE DIFFERENCE OF TWO RIGHT TRIANGLES 5/15

We know how to calculate sin(x) and cos(x) when x is 30°, 45°, or 60°. We know that sin(x) and cos(x) are defined for all acute angles, that is, for all real numbers x where $0^\circ < x < 90^\circ$. Given angle x, we draw a right triangle with that angle. If the hypotenuse has length 1, then we measure the length of the adjacent leg to get sin(x). We measure the length of the opposite leg to get cos(x).

sin(x) is increasing. cos(x) is decreasing. Both are one-to-one functions and both have inverses.

How do we extend the domain of these functions ? First we consider the "limiting cases" $x=0^\circ$ and $x = 90^\circ$. Then, as with exponentiation, we consider the addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$



$\sin(0^\circ) = 0$
$\cos(0^\circ) = 1$

$\sin(90^\circ) = 1$
$\cos(90^\circ) = 0$

$\sin(180^\circ) = \sin(90^\circ+90^\circ) = \sin 90^\circ \cos 90^\circ + \cos 90^\circ \sin 90^\circ = 0$
$\cos(180^\circ) = \cos(90^\circ+90^\circ) = \cos 90^\circ \cos 90^\circ - \sin 90^\circ \sin 90^\circ = -1$

$\sin(270^\circ) = \sin(180^\circ+90^\circ) = \sin 180^\circ \cos 90^\circ + \cos 180^\circ \sin 90^\circ = -1$
$\cos(270^\circ) = \cos(180^\circ+90^\circ) = \cos 180^\circ \cos 90^\circ - \sin 180^\circ \sin 90^\circ = 0$

$\sin(360^\circ) = \sin(270^\circ+90^\circ) = \sin 270^\circ \cos 90^\circ + \cos 270^\circ \sin 90^\circ = 0$
$\cos(360^\circ) = \cos(270^\circ+90^\circ) = \cos 270^\circ \cos 90^\circ - \sin 270^\circ \sin 90^\circ = 1$

Note that there is a period 360°:

$\sin(x+360^\circ) = \sin x \cos 360^\circ + \cos x \sin 360^\circ = \sin x$
$\cos(x+360^\circ) = \cos x \cos 360^\circ - \sin x \sin 360^\circ = \cos x$

EXTENDING THE DOMAIN OF TRIGONOMETRIC FUNCTIONS: THE CIRCLE 5/20

It suffices to look at the values from 0° to 360°: all other values repeat. We can therefore use a circle to describe the domain. Let the circle have radius 1. Each point x on the circle represents an angle. We know that if $0^\circ < x < 90^\circ$, then cos(x) gives the x-coordinate, sin(x) gives the y-coordinate, and $\sin^2(x) + \cos^2(x) = 1$ by the Pythagorean theorem.

Pairs of angles that are important:

Angles that add up to 0°:	x and -x.	
Angles that add up to 90°:	x and 90°-x	complementary angles
Angles that add up to 180°:	x and 180°-x	supplementary angles

How do we get sin(-x) and cos(-x) ?

$$0 = \sin(0^\circ) = \sin(x + (-x)) = \sin(x)\cos(-x) + \cos(x)\sin(-x)$$
$$1 = \cos(0^\circ) = \cos(x+(-x)) = \cos(x)\cos(-x) - \sin(x)\sin(-x)$$

We think of A = sin(x) and B = cos(x) as constants, and we think of sin(-x) and cos(-x) as the unknowns that we are solving for.

$$0 = A\cos(-x) + B\sin(-x)$$
$$1 = B\cos(-x) - A\sin(-x)$$

We get that $\sin(-x) = [-A]/[A^2 + B^2]$ and $\cos(-x) = [B]/[A^2 + B^2]$. Furthermore, by the Pythagorean theorem, $A^2 + B^2 = 1$. We therefore get:

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

We can now derive subtraction formulas:

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

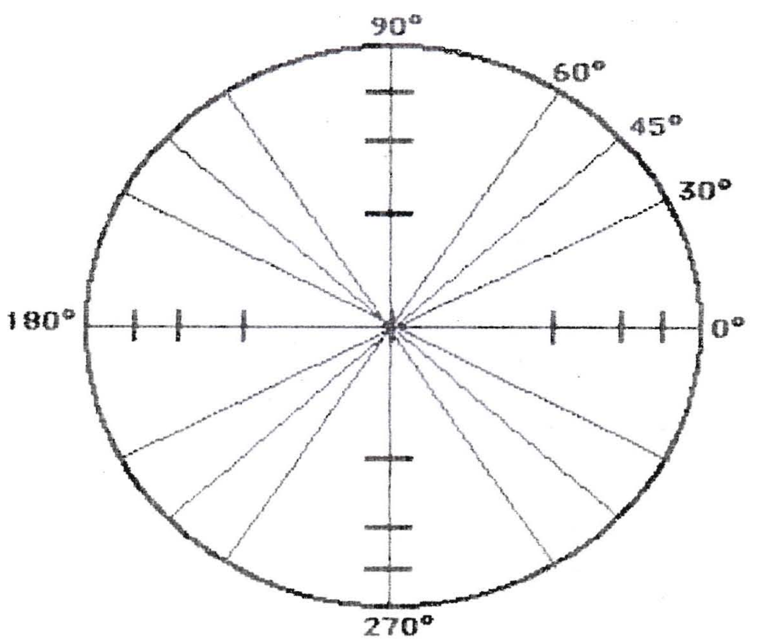
Therefore

$$\sin(90^\circ-x) = \sin(90^\circ)\cos(x) - \cos(90^\circ)\sin(x) = \cos(x)$$
$$\cos(90^\circ-x) = \cos(90^\circ)\cos(x) + \sin(90^\circ)\sin(x) = \sin(x)$$

$$\sin(180^\circ-x) = \sin(180^\circ)\cos(x) - \cos(180^\circ)\sin(x) = \sin(x)$$
$$\cos(180^\circ-x) = \cos(180^\circ)\cos(x) + \sin(180^\circ)\sin(x) = -\cos(x)$$

THE TRIGONOMETRIC FUNCTIONS

degrees				-90°	-60°	-45°	-30°	0°	30°	45°	60°	90°
radians								0				
sin(x)												
cos(x)												
tan(x)												
degrees	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
radians												2π
sin(x)												
cos(x)												
tan(x)												



$\frac{\sqrt{4}}{2}$

$\frac{\sqrt{3}}{2}$

$\frac{\sqrt{2}}{2}$

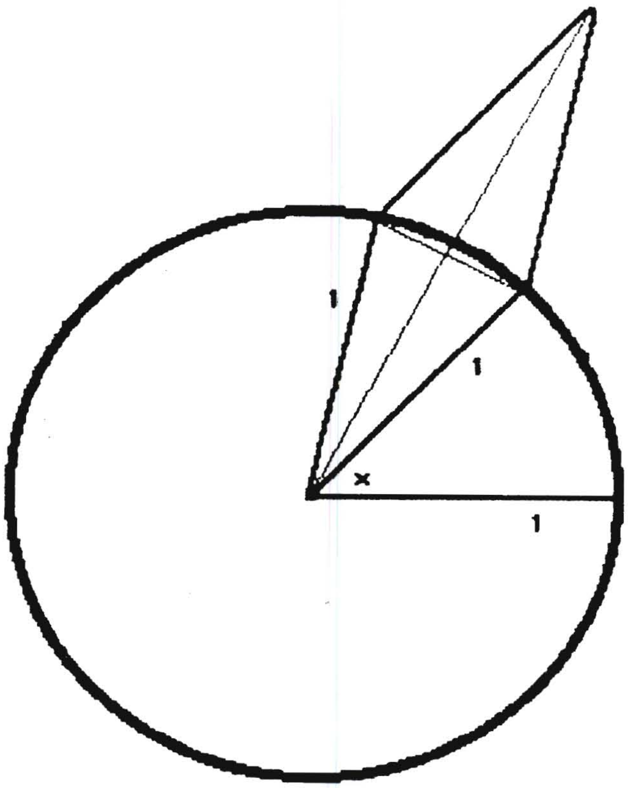
$\frac{\sqrt{1}}{2}$

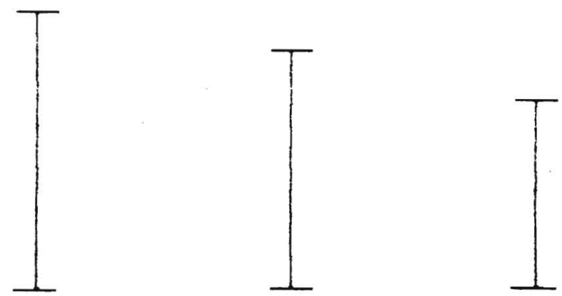
$\frac{\sqrt{0}}{2}$

PERIODIC FUNCTIONS 5/22

$y = A \sin[B(x - \frac{C}{B})]$ $\quad$ $y = A \cos[B(x - \frac{C}{B})]$

$|A|$ amplitude $\quad$ $\frac{2\pi}{B}$ period $\quad$ $\frac{C}{B}$ phase shift





$\sin(x+\partial)$ $\quad$ $\sin(x)$

$\frac{\sin(x) + \sin(x+\partial)}{2}$

$$\frac{\sin(x) + \sin(x+\partial)}{2} = \cos(\tfrac{\partial}{2}) \sin(x + \tfrac{\partial}{2})$$

$$\sin(x) + \sin(x+\partial) = 2\cos(\tfrac{\partial}{2}) \sin(x + \tfrac{\partial}{2})$$

x variable
∂ constant

the resulting function is a sine function !

amplitude $2\cos(\frac{\partial}{2})$

period 2π

phase shift $-\frac{\partial}{2}$

HOMEWORK DUE: 5/28
5.5 #26, #47 6.8 #24, #48
pg.382 #207 8.1 #48

I) Find the three exercises in the book that you think are the most interesting or significant. Explain why.
II) Consider a mathematical problem that arises in your work or in your field of study. Does it involve linear, quadratic, polynomial, exponential, trigonometric, or other functions ? Describe the problem - the quantities that it involves and the relation between them.

$y = \sin x + \sin x$

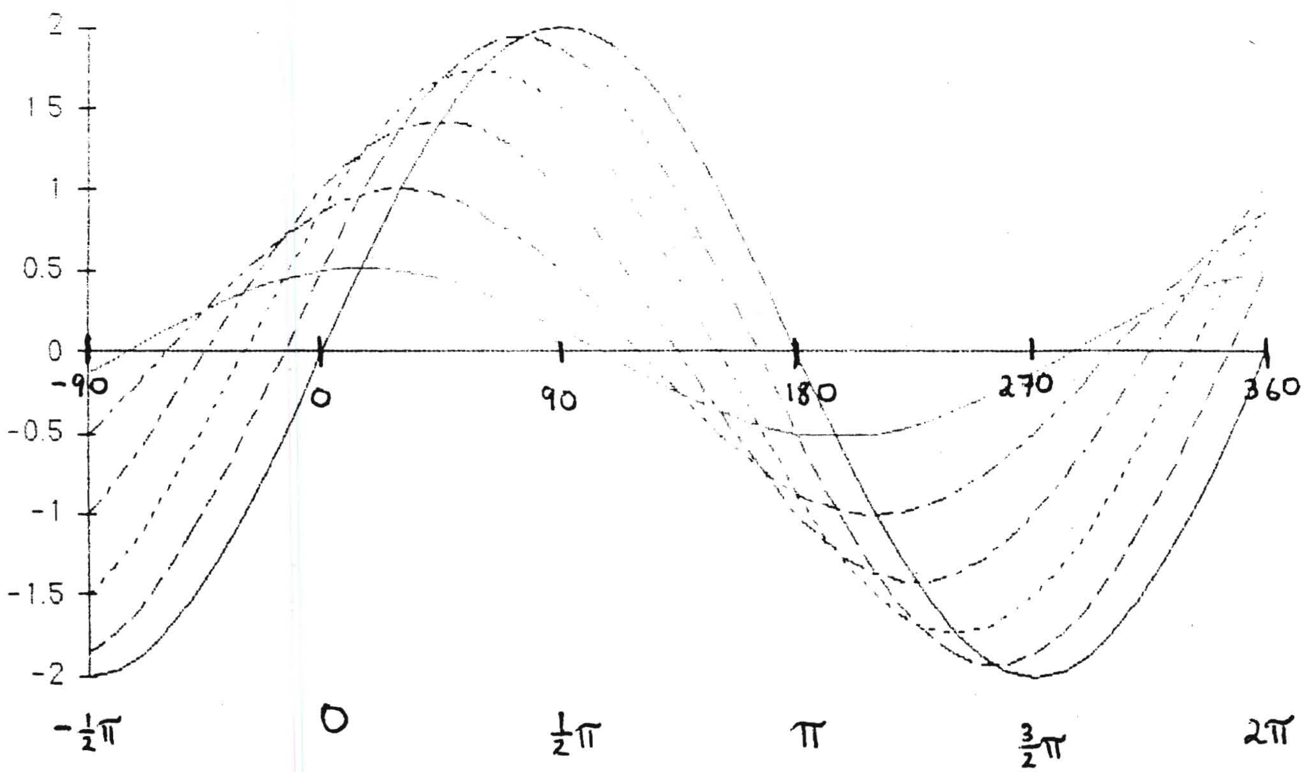
$y = \sin x + \sin(x+30)$

$y = \sin x + \sin(x+60)$

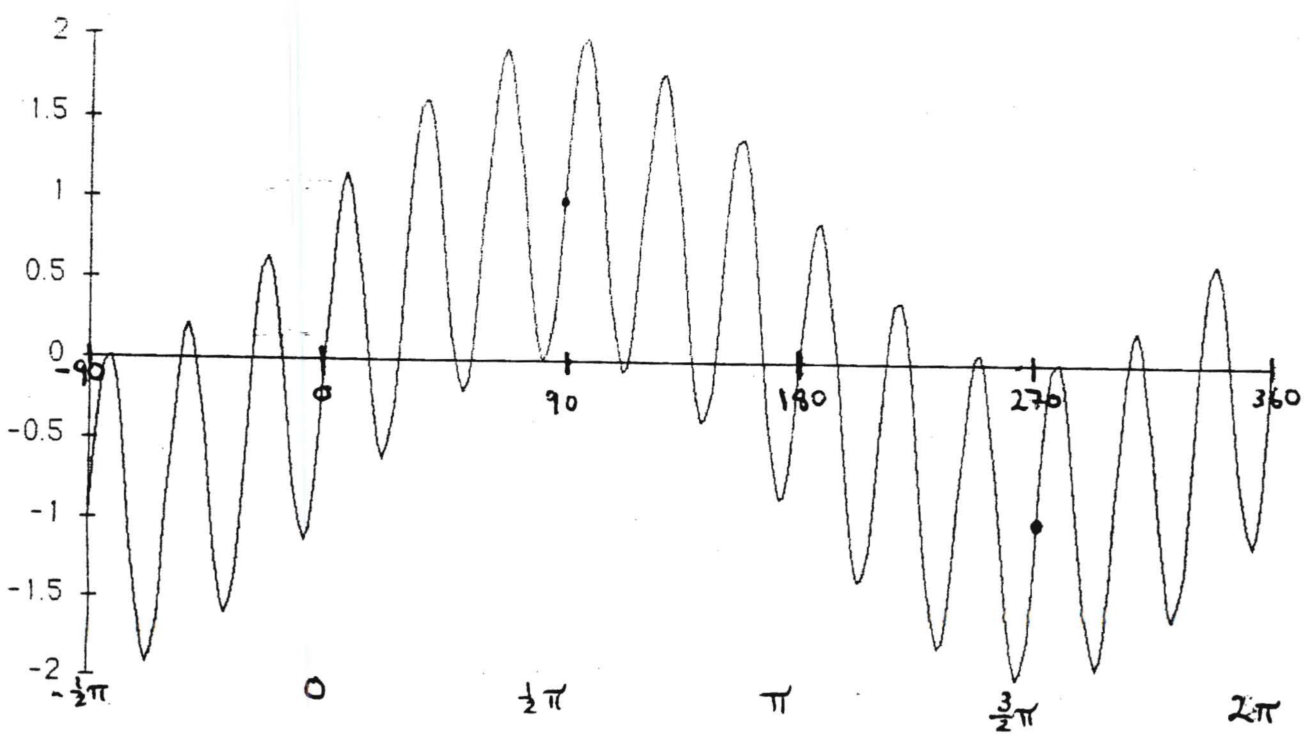
$y = \sin x + \sin(x+90)$

$y = \sin x + \sin(x+120)$

$y = \sin x + \sin(x+150)$

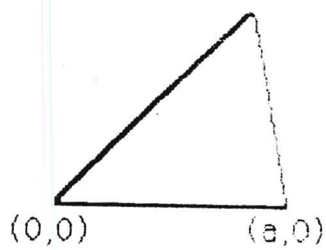


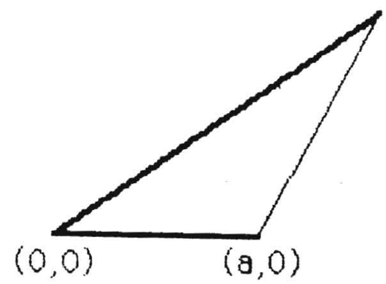
$y = \sin x + \sin 12x$



THE DISTANCE FORMULA 5/27

By the Pythagorean theorem we know that the distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.





Law of cosines: $c^2 = a^2 + b^2 - 2b\cos(C)$

Law of sines: $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

If we are looking for angle x, and $0° < x < 180°$, then $\cos x = d$ yields one solution, whereas $\sin x = d$ yields two solutions.

Therefore, when we are led to use the Law of Cosines, we find a unique angle x. When we are led to use the Law of Sines, we find two angles: one acute, and the other obtuse.

SIDE: SIDE: SIDE
find angle -> Law of Cosines

SIDE: ANGLE: SIDE
find missing side -> Law of Cosines
find missing angle -> first find missing side

SIDE: SIDE: ANGLE
find the angle across -> Law of Sines (should get two answers !)
find the angle in between the sides -> first find the angle across

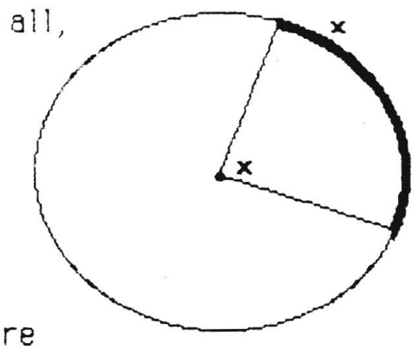
SIDE and two ANGLES
(know all three angles)
find missing side -> Law of Sines

AREA OF A CIRCLE 5/29

All circles have the same proportions. If a circle has radius r, then there is a number π such that $2\pi r$ is the circumference of the circle, that is, the distance around.

The "unit circle" has radius 1 and circumference 2π. We use this to measure angles: the total angle around is 2π "radians". Smaller angles are fractions of 2π.

What advantage does this way of measuring have ? First of all, it is easy to measure the length of a segment on the circle. The length x of the segment equals the angle x.

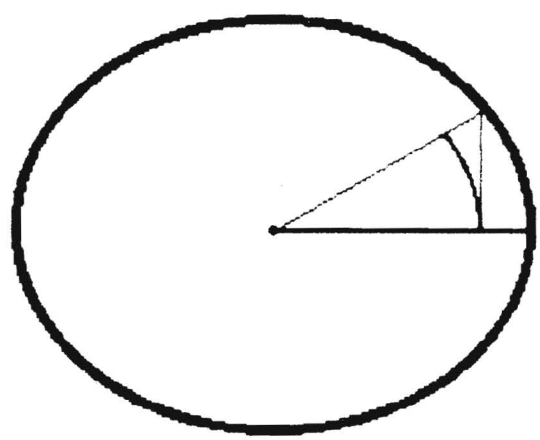


CALCULATOR TRICK: Let x be a number close to 0. Make sure that your calculator is using radians, then calculate sin(x). What do you notice ?
Is this still true when your calculator is using degrees ?

If x is in radians, then the approximation $x \approx \sin(x)$ becomes better and better as x approaches zero.
In other words, the ratio between the two approaches one

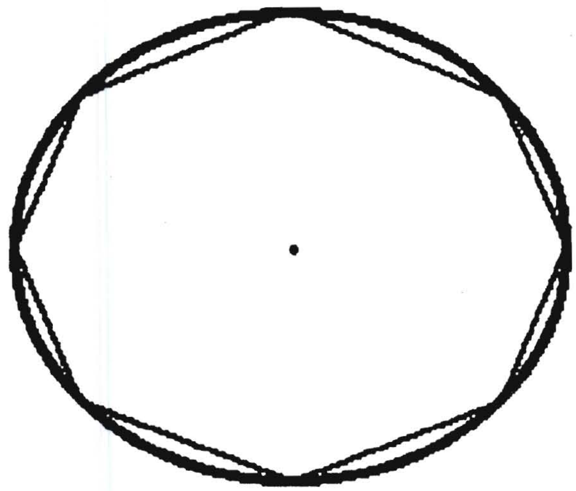
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

The above fact is only true if we calculate x using radians: it is the main reason why we use radians.

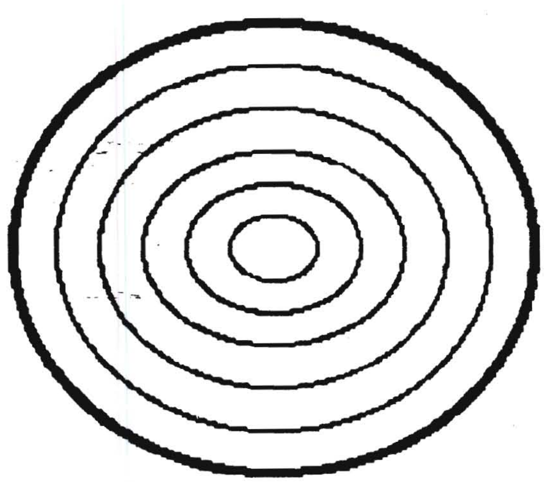


SANDWICH THEOREM: We prove the above formula by using the sandwich theorem.

AREA OF A CIRCLE I   Within a circle we inscribe a regular polygon with n sides. What is the area of the polygon ?
It depends on n.
What happens to the shape of the polygon as n approaches ∞ ?
What happens to its area ?



AREA OF A CIRCLE II   The circle has radius r. We think of it as consisting of n rings, each of thickness r/n.
If we take a ring, cut it, and think of it as a rectangle, then what is its area ? Does this do justice to the area of the ring ?
If there are more rings, does this approximation improve ? What happens to the total area of these rectangles as n approaches ∞ ?



EXAMINATION OF CASES This technique is good for attacking intractable problems where it may be fruitful to work on individual cases rather than with the whole thing at once. In our problem we may consider separately the cases when N is even and N is odd.

If N is even, then we can pair up all of the numbers from 1 to N. This makes for N/2 pairs. We pair up the numbers in the following way. First we pair 1 with N, then we pair 2 with N-1, then 3 with N-2 and so on, so that in general we pair up the number i with the number N - (i+1). We note that the sum of the numbers in each pair is i + N - (i+1) which equals N+1. Therefore the total is:

(number of pairs) (contribution from each pair)

If N is odd, then how many numbers are there from 0 to N ? There are N+1 in all, and this number is even. Therefore we can pair up all of the numbers from 0 to N. This makes for _____ pairs. We pair up the numbers in the following way. First we pair 0 with ____, then we pair 1 with ____, then 2 with ____ and so on, so that in general we pair up the number i with the number _________. We note that the sum of the numbers in each pair is ________ which equals _____. Therefore the total is:

(number of pairs) (contribution from each pair)

As we have shown the formula to be true when N is even and when N is odd, it is true for positive integers in general.

ALGEBRAIC MANIPULATION Suppose that we already know the following formula:

$$1 + 3 + 5 + 7 + \ldots + (2j - 1) + \ldots + (2T - 1) \quad = \quad T^2$$

There are T terms on the left hand side. If we want to add 1 to each term on the left hand side, then we must be adding T to each side of the equation. This gives us:

Dividing by two yields:

A CERTAIN INFINITE POLYNOMIAL

Calculate: $(1 + \frac{x}{1})^1$

$(1 + \frac{x}{2})^2$

$(1 + \frac{x}{3})^3$ Do you see any patterns ?

$(1 + \frac{x}{4})^4$

How many terms will $(1 + \frac{x}{n})^n$ have ? What happens to the number of terms as n approaches infinity ? What happens to the coefficients as n approaches infinity ?

Fix n. Using the binomial theorem, calculate the kth term of $(1 + \frac{x}{n})^n$

What happens to this term when n approaches infinity ?

Show that as n approaches infinity we end up with the infinite polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \cdots + \frac{x^n}{n!} + \cdots$$

Graph the following polynomials: What do they have in common, and how are they different ?

$(1 + \frac{x}{1})^1$

$(1 + \frac{x}{2})^2$

$(1 + \frac{x}{3})^3$

$(1 + \frac{x}{4})^4$

What do you think the graph of $(1 + \frac{x}{n})^n$ looks like in general ?

What happens to this graph as n approaches infinity ?

TESTING A CERTAIN IDENTITY

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \cdots + \frac{x^n}{n!} + \cdots$$

We will not try to prove the above formula, but will test it in various other ways.
Plug in $x = 1$. What do we get on either side ? How many terms do we need to include on the right hand side so that both sides are approximately equal ?
Plug in other values for x. How many terms on the right hand side do you use in each case ?

How can you add infinitely many things together and get a finite number ?

If the identity above is really true, then we should be able to check the following:

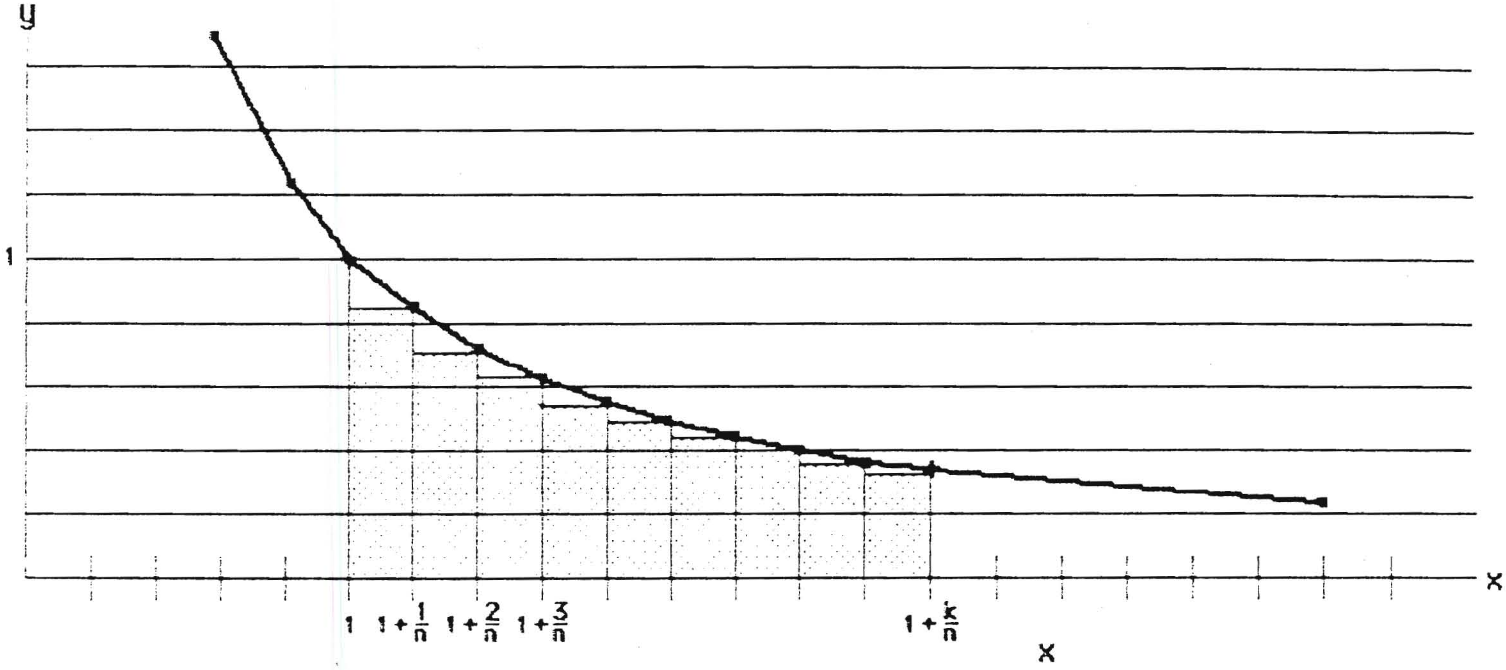
$$(e^x)^2 = e^{2x}$$

(Replace e^x and e^{2x} with the relevant infinite polynomials) Prove the equality by multiplying out and using the binomial theorem.

Suppose that the above infinite polynomial truly equals e^x . How would it then follow that for any fixed n, $\quad e^x > x^n \quad$ if x is large enough ?

Graph the functions 1, x, $\frac{x^2}{2!}$, $\frac{x^3}{3!}$, $\frac{x^4}{4!}$. Which of these is the greatest in which region ?

A CERTAIN REGION UNDER THE CURVE $y = \frac{1}{x}$



Consider the region beneath the graph $y = \frac{1}{x}$, above the x-axis, and to the ~~left~~ right of the line x = 1. How far to the right must we go along the x-axis before the area enclosed equals 1 ?

In order to do approximate the area beneath the curve, construct thin rectangles as shown in the picture. Let each rectangle have width $\frac{1}{n}$, where n is a big integer. Let k be the number of rectangles needed before the area equals 1. What is the height of each rectangle ? Show that the total area of the rectangles is given by:

$$\left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{1}{n}\right) + \left(\frac{1}{1+\frac{2}{n}}\right)\left(\frac{1}{n}\right) + \left(\frac{1}{1+\frac{3}{n}}\right)\left(\frac{1}{n}\right) + \ldots \ldots + \left(\frac{1}{1+\frac{k}{n}}\right)\left(\frac{1}{n}\right) \approx 1$$

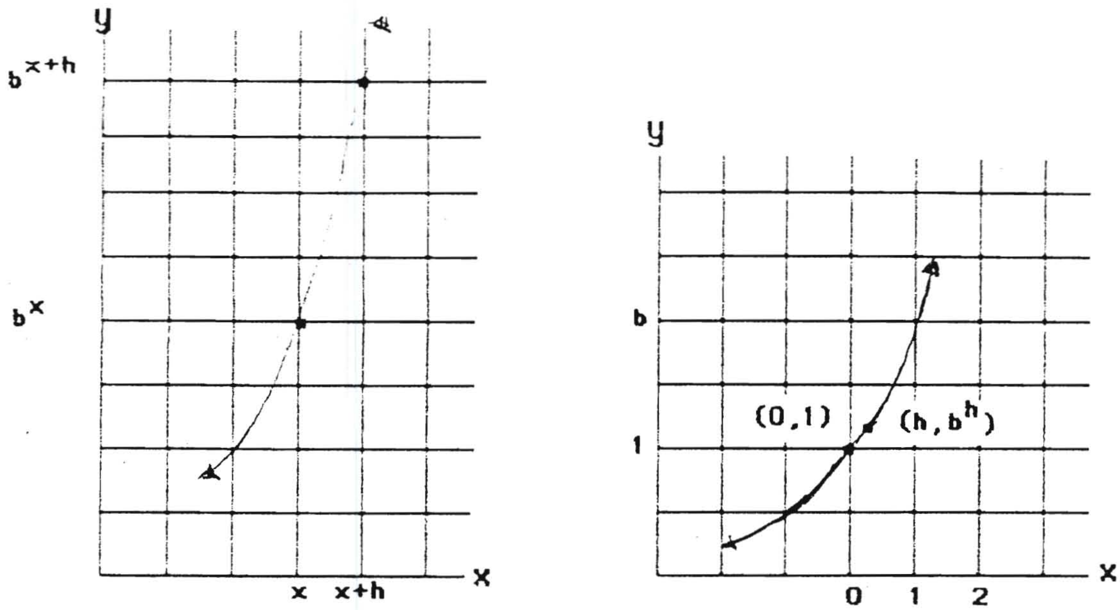
Show that this equation is equivalent to:

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots \ldots + \frac{1}{n+k} \approx 1$$

By trying out different integers, find the n and k such that the left hand side is as close to 1 as possible.

What value is $1 + \frac{k}{n}$?

THE STEEPNESS OF THE CURVE $y = b^x$



Let (x, b^x) and $(x+h, b^{x+h})$ be two points on the curve $y = b^x$
What is the slope of the line passing between these two points ?

Note that as h becomes small, the second point becomes closer to the first, until finally the line seems to touch the curve at only one point.
Show that the steepness at the point (x, b^x) is the product of b^x and the steepness at the point (0,1).
Therefore, we will look for a function b^x such that its steepness at the point (0,1) equals 1. Then we will have a function whose steepness at x will always equal its value at x.
For such a b, show that the following equation must be true.

$$b^h = 1 + h$$

If $h = 1$, solve for b. Likewise, if $h = \frac{1}{2}$, solve for b.
After that, show that in general, if you solve for b you get:

$$b = (1 + h)^{\frac{1}{h}}$$

Then let h be a very small number, and solve for b.
What do you get ?

THE RATE OF CHANGE OF AN INFINITE POLYNOMIAL

Recall how we once interpreted all of the terms of $(x+h)^3$ as relating to a cube which had grown by h inches in width, length, and height. The term x^3 gave the original volume, and the term $3(x)^2(h)$ gave the volume of the three faces that arose due to growth.

The rate of change in the volume (the rate of change in x^3) is given by

$$\frac{\text{change in volume}}{\text{change in h}} = \frac{(x+h)^3 - x^3}{h}$$

If h is very small, then show that the only relevant term left is $3x^2$ which is the growth due to the faces. Show that the other terms disappear as h approaches 0.

Now do the same for the function x^2. Give a geometric interpretation for $(x+h)^2$. Then calculate the change in volume per change in h. Finally, show what term survives as h approaches 0.

Do the same for the function x, and then for the function 1.

Now consider the function x^n, where n is fixed. What do you think will be the change in x^n per change in h ? Try to prove this.

Now consider the following function: the infinite polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \cdots + \frac{x^n}{n!} + \cdots$$

Show that if we find the change in this infinite polynomial per the change in h, we get back the exact same infinite polynomial !!

DEFINING e^x FOR COMPLEX NUMBERS X

We will accept the formula below as the definition for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \quad \dots + \frac{x^n}{n!} + \dots$$

We will consider what happens when x is a complex number of the form $a+bi$ where a, b are real numbers and $i = \sqrt{-1}$

First we consider the following identities:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

See if they make sense for various values of x (make sure that x is in radians).

Check to see if $\sin^2 x + \cos^2 x = 1$

by squaring the infinite polynomials and adding them together. Try to prove that it is indeed true.

Consider $i = \sqrt{-1}$. Calculate $i^2, i^3, i^4, i^5, i^6 \dots$

Using the identity at the top of the page, evaluate e^{ix}. Simplify and come up with a formula for e^{ix} in terms of sin x, cos x, and i.

Then come up with a formula for e^{a+bi} where a and b are real numbers

MATH 4C: SECOND MIDTERM

Name: ______________________ **May 18, 1992**

Andrius Kulikauskas

Do all eight problems. They are all worth the same number of points. Show all of your work. If you are presenting a proof or argument, make sure that each step is understandable. Feel free to use English. **No calculators.** If a question asks for a numerical answer, simplify the answer as much as is possible without the use of a calculator. **No notes. Relax. Good luck!**

1) You place money in a bank account. The interest rate is 6% and it is compounded annually. How long will it take for your money to double ?

2) It is a fact that $\log_{10}x^n = n\log_{10}x$ for all positive integers n. Without making use of this formula, prove that the following is true:

$$\log_{10}x^3 = 3\log_{10}x$$

3) A family has six children. With the birth of each child, there was a 50% chance that it would be a girl, and a 50% chance that it would be a boy. What is the probability that the family has exactly three girls and three boys?

4) Prove that the formula below is true:

$$1 + 2 + 3 + \dots + (N-1) + N = \frac{N(N+1)}{2}$$

Use the method of proof of your choice.

5) Suppose that $f(x)$ is an increasing function. That is, the following is always true: if a and b are in the domain of $f(x)$, and $a < b$, then $f(a) < f(b)$.

Is it necessarily the case that $f(x)$ is a one-to-one function ?

If yes, then prove that $f(x)$ is a one-to-one function.

If no, then provide an example of a function $f(x)$ which is an increasing function, but is not a one-to-one function.

6) Graph $y = -\left(\frac{1}{2}\right)^{x+\frac{1}{2}} + 1$

7) A 10 foot ladder is leaning against a wall. The ladder and the ground form an angle of 30°. At what height H is the ladder touching the wall ?

8) A triangle has one side of length a, another side of length b, and the angle in between is C. Derive a formula that gives the area of this triangle in terms of lengths a, b, and angle C. It is not enough to merely state the correct answer!

4/17 What is the chance that if we flip a coin five times, three of the times it will land on heads, and two of the times it will land on tails ?

H heads
T tails
addition means "either or"
multiplication means "and"

$$(H+T)^5 = \frac{5!}{5!\,0!}H^5 + \frac{5!}{4!\,1!}H^4T^1 + \underbrace{\frac{5!}{3!\,2!}H^3T^2}_{} + \frac{5!}{2!\,3!}H^2T^3 + \frac{5!}{1!\,4!}H^1T^4 + \frac{5!}{0!\,5!}T^5$$

H^3 means three heads

T^2 means two tails

$$\frac{5!}{3!\,2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(2\cdot1)} = 10$$

gives the number of ways of choosing three out of five, for example, the number of ways that exactly three out of five coins end up heads.

$H = \frac{1}{2}$ because there is a 50% chance of getting heads
$T = \frac{1}{2}$ because there is a 50% chance of getting tails

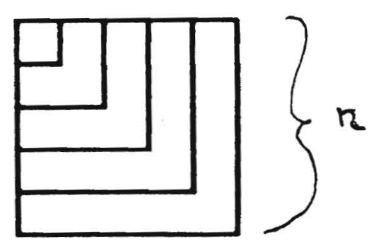
The chance of getting 3 heads and 2 tails is

$$10H^3T^2 = 10\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 = 31.25\,\%$$

4/20 Let Pn be the statement that $1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$
Prove that Pn is true for all positive integers n.
(Use the method of proof of your choice).

Note: This problem is wrong as originally stated.

BIJECTION: Note that the following n x n square has area n^2. It is built of bent shapes. The smallest of these has area 1. Each consecutive shape has an area that is two squares greater than the one before. The largest shape has area $n + n - 1 = 2n - 1$
The sum of the areas of the bent shapes equals the area of the square. Therefore $1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$



INDUCTION
Part I: We show that P_1 is true.
P_1 says that $1 = 1^2$. So it's true.
Part II: We assume that P_k is true. Using this assumption, we show that P_{k+1} must be true.
P_k says $1+3+5+\dots+(2k-1) = k^2$ We assume it's true.
Consequently $1+3+5+\dots+(2k-1)+(2(k+1)-1) = k^2 + (2(k+1)-1)$

$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

But this proves that P_{k+1} is true! Because P_{k+1} says:

$$1+3+\ldots+[2(k+1)-1] = (k+1)^2$$

Conclusion: We have shown that P_1 is true. We have also shown that if P_k is true, then P_{k+1} is true. Therefore, by induction, it is the case that P_n is true for all positive integers n.

4/27 The length of a bacteria is 10^{-6} meters. The length of the universe is 10^{26} meters. If the universe is shaped like a cube, and if a bacteria is shaped like a cube, and if we can stack bacteria in the way we stack cubes, then how many bacteria can fit in the universe ?

Volume of a bacteria: $(10^{-6})^3 = 10^{-18}$ cubic meters

Volume of the universe: $(10^{26})^3 = 10^{78}$ cubic meters

Number of bacterias: $\dfrac{\text{Size of universe}}{\text{Size of bacteria}} = \dfrac{10^{78}}{10^{-18}} = 10^{96}$.

4/24 We start with one bacteria. With the passing of every hour, the number of bacteria is double what it was before. When will bacteria fill the entire universe ? (Note: calculate as far as you can without a calculator. If you can get an approximate answer without a calculator, that is all the better - you can use the fact that $2^{10} \approx 10^3$.)

Exponential growth is described by the formula $N = N_0(R^t)$

N is amount at time t N_0 is amount at time 0.

R gives the way in which the amount is growing (or shrinking).

$N_0 = 1$ $R = 2$ (Why? because at time $t=1$, $N=2$ so $2 = 1R^1$ $2 = R^1$ so $R=2$) So we get 10^{96} bacteria at time t: solve for t. $10^{96} = 1(2^t)$.

$10^{96} = 2^t \qquad \log_{10}(10^{96}) = \log_{10}(2^t) \qquad \log_{10}(10^{96}) = t\log_{10}(2)$

$t = \dfrac{\log_{10}(10^{96})}{\log_{10}(2)}$ You can leave it in this form on the test.

But notice that $\log_{10}(10^{96}) = 96.$ How do we get $\log_{10}(2)$?

Well, $10^3 = 1{,}000 \approx 1{,}024 = 2^{10}$ so $\log_{10}(10^3) \approx \log_{10}(2^{10})$

so $3 \approx 10\log_{10} 2$ so $\log_{10} 2 \approx \frac{3}{10}$ where $\approx$ means "approximately equal to".

So $t \approx \dfrac{96}{\frac{3}{10}} = 320$ hours, about two weeks.

4/29 We come from Jupiter and we wear gloves. The number of fingers that we have is a positive integer. The actual number is a secret: let us say that we have b+1 fingers in all. This is why the number system that we use is base b+1, that is, we make use of b+1 different symbols in writing down numbers.

Starting with the number 1 0 0 0 (we have written it in Jupiter notation) we subtract the number 1 (written in Jupiter notation). What number do we get (in Jupiter notation) ?

If we use base b+1, then we have b+1 symbols in all, say they are: $0, 1, 2, 3, \ldots, b-1, b$.

(In base 10 the symbols are $0,1,2,3,4,5,6,7,8,9$

In base 2 they are $0, 1$).

Note that in "Earth notation", that is, base 10,

$$\begin{array}{r} 1{,}000 \\ -\quad 1 \\ \hline 999 \end{array}$$

this says that $1(10^3) + 0(10^2) + 0(10^1) + 0(1)$ is the same as

$9(10^2) + 9(10^1) + 10(1)$

we write it as

$$\begin{array}{r} \cancel{1},\overset{9}{\cancel{0}}\overset{9}{\cancel{0}}\overset{1}{0} \\ -\quad 1 \\ \hline 999 \end{array}$$

Subtracting 1 we get

$9(10^2) + 9(10^1) + 9(1)$.

$$1(b+1)^3 + 0(b+1)^2 + 0(b+1)^1 + 0(1)$$

$$= (b+1)(b+1)^2 + 0(b+1)^2 + 0(b+1)^1 + 0(1)$$

$$= (b+1)(b+1)^2 + 0(b+1)^1 + 0(1)$$

$$= b(b+1)^2 + (b+1)(b+1) + 0(1)$$

$$= b(b+1)^2 + b(b+1) + (b+1)(1)$$

Subtract 1, we get

$$b(b+1)^2 + b(b+1) + b(1)$$

which is written as bbb in Jupiter notation.

5/1 A computer uses the binary number system. That is, it uses only 1's and 0's to think about numbers and to write them down. How many different numbers can it write so that they have 6 or fewer digits ? Of these, how many have exactly three 1's ?

Six digits: _ _ _ _ _ _. Each one can be a 1 or 0. So for each one there are two choices. The total number of numbers is $2^6 = 64$. Suppose that exactly three are 1's, (so that exactly three are 0's as well). Choose three of the spaces (and fill them with 1's). Each such way gives a different number. How many ways can we choose three spaces out of six spaces? $\frac{6!}{3!(6-3)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(3\cdot2\cdot1)}$ $= 20$.

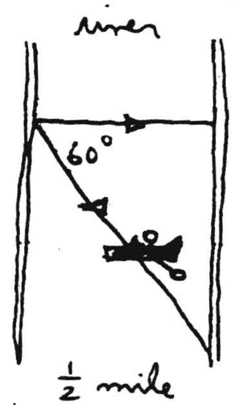
5/11 A river is 1/2 mile across. A rower is making her way across. In still waters she can row 8 miles per hour. She is rowing very hard in the direction perpendicular to the river bank. She does not see that the current is strong and that she is drifting 60° off course.
How fast is the river flowing and how far down the bank will she find herself ?
How fast is the boat moving and what is the total distance that she will have travelled ?
In what direction should she row if she wants to get across the river as fast as possible ?

$$\tan 60^\circ = \frac{s}{8}$$

$$\frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = \frac{s}{8}$$

$$s = 8\sqrt{3} \approx 13.9 \text{ miles per hour}$$

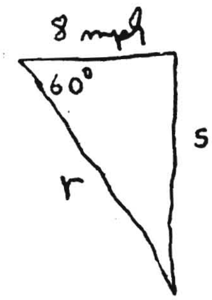
The river is flowing 13.9 miles per hour.



$$\cos 60^\circ = \frac{8}{r}$$

$$r = \frac{8}{\cos 60^\circ} = \frac{8}{\frac{1}{2}} = 16 \text{ miles per hour.}$$

is her total speed.



5/13 Calculate $\sin(75^\circ)$ without using your calculator.

$$\sin(75^\circ) = \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

How long does it take her to cross the river? This is not affected by the current. (The current makes her drift down, but does not keep her from moving steadily across).

$$8 \frac{\text{miles}}{\text{hour}} = \frac{\frac{1}{2} \text{ mile}}{t \text{ hours}} \qquad t = \frac{\frac{1}{2}}{8} = \frac{1}{16} \text{ hour}$$

Total distance travelled: $\left[16 \text{ miles per hour}\right]\left[\frac{1}{16} \text{ hour}\right] = 1 \text{ mile.}$

Distance drifted: $\left[13.9 \frac{\text{miles}}{\text{hour}}\right]\left[\frac{1}{16} \text{ hours}\right] = .869 \text{ mile}$

[illegible]

We multiply an even integer together with an odd integer. What kind of integer do we get, odd or even ? Prove your answer.

x even $= 2m$ where m, n are integers

y odd $= 2n+1$

$xy = (2m)(2n+1) = 4mn + 2m$

$= 2(2mn + m)$ Note that $2mn + m$ is an integer.

So xy is even.

(it is 2 times an integer).

Can we multiply together a nonzero integer X and an irrational number Y and get a rational number XY ? Prove your answer.

No we cannot.

PROOF: Suppose that we could. Then there would exist integers m, n such that $XY = \frac{m}{n}$. But then $Y = \frac{m}{Xn}$. The numerator is an integer. The demonminator is the product xn of two integers, so it is also an integer. Therefore Y is a rational number. This yields a contradiction. Therefore our assumption that XY can be rational must have been wrong.

What is the domain of the function $\sqrt{x}$? What is the range ? Is this a one-to-one function ?

$\sqrt{x}$ makes sense when $x \geq 0$. So the domain of $\sqrt{x}$ is the set $\{x \mid x \geq 0\}$.

What are the outputs y, where $y = \sqrt{x}$? All nonnegative numbers.

So the range of $\sqrt{x}$ is the set $\{y \mid y \geq 0\}$

We prove that it is a one-to-one function.

PROOF: Suppose that $\sqrt{a} = \sqrt{b}$. Then square both sides, we have $a = b$. So the function is one-to one.

ALTERNATE PROOF: Graph $y = \sqrt{x}$. Note that it passes the horizontal line test.

MATH 4C: FINAL EXAM

Name: ______________________ **June 10, 1992**

Andrius Kulikauskas

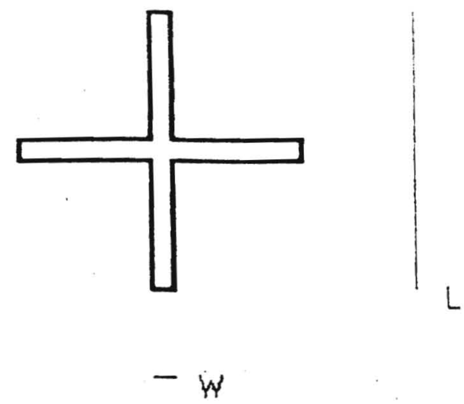
Do problems 1 through 14 first. Then attempt problems 15 through 18. Of these you may try all four but you will only receive credit for those two on which you do the best. **All problems are worth the same number of points. Show all of your work.** If you are presenting a proof or argument, make sure that each step is understandable. Feel free to use English. **No calculators.** If a question asks for a numerical answer, simplify the answer as much as is possible without the use of a calculator. **No notes. Relax. Good luck!**

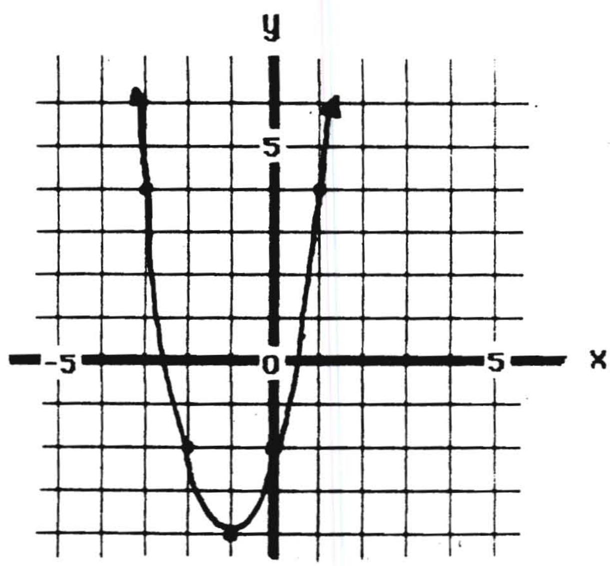
1) You have five dollars worth of change. It consists of nickels, dimes, and quarters. The nickels and dimes are worth one dollar if taken together. The number of quarters equals the number of nickels plus the number of dimes. How many quarters are there, how many dimes, and how many nickels ?

2) You are designing a square cross. It is to be the symbol for a relief o[illegible]nization.

[illegible] is the distance from one end of the cross to the other, whether up and down, or left to right. The two beams of the cross have width W.

The center of the cross consists of a square of area W^2. Let A be the area of the rest of the cross. Given L, what W yields a maximum for the area A ?





3) The graph in the picture is a parabola. It includes the points (-3,4), (-2,-2), (-1,-4), (0,-2), (1,4). Write down the quadratic function that is described by this graph.

4) You are designing license plates for the newly independent country of Lithuania. The first two symbols are to be Arabic numerals. All of the other symbols are to be letters from the Lithuanian alphabet - this alphabet has 32 letters in all. How many letters should there be on the license plate if we expect there to be no more than 2,000,000 motor vehicles?

5) Prove the change of base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

6) Prove that the formula below is true:

$$1 + 2 + 3 + 4 + \ldots + N = \frac{N(N+1)}{2}$$

Use the method of proof of your choice.

7) Graph the function $f(x) = \log_2 x$. Then answer the following questions. Justify your answers without referring to the graph.

What is the domain of $\log_2 x$?

What is the range of $\log_2 x$?

What happens to $\log_2 x$ when $x \to \infty$?

8) Suppose that $f(x)$ is a periodic function with period P. Can $f(x)$ be a one-to-one function ? Write down yes or no.

If yes, provide an example of such a function $f(x)$.
If no, give a proof of why there can be no such $f(x)$.

9) The sun rises at 6:00 AM. At noon it is directly overhead. It sets at 6:00 PM. You are 5 feet tall. How long is your shadow at 10:00 AM ?

10) The star Alpha Beta is 4 light years away from us. The star Vega Omega is 6 light years away from us. The two stars are $\sqrt{52}$ light years away from each other.

When we look up into the heavens above, what angle separates the two stars ?

11) Graph $y = \sin(2x + \pi)$

12) Consider the equation $y = (A)(B^t)$. Create and solve a word problem where you know y, A, t, and you are solving for B.

You need not be original, but you must indicate what y, A, t, and B mean with regard to the word problem. You must assign concrete numbers and units to y, A, t, and arrive at a concrete number for B.

You need not simplify any expression for which a calculator would be needed.

13) Consider the Law of sines. Create and solve a word problem where you know sides a and b and angle A, and you are solving for angle B.

You need not be original, but you must draw a picture and indicate what the two sides and the two angles mean with regard to the word problem. You must assign concrete numbers to a, b, A, and arrive at a concrete number for B.

How many answers do you get for B ? Answer this question by referring to any constraints in your problem.

You need not simplify any expression for which a calculator would be needed.

14) Essay: Suppose that $b > 1$ and $n > 1$ are fixed. Why is it that, given b and n, we can always find an x large enough such that the exponential function b^x is greater than the polynomial function x^n ? You may focus on a particular exponential function, such as e^x or 2^x. Make your arguments as mathematical as possible.

15) Let x and y be variables. Let a, b, c, d, e, f be constants.
You are given the equations $ax + by = e$ and $cx + dy = f$.
Solve for one of the variables, either x or y.

Having in mind the constants a, b, c, d, e, f, describe the conditions under which:

I) there is exactly one solution

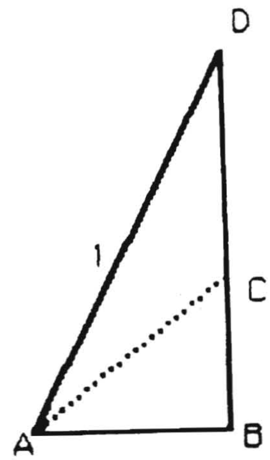
II) there are infinitely many solutions

III) there is no solution.

16) Derive the quadratic formula. That is, without using the quadratic formula, solve for x in the equation $ax^2+bx+c = 0$, where a, b, c are constants and $a \neq 0$.

17) Give a formula for sin(x+y) in terms of sin x, sin y, cos x, and cos y. Then use the geometry of right triangles to prove this formula.

You may find the diagram below helpful. As in the diagram, you may assume that x, y, and x+y are all acute angles. Feel free to add more lines to the diagram.



18) Prove that $\sqrt{2}$ is not a rational number. ($\sqrt{2}$ is defined to be the positive number such that its square is equal to 2.)